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On ϕ- Classes of Submodules

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Abstract

Let R be a commutative ring with identity and let M be a unitary R-module. Let S(M) be the set of all submodules of M and ϕ :S(M)$\rightarrow$ S(M) $\bigcup$ $\{\emptyset\}$ be a function. A proper submodule N of M is said to be a ϕ-prime (resp. a ϕ-primary) submodule if am $\in$ N-ϕ(N) for a $\in$ R, m $\in$ M implies that either m $\in$ N or a $\in$ (N : M) (resp. a $\in$ $\sqrt{(N:M)}$). These concepts were introduced by N. Zamani and M. Bataineh, in this paper, we study the concept of ϕ-primary submodule in details. Also, we introduce the concepts of ϕ-primal submodules and ϕ-2-absorbing submodules.

Keywords: ϕ-prime submodules, ϕ-primary submodules, ϕ-primal submodules, ϕ-prime to submodule, ϕ-2-absorbing submodules.

1 Introduction

Let R be a commutative ring with identity and let M be a unitary R - module. Let S(M) be the set of all submodules of M and ϕ :S(M)$\rightarrow$ S(M)$\bigcup$ $\{\emptyset\}$ be a function. A proper submodule N of M is said to be a ϕ-prime submodule if am $\in$ N - ϕ(N) for a $\in$ R, m $\in$ M implies that either m $\in$ N or a $\in$ (N : M). This definition was introduced by Zamani and Khaksari as a generalization of prime submodules that covers the definitions of prime, weakly prime, almost prime and m-almost prime submodules, see [14] and [20]. In our work, we study the concept of ϕ-primary submodule that was introduced in [15] in more details. We clarify that this definition is a generalized of primary submodules that covers the definition of primary, weakly primary, almost primary and m-almost primary submodules.
Let ϕ : J(R)$\longrightarrow$ J(R) $\bigcup$ $\{\emptyset\}$ be a function with J(R) the set of all ideals of R. Let I be an ideal of R, an element a $\in$ R is called ϕ-prime to I if ra $\in$ I -ϕ(I) (with r $\in$ R) implies that r $\in$ I. We denote by S_ϕ(I) the set of all elements of R that are not ϕ-prime to I. I is called a ϕ-primal ideal of R if the set P = S_ϕ(I) $\bigcup$ ϕ(I) forms an ideal of R. The concept of ϕ-primal ideal over commutative ring was introduced by Darani (see[8]). In our work, we generalize the concept of ϕ-primal ideal to ϕ-primal submodule. We also, introduce the concept of ϕ-2-absorbing submodules which is a generalization to 2-absorbing submodules.

2 Basic Concepts

In this section, we recall some basic definitions and study some important results that we need throughout this paper.

Definition 2.1. [17] Let M be an R-module. A proper submodule N of M is said to be a *prime submodule* if whenever rm $\in$ N for r $\in$ R and m $\in$ M we get either m $\in$ N or rM $\subseteq$ N (equivalent r $\in$ (N : M)).

Definition 2.2. [4] Let M be an R-module and N be a proper submodule of M. N is called a *weakly prime submodule* of M if, whenever r $\in$ R and m $\in$ M such that $0 \neq$ rm $\in$ N, then either m $\in$ N or r $\in$ (N : M).

Definition 2.3. [15] Let M be an R-module. A proper submodule N of M is called an *almost prime submodule* of M if, whenever r $\in$ R and m $\in$ M such that rm $\in$ N - (N : M)N, then either m $\in$ N or r $\in$ (N : M).

Definition 2.4. [18] Let M be an R-module. A proper submodule N of M is said to be a *primary submodule* if $rm \in N$ for $r \in R$ and $m \in M$ implies that either $m \in N$ or $r^n M \subseteq N$ for some positive integer n.

Definition 2.5. [3] A proper submodule N of a module M over a commutative ring R is said to be a *weakly primary submodule* if whenever $0 \neq rm \in N$, for some $r \in R$, $m \in M$, then $m \in N$ or $r^n M \subseteq N$ for some n $\in \mathbb{N}$.

Definition 2.6. [16] Let M be an R-module and N a proper submodule of M, N is called an *almost primary submodule* of M if, whenever r $\in$ R, m $\in$ M such that rm $\in N - (N : M)N$, then either m $\in$ N or r$\in \sqrt{(N : M)}$.

Definition 2.7. [12] Let M be an R-module and N a submodule of M. The element a $\in$ R is *(left) prime to N* if am $\in$ N (m $\in$ M) implies m $\in$ N. The subset A of R is *uniformly not prime to N*, if there exists an element u $\in M - N$ with Au $\subseteq$ N.

Definition 2.8. [12] Let M be an R-module and N a submodule of M. The *adjoint of N* is the set of all elements of R that are not prime to N and denoted by $adj(N)$. On other words, $adj(N) = \{$r $\in$ R : rm $\in$ N for some m $\in M - N\}$.

Definition 2.9. [12] Let M be an R-module. A proper submodule N of M is said to be *primal* if $adj(N)$ forms an ideal of R. In this case the adjoint of N will also be called the *adjoint ideal of N*.

Definition 2.10. [5] Let N be a submodule of an R-module M. An element r $\in$ R is called *weakly prime (simply wp) to N* if $0 \neq$ rm $\in$ N (m $\in$ M) implies that m $\in$ N. Otherwise r is not weakly prime (simply nwp) to N. Denote by W(N) the set of elements of R that are nwp to N.

Definition 2.11. [5] Let R be a commutative ring and let N be a proper submodule of an R-module M. N is called *weakly primal* if the set P = W(N) $\bigcup \{0\}$ forms an ideal of R. P is called the *(weakly) adjoint ideal of N* and we also say that N is a *P-weakly primal submodule of M.*

The concept of almost primal ideals in a commutative ring was introduced by A.Y. Darani in [11]. Let R be a ring and let I be a proper ideal of R. An element a $\in$ R is called almost prime to I if ra $\in$ I-I^2 (with r $\in$ R) implies that r $\in$ I. We denote by A(I) the set of all elements of R that are not almost prime to I. A proper ideal I is called almost primal if the set P = A(I) $\bigcup$ I^2 forms an ideal of R. This ideal P is an almost prime ideal of R, called the almost prime adjoint ideal of I. In this case we also say that I is a P-almost primal ideal. Now we give some definitions and result in almost primal submodules.

Definition 2.12. Let M be an R-module and N a submodule of M. The element a $\in$ R is *(left) almost prime to N* if am $\in$ N -(N : M)N (m $\in$ M) implies m $\in$ N. Denote by A(N) the set of elements of R that are not almost prime to N.

Definition 2.13. Let R be a commutative ring and let N be a proper submodule of an R-module M. N is called *almost primal* if the set P = A(N) $\bigcup$ $(N : M)N$ forms an ideal of R. P is called the *(almost) adjoint ideal of N* and we also say that N is a *P-almost primal submodule of M.*

Theorem 2.14. *Let P be an ideal of a commutative ring R. Let N be a proper submodule of R-module M. The following are equivalent:*
(1) N is P-almost primal.
(2) For every $x \notin P$ - $(N : M)N$, $(N : x) = N \bigcup ((N : M)N : x)$ and for $x \in P$ - $(N : M)N$, $(N : x) \supsetneqq N \bigcup((N : M)N : x)$.

Proof. (1) $\Longrightarrow$ (2) Assume that N is P-almost primal then P-(N : M)N = A(N). Let x $\notin$ P - (N : M)N then x is almost prime to N. Clearly N $\cup$ ((N : M)N : x) $\subseteq$ (N : x). For every m $\in$ (N : x), if mx $\in$ (N : M)N then m $\in$ ((N : M)N : x) and if mx $\notin$ (N : M)N then x is almost prime to N, gives m $\in$ N. Hence m $\in$ N $\bigcup$ ((N : M)N : x), that is (N : x)$\subseteq$ N $\cup$ ((N : M)N : x). Therefor (N : x) = N $\cup$ ((N : M)N : x). Now assume that x $\in P - (N : M)N$ then x is not almost prime to N so $\exists$ m $\in$ M-N such that xm $\in$ N - (N : M)N. So m $\in$ (N : x), but m $\notin$ ((N : M)N : x) nor m $\in$ N. Hence, (N : x) $\neq$ N $\bigcup$ ((N : M)N : x). However, it is clear that N $\bigcup$ ((N : M)N : x) $\subsetneqq$ (N : x).
(2) $\Longrightarrow$ (1) We want to prove that P-(N : M)N consists exactly of all elements of R that are not almost prime to N. Hence N is P-almost primal.
Let $x \notin P - (N : M)N$, then (N : x) = N $\cup$ ((N : M)N : x). We want to prove that x $\notin$ A(N). Let xm $\in$ N - (N : M)N with m $\in$ M. So, m $\in$ (N : x). By assumption, either (N : x) = N or (N : x) = ((N : M)N :x). As xm $\in$ N - (N : M)N, so m $\notin$ ((N : M)N : x). Thus, m $\in$ N and hence, x $\notin$ A(N). Conversely, let $x \in P - (N : M)N$, then (N :x) $\supsetneqq$ N $\bigcup$ ((N : M)N : x), so, $\exists$ m $\in$ (N : x) such that m $\notin$ (N $\bigcup$ ((N : M)N :x)). Therefore, m $\notin$ N and m $\notin$ ((N : M)N : x). Thus xm $\in N - (N : M)N$ with m $\notin$ N, so x is not almost prime to N and hence x $\in$ A(N). □

Proposition 2.15. *Let N be asubmodule of R-module M. If N is almost primal submodule, then $P = A(N) \bigcup (N : M)N$ is almost prime ideal of R.*

Proof. Suppose that r,s $\notin$ P, we show that either rs $\in P^2$ or rs $\notin$ P. Assume that rs $\notin P^2$. Let rsm $\in$ N-(N : M)N for some m $\in$ M. Then, by Theorem 2.14 gives that rm $\in$ (N : s)= N $\bigcup$((N : M)N : s) where rm $\notin$ ((N : M)N : s); hence rm $\in$ N which implies that rm $\in$ N -(N : M)N. Thus m $\in$ (N : r) = N $\bigcup$ ((N : M)N : r), and so m $\in$ N. Therefore, rs is almost prime to N and rs $\notin$ P as required. □

Definition 2.16. [1] Let R be ring. Let ϕ : I(R) $\to$ I(R) $\cup$ {$\emptyset$} be a function where I(R) is the set of all ideals of R. A proper ideal I of R is a *ϕ-prime ideal* if a, b $\in$ R with ab $\in$ I - ϕ(I) implies a $\in$ I or b $\in$ I.

Definition 2.17. [20] Let R be a commutative ring with identity and M be a unitary R-module. Let S(M) be the set of all submodules of M, and let ϕ : S(M) $\to$ S(M) $\cup$ {$\emptyset$} be a function. A proper submodule N of M is called *ϕ-prime submodule* if a $\in$ R, x $\in$ M with ax $\in$ N - $\phi(N)$ implies that a $\in$ (N : M) or x $\in$ N.

Definition 2.18. [9] Let R be a commutative ring with unity and M an R-module. A proper submodule N of M is said to be a *2-absorbing submodule* if whenever a, b $\in$ R and m $\in$ M with abm $\in$ N then ab $\in$ (N : M) or am $\in$ N or bm $\in$ N.

Definition 2.19. [9] Let R be a commutative ring and M an R-module. A proper submodule N of M is said to be *weakly 2-absorbing submodule* if whenever a, b $\in$ R, m $\in$ M with 0 $\neq$ abm $\in$ N then ab $\in$ (N : M) or am $\in$ N or bm $\in$ N.

The following proposition study the relations between the previous submodules, which were proved in [7], [3], [4], [16], [10], [9].

Proposition 2.20. *Let M be a module over a commutative ring and N a submodule of M. Then*
(1) N is prime $\to$ N is weakly prime submodule $\to$ N is almost prime submodule.
(2) N is primary $\to$ N is weakly primary submodule $\to$ N is almost primary submodule.

(3) N is almost prime submodule → N is almost primary submodule.
(4) N is prime submodule → N is primary submodule → N is primal submodule.
(5) N is prime submodule → N is 2-absorbing submodule → N is weakly 2-absorbing submodule.
(6) N is weakly prime submodule → N is weakly 2-absorbing submodule.

3 ϕ - Primary Submodules

Let S(M) be the set of all submodules of M, and ϕ : S(M) $\longrightarrow$ S(M) $\cup$ $\{\emptyset\}$ be a function. Then we have the following definition.

Definition 3.1. [7] A proper submodule N of M is called ϕ - *primary submodule* if a $\in$ R, x $\in$ M with ax $\in$ N - $\phi(N)$ implies that x $\in$ N or a^k $\in$ (N : M), for some positive integer k. In other word, x $\in$ N or a $\in$ $\sqrt{(N:M)}$.

Example 3.2. *Let R be a commutative ring. Let M be an R-module. Let S(M) be the set of all submodules of M. Define the following type of the functions ϕ_α : S(M) $\longrightarrow$ S(M) $\cup$ $\{\emptyset\}$ and the corresponding ϕ_α - primary submodules as follows :*
1) $\phi_\emptyset$: $\phi_\emptyset(N) = \emptyset$, $\forall$ $N \in$ S(M), defines primary submodules.
2) ϕ_0 : $\phi_0(N) = \{ 0 \}$, $\forall$ $N \in$ S(M), defines weakly primary submodules.
3) ϕ_1 : $\phi_1(N) = N$, $\forall$ $N \in S(M)$, defines any submodule N.
4) ϕ_2 : $\phi_2(N) = (N : M)N$, $\forall$ $N \in$ S(M), defines almost primary submodules.
5) ϕ_w : $\phi_w(N) = \bigcap_{i=1}^{\infty}(N : M)^i N$, $\forall$ $N \in$ S(M), defines ϕ_w-primary submodules.
6) ϕ_n : $\phi_n(N) = (N : M)^{n-1}N$, $n \geq 2$, $\forall$ $N \in$ S(M), defines n-almost primary submodules.

Remarks 3.3. (1) Since $N - \phi(N) = N - (N \cap \phi(N))$, so without loss of generality, throughout this thesis we will consider $\phi(N) \subseteq N$ for any N $\in$ S(M).
(2) For functions ϕ, ψ :S(M)$\longrightarrow$ S(M)$\cup$ $\{\emptyset\}$, we write $\phi \leq \psi$ if $\phi(N) \subseteq \psi(N)$ $\forall$ N $\in$ S(M).
(3)Observe that $\phi_\emptyset \leq \phi_0 \leq \phi_w \leq ... \leq \phi_{n+1} \leq \phi_n \leq ... \leq \phi_2 \leq \phi_1$.

Proposition 3.4. *Let R be a commutative ring and N be a submodule of R-module M.*
(1)Let ψ_1, ψ_2 : S(M)$\rightarrow$ S(M) $\cup$ $\{\emptyset\}$ be functions with $\psi_1 \leq \psi_2$. Then N is ψ_1-primary implies N is ψ_2-primary.
(2)Let ϕ : S(M)$\rightarrow$ S(M) $\cup$ $\{\emptyset\}$ be functions. If N is ϕ-prime then N is ϕ-primary.
(3)N is primary $\Longrightarrow$ N is weakly primary $\Longrightarrow$ N is ϕ_w-primary $\Longrightarrow$ N is ϕ_{n+1}-primary $\Longrightarrow$ ϕ_n-primary $(n \geq 2)$ $\Longrightarrow$ N is almost primary.

Proof. (1)Assume that N is ψ_1-primary. Let rm $\in N - \psi_2(N)$ for r $\in$ R, m $\in$ M then rm $\in N - \psi_1(N)$. Since N is ψ_1-primary, r^k $\in$ (N : M) for some $k \in \mathbb{N}$ or m $\in$ N. Hence N is ψ_2-primary.
(2) Is trivial and follows immediately from the definition.
(3) This follows from (1) and the ordering of the ϕ_α' s given in Remark 3.3. □

Theorem 3.5. *Let R be a commutative ring and M be an R-module. Let ϕ: S(M)$\rightarrow$ S(M) $\cup$ $\{\emptyset\}$ be a function. Let N be a ϕ-primary submodule of M. If (N : M)N $\nsubseteq \phi(N)$ then N is a primary submodule of M.*

Proof. Let a $\in$ R and x $\in$ M be such that ax $\in$ N. If ax $\notin$ ϕ(N), then since N is ϕ-primary, we have a^k $\in (N : M)$ for some k $\in \mathbb{N}$ or x $\in$ N. So let ax $\in \phi(N)$. In this case we may assume that aN $\subseteq \phi(N)$, because if aN $\nsubseteq \phi(N)$ then there exists p $\in$ N such that ap $\notin$ $\phi(N)$, so that a(x+p) $\in$ N - $\phi(N)$. Therefore a $\in \sqrt{(N:M)}$ or x + p $\in$ N and hence a $\in \sqrt{(N:M)}$ or x $\in$ N. Second we may assume that $(N : M)x \in \phi(N)$. If this is not the case, there exists u $\in (N : M)$ such that ux $\notin \phi(N)$ and so (a+ u)x $\in$ N - $\phi(N)$. Since N is a ϕ -primary submodule, we have a + u $\in \sqrt{(N:M)}$ or x $\in$ N. So a $\in \sqrt{(N:M)}$ or x $\in$ N. Now since $(N : M)$N $\nsubseteq \phi(N)$, there exist r $\in (N : M)$ and p $\in$ N such that rp $\notin \phi(N)$.

So (a + r)(x + p) $\in$ N - $\phi(N)$, and hence a + r $\in \sqrt{(N:M)}$ or x + p $\in$ N. Therefore a $\in \sqrt{(N:M)}$ or x $\in$ N. Thus N is primary submodule. □

Corollary 3.6. *Let N be a weakly primary submodule of M such that $(N : M)N \neq 0$. Then N is a primary submodule of M.*

Proof. In the above theorem, set $\phi = \phi_0$. □

Remark 3.7. Suppose that N is a ϕ-primary submodule of M such that $\phi(N) \subseteq (N : M)N$ (resp. $\phi(N) \subseteq (N : M)^2N$) and that N is not a primary submodule. Then by Theorem 3.5, we have $\phi(N) = (N : M)N$ (resp. $\phi(N) = (N : M)^2N$). In particular if N is a weakly primary (resp. ϕ_3 - primary) submodule but not primary submodule then $(N : M)N = 0$ (resp. $(N : M)N = (N : M)^2N$).

Theorem 3.8. [3] *Let $R = R_1 \times R_2$ where each R_i is a commutative ring with identity. Let M_i be R_i-module $\forall\ i \in \{1,2\}$, and $M = M_1 \times M_2$ be an R-module with $(r_1, r_2)(m_1, m_2) = (r_1m_1, r_2m_2)$, where $r_i \in R_i$, $m_i \in M_i$. Then,*
(1)If N_1 is a primary submodule of M_1, then $N_1 \times M_2$ is a primary submodule of M.
(2) If N_2 is a primary submodule of M_2, then $M_1 \times N_2$ is a primary submodule of M.

Remark 3.9. The above theorem is not true for correspondence ϕ - primary submodules in general, for example if N_1 is a ϕ_0-primary submodule of M_1 then $N_1 \times M_2$ is not necessarily a ϕ_0-primary submodule of $M_1 \times M_2$. Let $R_1 = R_2 = M_1 = M_2 = Z_{14}$, and suppose $N_1 = 0$. Then evidently N_1 is a ϕ_0-primary submodule of M_1. However, (2, 1)(7, 1) $\in N_1 \times M_2$, and (7, 1) $\notin N_1 \times M_2$. Also $(2,1)^k(2,1) \notin N_1 \times M_2$ for any k $\in \mathbb{N}$, $(2,1)^k$ M $\nsubseteq N_1 \times M_2$.

Proposition 3.10. *Let R_1 and R_2 be two commutative rings, with $R = R_1 \times R_2$, M_1 and M_2 be R_1 and R_2 - modules respectively. Let $M = M_1 \times M_2$ be an R-modules with $(r_1, r_2)(m_1, m_2) = (r_1m_1, r_2m_2)$ where $r_i \in R_i$, $m_i \in M_i$. Let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. Suppose that N_1 is a weakly primary submodule of M_1 such that $\{0\} \times M_2 \subseteq \phi(N_1 \times M_2)$. Then $N_1 \times M_2$ is a ϕ-primary submodule of $M_1 \times M_2$.*

Proof. Let $(r_1, r_2)(x_1, x_2) = (r_1x_1, r_2x_2) \in N_1 \times M_2$ - $\phi(N_1 \times M_2)$, but $N_1 \times M_2$ - $\phi(N_1 \times M_2)$ $\subseteq N_1 \times M_2$ - $\{0\} \times M_2 = (N_1 - \{0\}) \times M_2$. We have $r_1x_1 \in N_1 - \{0\}$ and by the assumption on N_1 we have $r_1^k \in (N_1 :_{R1} M_1)$ for some positive integer k or $x_1 \in N_1$. This gives that $(r_1, r_2)^k = (r_1^k, r_2^k) \in (N_1 :_{R_1} M_1) \times R_2 = (N_1 \times M_2 :_{R_1 \times R_2} M_1 \times M_2)$ for some positive integer k or $(x_1, x_2) \in N_1 \times M_2$. Therefore $N_1 \times M_2$ is a ϕ-primary submodule of $M_1 \times M_2$. □

Proposition 3.11. *Let R_1 and R_2 be two commutative rings, M_1 and M_2 be R_1 and R_2 - modules respectively. Let $M = M_1 \times M_2$ and $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. such that $\phi_w \leq \phi$. Then for any weakly primary submodule N_1 of M_1, $N_1 \times M_2$ is a ϕ - primary submodule of $M_1 \times M_2$.*

Proof. If N_1 is a primary submodule of M_1, then $N_1 \times M_2$ is primary submodule of M, (see Theorem 3.8), and so a ϕ - primary submodule of $M_1 \times M_2$. Suppose that N_1 is not a primary submodule of M_1. Then by Remark 3.7, we have $(N_1 :_{R_1} M_1)N_1 = \{0\}$. This gives that $(N_1 \times M_2 :_{R_1 \times R_2} M_1 \times M_2)^i(N_1 \times M_2) = [(N_1 :_{R_1} M_1)^i N_1] \times M_2 = \{0\} \times M_2$, for all i $\geq$ 1 and hence we have $\{0\} \times M_2 = \cap_{i=1}^{\infty}(N_1 \times M_2 :_{R_1 \times R_2} M_1 \times M_2)^i(N_1 \times M_2)$ $= \phi_w(N_1 \times M_2) \subseteq \phi(N_1 \times M_2)$, and by Proposition 3.10, we have $N_1 \times M_2$ is a ϕ-primary submodule of $M_1 \times M_2$. □

Theorem 3.12. *Let $R = R_1 \times R_2$ such that each R_i is a commutative ring with identity. Let M_i be R_i-module $\forall\ i \in \{1,2\}$, and $M = M_1 \times M_2$ with $(r_1, r_2)(m_1, m_2) = (r_1m_1, r_2m_2)$, be an R-module, where $r_i \in R_i$, $m_i \in M_i$ $\forall\ i \in \{1,2\}$, and let $\psi_i : S(M) \to S(M) \cup \{\emptyset\}$ be a functions, $\phi = \psi_1 \times \psi_2$.Then each of the following types are ϕ-primary submodules of $M_1 \times M_2$,*
(i) $N_1 \times N_2$ where N_i is a proper submodule of M_i, with $\psi_i(N_i) = N_i$.
(ii) $P_1 \times M_2$ where P_1 is a primary submodule of M_1.

(iii) $P_1 \times M_2$ *where* P_1 *is a* ψ_1*-primary submodule of* M_1 *and* $\psi_2(M_2) = M_2$.
(iv) $M_1 \times P_2$ *where* P_2 *is a primary submodule of* M_2.
(v) $M_1 \times P_2$ *where* P_2 *is a* ψ_2*-primary submodule of* M_2 *and* $\psi_1(M_1) = M_1$.

Proof. (i) is clear, since $N_1 \times N_2$ - $\phi(N_1 \times N_2) = \emptyset$
(ii) If P_1 is a primary submodule of M_1, then by Theorem 3.8, $P_1 \times M_2$ a primary submodule of $M_1 \times M_2$, and thus $P_1 \times M_2$ is ϕ-primary submodule of M.
(iii) Let P_1 be a ψ_1-primary submodule of M_1 and $\psi_2(M_2) = M_2$. Let $(r_1, r_2) \in R$ and $(x_1, x_2) \in M$ be such that $(r_1, r_2)(x_1, x_2) = (r_1x_1, r_2x_2) \in P_1 \times M_2$ - $\phi(P_1 \times M_2) = P_1 \times M_2$ - $\psi_1(P_1) \times \psi_2(M_2) = P_1 \times M_2$ - $\psi_1(P_1) \times M_2 = (P_1 - \psi_1(P_1)) \times M_2$. So $r_1x_1 \in P_1$ - $\psi_1(P_1)$ but P_1 is ψ_1 - primary submodule, so $r_1^k \in (P_1 :_{R_1} M_1)$ for some k $\in \mathbb{N}$ or $x_1 \in P_1$. Therefore $(r_1, r_2)^k \in (P_1 :_{R_1} M_1) \times R_2 = (P_1 \times M_2 :_{R_1 \times R_2} M_1 \times M_2)$ or $(x_1, x_2) \in P_1 \times M_2$. So $P_1 \times M_2$ is a ϕ-primary submodule of $M_1 \times M_2$.
Parts (iv), (v) are proved similar to (ii), (iii) respectively. □

Theorem 3.13. *Let N be a proper submodule of M and let* $\phi : S(M) \to S(M) \cup \{\emptyset\}$ *be a function. Then the following are equivalent:*
(i) N is ϕ *- primary submodule of M.*
(ii) For $r \in R - \sqrt{(N : M)}$, $(N : (r)) = N \cup (\phi(N) : (r))$.
(iii) For $r \in R - \sqrt{(N : M)}$, $(N : (r)) = N$ *or* $(N : (r)) = (\phi(N) : (r))$.

Proof. (i)$\Longrightarrow$(ii) Suppose that N is ϕ - primary such that r $\notin \sqrt{(N : M)}$. Let m $\in$ (N : (r)). So rm $\in$ N. If rm $\notin \phi(N)$, then N is ϕ - primary implies m $\in$ N, and if rm $\in \phi(N)$, then m $\in (\phi(N)$: (r)). Hence (N : (r)) $\subseteq$ N $\cup$ ($\phi(N)$: (r)). The other inclusion hold trivially, since $\phi(N) \subseteq$ N.
(ii) $\Longrightarrow$ (iii) It is clear because (N : (r)) is an ideal of R.
(iii) $\Longrightarrow$(i) Let r $\in$ R, m $\in$ M such that rm $\in$ N - $\phi(N)$. If r $\notin \sqrt{(N : M)}$, then by assumption, either (N : (r)) = N or (N : (r)) = ($\phi(N)$: (r)). As rm $\notin \phi(N)$, then m $\notin$ ($\phi(N)$: (r)) and as rm $\in$ N, then m $\in$ (N : (r)). Hence (N : (r)) = N, and so m $\in$ N as required. □

Theorem 3.14. *Let M be an R-module and let N be a proper submodule of M. If for any ideal I of R and submodule K of M with* $IK \subseteq N$ *and* $IK \nsubseteq \phi(N)$, *we have* $I \subseteq \sqrt{(N : M)}$ *or* $K \subseteq N$, *then N is* ϕ *- primary submodule of M.*

Proof. Suppose that rm $\in$ N - $\phi(N)$ for r $\in$ R and m $\in$ N. Then $(r)(m) = (rm) \subseteq N - \phi(N)$. By the assumption, either $(m) \subseteq N$ or $(r) \subseteq \sqrt{(N : M)}$. Therefore, $m \in N$ or $r \in \sqrt{(N : M)}$ and N is ϕ-primary submodule of M. □

Proposition 3.15. *Let N be a submodule of M with* $(N : M) = \sqrt{(N : M)}$, *then N is* ϕ*-primary if and only if N is* ϕ*-prime.*

Proof. Trivial from the definitions of ϕ-prime and ϕ-primary submodules. □

Theorem 3.16. *Let M be an R-module and let* $\phi : S(M) \longrightarrow S(M) \cup \{\emptyset\}$. *Let P be a* ϕ*-primary submodule of M.*
(i) If $L \subseteq P$ *is a submodule of M, then* P/L *is a* ϕ_L*-primary submodule of* M/L.
(ii) Suppose that S is a multiplicatively closed subset of R such that $S^{-1}P \neq S^{-1}M$ *and* $S^{-1}(\phi(P)) \subseteq (S^{-1}\phi)(S^{-1}P)$ *and* $(P : M) \cap S = \emptyset$. *Then* $S^{-1}P$ *is an* $(S^{-1}\phi)$*-primary submodule of* $S^{-1}M$.

Proof. (i) Let a $\in$ R and $\bar{x} \in$ M/L with a $\bar{x} \in$ P/L - $\phi_L(P/L)$, where $\bar{x}$ = x + L, for some x $\in$ M. By the definition of ϕ_L, this gives that ax $\in$ P - $\phi(P)$, which gives that $a^k \in (P : M)$ for some k $\in \mathbb{N}$ or x $\in$ P. Therefor $a^k \in (P/L : M/L)$ for some k $\in \mathbb{N}$ or $\overline{x} \in$ P/L and so P/L is ϕ_L-primary submodule.
(ii) Let a/s $\in S^{-1}R$ and x/t $\in S^{-1}M$ with ax/st $\in S^{-1}P$ - $(S^{-1}\phi)(S^{-1}P)$. Then by our assumption ax/st $\in S^{-1}P$ - $S^{-1}(\phi(P))$. Therefore there exists u $\in$ S such that uax $\in$ P -

$\phi(P)$, (note that for each v $\in$ S, vax $\notin \phi(P)$). Since P is ϕ - primary and $(P:M)\bigcap S = \emptyset$, we have $(ua)^k \in (P:M)$ for some k $\in \mathbb{N}$ or x $\in$ P. Therefore $(a/s)^k \in S^{-1}((P:_R M)) \subseteq (S^{-1}P :_{S^{-1}R} S^{-1}M)$ for some k $\in \mathbb{N}$ (because (P : M) $\subseteq (S^{-1}P : S^{-1}M)$) or x/t $\in S^{-1}P$. Hence $S^{-1}P$ is an $(S^{-1}\phi)$-primary submodule of $S^{-1}M$. □

4 ϕ - Primal Submodules

The concept of ϕ - primal ideals in a commutative ring was introduced by A.Y. Darani in [8]. Let R be a commutative ring with identity. Let $\phi : \mathbb{J}(R) \to \mathbb{J}(R) \cup \{\emptyset\}$ be a function where $\mathbb{J}$(R) denotes the set of all ideals of R. Let I be an ideal of R. An element a $\in$ R is called ϕ - prime to I if ra $\in$ I - $\phi(I)$ (with r $\in$ R) implies that r $\in$ I. We denote by S_ϕ(I) the set of all elements of R that are not ϕ - prime to I. I is called a ϕ - primal ideal of R if the set P = S_ϕ(I) $\bigcup \phi$(I) forms an ideal of R. In this case P is called the ϕ - prime adjoint ideal (simply adjoint ideal) of I, and I is called a P-ϕ-primal ideal of R.
Now we generalize the concept of ϕ -primal ideals to ϕ - primal submodules. Let M be R-module, let S(M) be the set of all submodule of M and ϕ : S(M) $\to$ S(M) $\cup \{\emptyset\}$ be a function.

Definition 4.1. Let N be a submodule of R-module M and ϕ : S(M) $\to$ S(M) $\cup \{\emptyset\}$ be a function. An element r $\in$ R is called *ϕ-prime to N* if rm $\in$ N - $\phi(N)$ (with m $\in$ M) implies that m $\in$ N. Otherwise r is not ϕ-prime to N.

Remarks 4.2. Let N be a submodule of R-module M. Denote by $S_\emptyset$(N) the set of all elements of R that are not ϕ- prime to N, then
(1) If an element of R is prime to N then it is ϕ - prime to N, so $S_\emptyset$(N) $\subseteq adj(N)$ = S(N).
(2) The converse of (1) is not necessarily true in general. For example consider the $\mathbb{Z}$-module M = $\mathbb{Z}/24\mathbb{Z}$, its submodule N = $8\mathbb{Z}/24\mathbb{Z}$ and assume that $\phi = \phi_0$ where ϕ_0(N) = 0. Denote each coset a + $24\mathbb{Z}$ in M by $\overline{a}$. Then, as $6.\overline{12} = \overline{0} \in$ N and $\overline{12} \in$ M - N, so 6 is not prime to N. But if $6.\overline{a} \in$ N for some $\overline{a} \in$ M, then 4 divides a. Hence $6.\overline{a} = \overline{0}$. This implies that 6 is ϕ_0-prime to N. Thus, we have $adj(N) \nsubseteq S_\emptyset$(N).

Definition 4.3. Let R be a commutative ring and let ϕ : S(M) $\to$ S(M) $\cup \{\emptyset\}$ be a function. A proper submodule N of M is said to be a *ϕ-primal* if the set P = S_ϕ(N) $\bigcup \phi$(N) forms an ideal of R.

Example 4.4. *Let R be a commutative ring. Let M be an R-module. Let S(M) be the set of all submodules of M. Define the following type of the functions ϕ_α : S(M) $\longrightarrow$ S(M) $\cup$ $\{\emptyset\}$ and the corresponding ϕ_α - primal submodules as follows :*
1) $\phi_\emptyset : \phi_\emptyset(N) = \emptyset$, $\forall$ $N \in$ S(M), defines primal submodules.
2) $\phi_0 : \phi_0(N) = \{ 0 \}$, $\forall$ $N \in$ S(M), then defines weakly primal submodules.
3) $\phi_1 : \phi_1(N) = N$, $\forall$ $N \in S(M)$, defines any submodule N.
4) $\phi_2 : \phi_2(N) = (N:M)N$, $\forall$ $N \in$ S(M), defines almost primal submodules.
5) $\phi_w : \phi_w(N) = \bigcap_{i=1}^{\infty}(N:M)^iN$, $\forall$ $N \in$ S(M), defines ϕ_w-primal submodule.
6) $\phi_n : \phi_n(N) = (N:M)^{n-1}N$, $\forall$ $n \geq 2$, $\forall$ $N \in$ S(M), defines n-almost primal submodules.

Theorem 4.5. *Let P be an ideal of a commutative ring R. Let N be a proper submodule of R-module M. Let ϕ : S(M) $\to$ S(M) $\cup \{\emptyset\}$ be a function. Then the following are equivalent:*
(1) N is P-ϕ-primal submodule.
(2) For every $x \notin P$ - $\phi(N)$, $(N:x) = N \bigcup (\phi(N):x)$ and for $x \in P$ - $\phi(N)$, $(N:x) \supsetneqq N \bigcup (\phi(N):x)$.
(3) For every $x \notin P$ - $\phi(N)$, $(N:x) = N$ or $(N:x) = (\phi(N):x)$ and for $x \in P$ - $\phi(N)$, $(N:x) \supsetneqq N$ and $(N:x) \supsetneqq (\phi(N):x)$.

Proof. (1 $\to$ 2) Assume that N is P-ϕ-primal submodule then P - ϕ(N) consists entirely of elements of R that are not ϕ - prime to N. Let x $\notin$ P - ϕ(N) then x is ϕ-prime to N. Clearly N $\cup$ $(\phi(N):x) \subseteq$ (N : x). On the other hand, for every m $\in$ (N : x), if mx $\in \phi(N)$ then m

$\in (\phi(N) : x)$ and if mx $\notin \phi(N)$ then x is ϕ - prime to N gives m $\in$ N. Hence m $\in$ N $\bigcup (\phi(N) : x)$, that is (N : x) $\subseteq$ N $\bigcup (\phi(N) : x)$. Therefor (N : x) = N $\bigcup (\phi(N) : x)$. Now assume that x $\in$ P - $\phi(N)$ then x is not ϕ - prime to N, so $\exists$ m $\in$ M - N such that mx $\in$ N-ϕ(N). Hence m $\in$ (N : x) - (N $\bigcup$ ($\phi(N)$: x)). Thus (N : x) $\supsetneqq$ N $\bigcup (\phi(N) : x)$
(2 $\rightarrow$ 3) It is clear because (N : x) is an ideal in R.
(3 $\rightarrow$ 1) We want to prove that P-$\phi(N)$ consists exactly of all elements of R that are not ϕ-prime to N. Hence N is P-ϕ-primal.
Let x $\notin$ P-$\phi(N)$, then (N : x) = N $\cup$ ($\phi(N)$: x). We want to prove that x $\notin S_\phi$(N). Let xm $\in$ N - $\phi(N)$ with m $\in$ M. So, m $\in$ (N : x). By assumption, either (N : x) = N or (N : x) = ($\phi(N)$:x). As xm $\in$ N-$\phi(N)$, so m $\notin$ ($\phi(N)$: x). Thus, m $\in$ N and hence, x $\notin S_\phi$(N). Conversely, let x $\in$ P - $\phi(N)$, then (N :x) $\supsetneqq$ N $\bigcup$ ($\phi(N)$: x), so, $\exists$ m $\in$ (N : x) such that m $\notin$ (N $\bigcup$ ($\phi(N)$: x)). Therefore, m $\notin$ N and m $\notin$ ($\phi(N)$: x). Thus xm $\in$ N-$\phi(N)$ with m $\notin$ N, so x is not ϕ-prime to N and hence x $\in S_\phi$(N). Hence N is P-ϕ-primal submodule. □

Proposition 4.6. *Let R be a commutative ring and M be R-module. Let $\phi : S(M) \rightarrow S(M) \cup \{\emptyset\}$ be a function. If N is ϕ-primal submodule of M then $P = S_\emptyset(N) \bigcup \phi(N)$ is ϕ-prime ideal of R.*

Proof. Suppose that r,s $\notin$ P, we show that either rs $\in \phi(P)$ or rs $\notin$ P. Assume that rs $\notin \phi(P)$. Let rsm $\in$ N - $\phi(N)$ for some m $\in$ M. Then, by Theorem 4.5 gives that rm $\in$ (N : s)= N $\bigcup$ ($\phi(N)$: s) where rm $\notin$ ($\phi(N)$: s); hence rm $\in$ N which implies that rm $\in$ N -$\phi(N)$. Thus m $\in$ (N : r) = N $\bigcup$ ($\phi(N)$: r), and so m $\in$ N. Therefore, rs is ϕ-prime to N and rs $\notin$ P as required. □

Notation 4.7. Let N be a ϕ-primal submodule of R-module M. By Proposition 4.6, P = $S_\emptyset(N) \bigcup \phi(N)$ is ϕ-prime ideal of R. In this case P is called the ϕ-prime adjoint ideal and N is called a *P-ϕ-primal submodule* of M.

The concepts " primal submodule" and " ϕ-primal submodule " are different. In fact, neither implies the other. We will show this by the following examples, in Example 4.8 below we give a primal submodule that is not ϕ - primal. An example of ϕ- primal submodule which is not primal is given in Example 4.9.

Example 4.8. [5],[8] *Consider the submodule $N = 8\mathbb{Z}/24\mathbb{Z}$ of $\mathbb{Z}$-module $M = \mathbb{Z}/24\mathbb{Z}$. Denote each coset a + 24$\mathbb{Z}$ in M by $\overline{a}$. Let $\phi = \phi_0$ (weakly primal).*
(1) since $0 \neq 2.\overline{4} \in N$ and $0 \neq 4.\overline{2} \in N$ with $\overline{2}, \overline{4} \in$ M-N we have 2,4 $\in S_{\phi_0}(N)$. If $6.\overline{a} \in N$ for some $\overline{a} \in M$ then 4 divides a and hence $6.\overline{a} = 0$. This shows that 2+4 = 6 is ϕ_0- prime to N so 6 $\notin S_{\phi_0}(N)$. Therefor $S_\phi(N) \bigcup \phi(N)$ is not an ideal of $\mathbb{Z}$, that is N is not ϕ-primal submodule of M.
(2) Now set $P = 2\mathbb{Z}/24\mathbb{Z}$ then every element of P is not prime to N. Assume that $\overline{a} \notin P$, if $\overline{a}.\overline{n} \in N$ for some $\overline{n} \in M$ then 8 divides n, that is $\overline{n} \in N$. Hence $\overline{a}$ is prime to N so $\overline{a} \notin S(N) = adj(N)$. So we have S(N) = P, that is N is P-primal submodule. This example show that a primal submodule need not necessarily be ϕ-primal.

Example 4.9. [5] *Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_6$ and denote every integer a modulo 6 by $\overline{a}$. Consider the submodule N = {0} of M and let $\phi = \phi_0$ then:*
(1) 0 is weakly prime to N so $S_{\phi_0}(N) = \emptyset$. Thus N is weakly primal submodule of M.
(2) Since $2.\overline{3} = \overline{0} \in N$ and $3.\overline{2} = \overline{0} \in N$, so 2,3 $\in S_\emptyset(N)$ while 3-2 = 1 is prime to N, so we have 1 $\notin$ S(N). Therefore N is not a primal submodule of M.
This example shows that ϕ -primal submodule need not necessarily be primal.

Theorem 4.10. *Let M be R-module and $\phi : S(M) \rightarrow S(M) \cup \{\emptyset\}$ be a function. Let N and L be submodules of M with $L \subseteq \phi(N)$ then N is a ϕ-primal submodule of M iff N/L is a ϕ_L-primal submodule of M/L.*

Proof. Assume N is P-ϕ-primal submodule. Suppose that a + L is an element of M/L that is not ϕ_L - prime to N/L, so there exists b $\in$ M - N with (a + L)(b + L) $\in$ N/L - ϕ_L(N/L).

In this case ab $\in$ N - ϕ(N) with b $\in$ M - N implies that a is not ϕ - prime to N. Hence a $\in S_\phi$(N) $\subseteq$ P and so a + L $\in$ P/L. Now assume that c + L $\in$ P/L then c $\in$ P = S_ϕ(N) $\cup$ ϕ(N). If c $\in \phi$(N) then c + L $\in \phi_L$(N/L), so assume that c $\in S_\phi$(N), that is c is not ϕ - prime to N then cd $\in$ N - ϕ(N) for some d $\in$ M - N. Consequently, (c + L)(d + L) $\in$ N/L -(ϕ(N)/L) = N/L - ϕ_L(N/L) with d + L $\in$ M/L - N/L. This implies that c + L is not ϕ_L - prime to N/L, so c + L $\in S_{\phi_L}$(N/L). We have already shown that P/L = S_{ϕ_L}(N/L) $\bigcup$ ϕ_L(N/L). Therefore N/L is ϕ_L - primal.
Conversely, suppose that N/L is ϕ_L - primal in M/L with the adjoint ideal P/L. For every a $\in$ P - ϕ(N) we have a + L $\in$ P/L - ϕ_L(N/L) = S_{ϕ_L}(N/L), so a + L is not ϕ_L-prime to N/L, thus (a + L)(b + L) $\in$ N/L - ϕ_L(N/L) for some b + L $\in$ M/L - N/L. In this case b $\in$ M - N and ab $\in$ N - ϕ(N) implies that a is not ϕ-prime to N. On the other hand, assume that c $\in$ R is not ϕ-prime to N then cd $\in$ N - ϕ(N) for some d $\in$ M - N so we have (c + L)(d + L) $\in$ N/L - ϕ_L(N/L) with d + L $\notin$ N/L, that is c + L is not ϕ_L-prime to N/L. Hence c + L $\in$ P/L - ϕ_L(N/L), so we have c $\in$ P - ϕ(N). It follows that P = S_ϕ(N) $\bigcup$ ϕ(N) which implies that N is P-ϕ-primal submodule of M. □

Remark 4.11. [5] Let R be a commutative ring, M an R-module and S a multiplicatively closed set in R. If K is a submodule of S^{-1}M, define K $\cap$ M = v^{-1}(K) = {m $\in$ M : m/1 $\in$ K}, where v : M $\to S^{-1}$M is the natural mapping m $\mapsto$ m/1. Clearly, K $\cap$ M is a submodule of M.

Proposition 4.12. *Let R be a commutative ring and S a multiplicatively closed subset of R. Let ϕ : S(M) $\longrightarrow$ S(M) $\cup$ {$\emptyset$} be a function. Let N be a P-ϕ-primal submodule of an R-module M with P $\bigcap$ S = $\emptyset$.*
(1) Let $\lambda = a/s \in S^{-1}N$ - $S^{-1}(\phi(N))$ (with a $\in$ M, s $\in$ S), then a $\in$ N.
(2) If $S^{-1}(\phi(N)) \neq S^{-1}N$ then N = $S^{-1}N \bigcap M$.

Proof. (1) Assume that λ = a/s $\in S^{-1}N$ - $S^{-1}(\phi(N))$ then a/s = b/t for some b $\in$ N, t $\in$ S. In this case, since us $\in$ S and b $\in$ N, then uta = usb $\in$ N for some u $\in$ S. If uta $\in \phi(N)$ then a/s = uta/uts $\in S^{-1}(\phi(N))$ which is a contradiction, so we have uta $\in$ N - $\phi(N)$. If a $\notin$ N then ut is not ϕ-prime to N, so ut $\in$ P $\bigcap$ S which contradicts the hypothesis. Therefore a $\in$ N.
(2) Let m $\in S^{-1}$N $\bigcap$ M then m/1 $\in S^{-1}N$, so $\exists$ s $\in$ S such that sm $\in$ N. If sm $\notin \phi$(N) and m $\notin$ N then s is not ϕ-prime to N, so s $\in$ P $\bigcap$ S, which a contradiction. Thus m $\in$ N. If sm $\in \phi$(N) then m/1 = sm/s $\in S^{-1}(\phi$(N)) which implies that m $\in S^{-1}(\phi$(N)) $\bigcap$ M. Therefore (S^{-1}N $\bigcap$ M) = N $\bigcup$ (S^{-1} (ϕ(N) $\bigcap$ M), so (S^{-1}N $\bigcap$ M) = N or (S^{-1} N $\bigcap$ M) = (($S^{-1}\phi$(N)) $\bigcap$ M). But S^{-1} N $\neq S^{-1}(\phi$(N)), so S^{-1} N $\bigcap$ M $\neq S^{-1}$ (ϕ(N)) $\bigcap$ M. Thus S^{-1}N $\bigcap$ M = N. □

5 ϕ-2-Absorbing Submodules

In this section, we introduce the concept of ϕ-2-absorbing submodules which is a generalization to concept of 2-absorbing submodules. Let R be a commutative ring with identity and M be a unitary R-module. Let S(M) be the set of all submodules of M, and ϕ : S(M) $\to$ S(M) $\cup$ {$\emptyset$} be a function.

Definition 5.1. A proper submodule N of M is called *ϕ-2-absorbing submodule* if r,s $\in$ R, m $\in$ M with rsm $\in$ N - $\phi(N)$ implies that rs $\in$ (N : M) or rm $\in$ N or sm $\in$ N.

Example 5.2. *Let R be a commutative ring. Let M be an R-module. Let S(M) be the set of all submodules of M. Define the following type of the functions ϕ_α : S(M) $\longrightarrow$ S(M) $\cup$ {$\emptyset$} and the corresponding ϕ_α - primal submodules as follows :*
1) $\phi_\emptyset$: $\phi_\emptyset(N) = \emptyset$, $\forall$ N $\in$ S(M), defines 2-absorbing submodules.
2) ϕ_0 : $\phi_0(N)$ = { 0 }, $\forall$ N $\in$ S(M), then defines weakly 2-absorbing submodules.
3) ϕ_1 : $\phi_1(N)$ = N, $\forall$ N $\in$ S(M), defines any submodule N.

4) $\phi_2 : \phi_2(N) = (N : M)N, \forall N \in S(M)$, *defines almost 2-absorbing submodules.*
5) $\phi_w : \phi_w(N) = \bigcap_{i=1}^{\infty}(N : M)^i N, \forall N \in S(M)$, *defines* ϕ_w*-2-absorbing submodule.*
6) $\phi_n : \phi_n(N) = (N : M)^{n-1}N, \forall n \geq 2, \forall N \in S(M)$, *defines n-almost 2-absorbing submodules.*

Remarks 5.3. (1) every 2-absorbing submodule is ϕ-2-absorbing submodule but the converse need not be true in general. For example, let M = $\mathbb{Z}_8$ be a $\mathbb{Z}$ module and let N = $\{0\}$. N is ϕ_0(weakly)-2-absorbing submodule but not 2-absorbing submodule because 2.2.2 = 0 $\in$ N and 4 $\notin$ N and 4 $\notin$ (N : M) = $\{8n : n \in \mathbb{Z}\}$.
(2) Observe that $\phi_\emptyset \leq \phi_0 \leq \phi_w \leq ... \leq \phi_{n+1} \leq \phi_n \leq ... \leq \phi_2 \leq \phi_1$.

Proposition 5.4. *Let R be a commutative ring and N be a submodule of R - module M.*
(1) Let $\psi_1, \psi_2 : S(M) \longrightarrow S(M) \cup \{\emptyset\}$ *be functions with* $\psi_1 \leq \psi_2$*. Then N is* ψ_1*-2-absorbing submodule implies N is* ψ_2*-2-absorbing.*
(2) N is 2-absorbing $\Longrightarrow$ *N is weakly 2-absorbing* $\Longrightarrow$ *N is* ϕ_w*- 2-absorbing* $\Longrightarrow$ *N is* ϕ_{n+1}*-2-absorbing* $\Longrightarrow$ ϕ_n*-2-absorbing(n≥2)*$\Longrightarrow$ *N is* ϕ_2*-2-absorbing.*

Proof. (1) Assume that N is ϕ_1-2-absorbing submodule of M. Let rsm $\in N - \phi_2(N)$ for r,s $\in$ R, m $\in$ M then rsm $\in N - \phi_1(N)$. Since N is ϕ_1 -2-absorbing, rs $\in$ (N : M) or rm $\in$ N or sm $\in$ N. Hence N is ϕ_2-2-absorbing submodule of M.
(2) This follows from (1) and the ordering of the $\phi_{\alpha}\prime$ s given in Example 5.2 and Remarks 5.3. ☐

Theorem 5.5. *Let* $\phi : S(M) \rightarrow S(M) \cup \{\emptyset\}$ *be function. Let N be a* ϕ*-2-absorbing submodule of M. If* $(N : M)N \nsubseteq \phi(N)$*, then N is a 2-absorbing submodule of M.*

Proof. Let r,s $\in$ R and m $\in$ M be such that rsm $\in$ N. If rsm $\notin \phi$(N) and since N is ϕ -2-absorbing then we have rs $\in (N :_R M)$ or rm $\in$ N or sm $\in$ N. So let rsm $\in \phi(N)$. In this case we may assume that rsN $\subseteq \phi(N)$. Because if rsN $\nsubseteq \phi(N)$, then there exists p $\in$ N such that rsp $\notin \phi(N)$, so that rs(m + p) $\in$ N - $\phi(N)$. Therefore rs $\in (N : M)$ or r(m + p) $\in$ N or s(m +p) $\in$ N and hence rs $\in (N : M)$ or rm $\in$ N or sm $\in$ N. Second we may assume that $(N : M)m \in \phi(N)$. If this is not the case, there exists u $\in (N : M)$ such that um $\notin \phi(N)$ and so (r + u)sm $\in N - \phi(N)$. Since N is a ϕ -2-absorbing submodule, we have (r + u)s $\in (N : M)$ or (r + u)m $\in$ N or sm $\in$ N. Thus rs $\in (N : M)$ or rm $\in$ N or sm $\in$ N. Now since $(N :_R M)$N $\nsubseteq \phi(N)$, there exist v $\in (N : M)$ and p $\in$ N such that vp $\notin \phi(N)$. So (r + v)s(m + p) $\in$ N - $\phi(N)$, and hence (r +v)s $\in (N : M)$ or (r + v)(m + p) $\in$ N or s(m + p) $\in$ N. Therefore rs $\in (N : M)$ or rm $\in$ N or sm $\in$ N. Thus N is 2-absorbing submodule. ☐

Corollary 5.6. *Let N be a weakly 2-absorbing submodule of M such that* $(N :_R M)N \neq 0$*. Then N is a 2-absorbing submodule of M.*

Proof. In the above Theorem set $\phi = \phi_0$. ☐

Remark 5.7. Suppose that N is a ϕ-2-absorbing submodule of M such that $\phi(N) \subseteq$ (N : M)N and N is not 2-absorbing submodule then by Theorem 5.5, we have $\phi(N)$ = (N : M)N. In particular if N is weakly 2-absorbing submodule but not 2-absorbing then (N : M)N = 0.

Theorem 5.8. [13] *Let* $R = R_1 \times R_2$ *such that each* R_i *is a commutative ring with identity. Let* M_i *be* R_i*-module* $\forall i \in \{1,2\}$ *and* $M = M_1 \times M_2$ *be an R-module with* $(r_1, r_2)(m_1, m_2) = (r_1m_1, r_2m_2)$*, where* $r_i \in R_i$*,* $m_i \in M_i \forall i \in \{1,2\}$*. Then we have:*
(1) If N_1 *is a 2-absorbing submodule of* M_1*, then* $N_1 \times M_2$ *is a 2-absorbing submodule of M.*
(2) If N_2 *is a 2-absorbing submodule of* M_2*, then* $M_1 \times N_2$ *is a 2-absorbing submodule of M.*

Proof. Because the proof of (1) and (2) are similar, So we only prove (1). Hence suppose that N_1 is a 2-absorbing submodule of M_1 and let $r_1, s_1 \in R_1$, r_2, $s_2 \in R_2$, $m_1 \in M_1$ and m_2

$\in M_2$ such that $(r_1, r_2)(s_1, s_2)(m_1, m_2) = (r_1s_1m_1, r_2s_2m_2) \in N_1 \times M_2$. then $r_1s_1m_1 \in N_1$. Since N_1 is 2-absorbing submodule of M_1, So $r_1s_1 \in (N_1 : M_1)$ or $r_1m_1 \in N_1$ or $s_1m_1 \in N_1$. So $(r_1, r_2)(s_1, s_2) = (r_1s_1, r_2s_2) \in (N_1 : M_1) \times (M_2 : M_2) = (N_1 \times M_2 : M_1 \times M_2)$ or $(r_1, r_{2)}(m_1, m_2) \in N_1 \times M_2$ or $(s_1, s_{2)}(m_1, m_2) \in N_1 \times M_2$. Hence $N_1 \times M_2$ is 2-absorbing submodule of M. □

Example 5.9. *The above theorem is not true for correspondence ϕ - 2-absorbing submodules in general, for example if N_1 is a ϕ_0-2-absorbing submodule of M_1 then $N_1 \times M_2$ is not necessarily a ϕ_0 - 2-absorbing submodule of $M_1 \times M_2$. Let $R_1 = R_2 = M_1 = M_2 = \mathbb{Z}_8$ and suppose that $N_1 = \{0\}$ then evidently N_1 is a ϕ_0-2-absorbing submodule of M_1. However, 0 $\neq$ (2,1)(2,1)(2,1) $\in N_1 \times M_2$ and (2, 1)(2,1) = (4,1) $\notin (N_1 \times M_2 : M_1 \times M_2)$ and (2,1)(2,1) $\notin N_1 \times M_2$. Thus $N_1 \times M_2$ is not ϕ_0-absorbing submodule of M.*

Proposition 5.10. *Let R_1 and R_2 be two commutative rings, M_1 and M_2 be R_1 and R_2 - modules respectively. Let $M = M_1 \times M_2$ and define $\phi : S(M) \longrightarrow S(M) \cup \{\emptyset\}$ be a function. Suppose that N_1 is a weakly 2-absorbing submodule of M_1 such that $\{0\} \times M_2 \subseteq \phi(N_1 \times M_2)$. Then $N_1 \times M_2$ is a ϕ-2-absorbing submodule of $M_1 \times M_2$.*

Proof. Let $r_1, s_1 \in R_1$, $r_2, s_2 \in R_2$, $x_1 \in M_1$ and $x_2 \in M_2$. Let $(r_1, r_2)(s_1, s_2)(x_1, x_2) = (r_1s_1x_1, r_2s_2x_2) \in N_1 \times M_2$ - $\phi(N_1 \times M_2)$. Since $N_1 \times M_2$ - $\phi(N_1 \times M_2) \subseteq N_1 \times M_2$ - $\{0\} \times M_2 = (N_1 - \{0\}) \times M_2$, so we have $r_1s_1x_1 \in N_1 - \{0\}$ and by the assumption on N_1 we have $r_1s_1 \in (N_1 :_{R1} M_1)$ or $r_1x_1 \in N_1$ or $s_1x_1 \in N_1$. If $r_1s_1 \in (N_1 :_{R_1} M_1)$ then $(r_1, r_2)(s_1, s_2) = (r_1s_1, r_2s_2) \in (N_1 :_{R_1} M_1) \times R_2 = (N_1 \times M_2 :_{R_1 \times R_2} M_1 \times M_2)$. If $r_1x_2 \in N_1$ then $(r_1, r_2)(x_1, x_2) = (r_1x_1, r_2x_2) \in N_1 \times M_2$. If $s_1x_1 \in N_1$ then $(s_1, s_2)(x_{1,2}) = (s_1x_1, s_2x_2) \in N_1 \times M_2$. Therefore $N_1 \times M_2$ is ϕ-2-absorbing submodule of M. □

In the next theorem we give characterizations of ϕ -2-absorbing submodules.

Theorem 5.11. *Let N be a proper submodule of M and let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. Then the following are equivalent:*
(i) N is a ϕ-2-absorbing submodule of M;
(ii) for any r,s $\in$ R, with rs $\notin$ (N : M), we have $(N : rs) = (N : r) \bigcup (N : s) \bigcup (\phi(N) : rs)$;
(iii) for any r,s $\in$ R, with rs $\notin$ (N : M), we have, $(N : rs) = (N : r)$ or $(N : rs) = (\phi(N) : s)$ or (N : rs) = $(\phi(N)$: rs).

Proof. (i)$\Longrightarrow$(ii) Let m $\in$ (N : rs) then rsm $\in$ N. If rsm $\notin \phi$(N) then N is a ϕ-2-absorbing submodule of M implies rm $\in$ N or sm $\in$ N, that is m $\in$ (N : r) or m $\in$ (N : s). If rsm $\in$ ϕ(N) then m $\in$ (ϕ(N) : rs). As we may assume that ϕ(N) $\subseteq$ N, the other inclusion always hold.
(ii) $\longrightarrow$ (iii) If an ideal is the union of two ideals, it is equal to one of them.
(iii)$\Longrightarrow$(i) Let rsm $\in$ N - ϕ(N) with rs $\notin$ (N : M) then m $\in$ (N : rs) and m $\notin$ (ϕ(N) :rs), so m $\in$ (N : r) or m $\in$ (N : s) that is, rm $\in$ N or sm $\in$ N. □

Theorem 5.12. *Let M be an R-module and let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. Let N be a ϕ-2-absorbing submodule of M.*
(i) If $L \subseteq N$ is a submodule of M, then N/L is a ϕ_L-2-absorbing submodule of M/L.
(ii) Suppose that S is a multiplicatively closed subset of R such that $S^{-1}N \neq S^{-1}M$ and $S^{-1}(\phi(N)) \subseteq (S^{-1}\phi)(S^{-1}N)$ with $(N :_R M) \cap S = \emptyset$. Let $S^{-1}\phi : S(S^{-1}M) \longrightarrow S(S^{-1}M) \bigcup \{\emptyset\}$. Then $S^{-1}N$ is an $(S^{-1}\phi)$-2-absorbing submodule of $S^{-1}M$.

Proof. (i) Let r,s $\in$ R and $\bar{x} \in$ M/L with rs$\bar{x} \in$ N/L - $\phi_L(N/L)$, where $\bar{x}$ = x + L, for some x $\in$ M. By the definition of ϕ_L, this gives that rsx $\in$ N - $(\phi(N) + L)$. So we have rsx $\in$ N - $\phi(N)$, which gives that rs $\in (N : M)$ or rx$\in$ N or sx $\in$ N. Therefore rs $\in (N/L : M/L)$ or $r\bar{x} \in$ N/L or $s\bar{x} \in$ N/L and so N/L is ϕ_L - 2-absorbing submodule.
(ii) Let a/s , b/w $\in S^{-1}R$ and x/t $\in S^{-1}M$ with abx/swt $\in S^{-1}N$ - $(S^{-1}\phi)(S^{-1}N)$. Then by our assumption abx/swt $\in S^{-1}N$ - $S^{-1}(\phi(N))$. Therefore there exists u $\in$ S such that

uabx $\in$ N - $\phi(N)$, (note that for each v $\in$ S, vabx $\notin \phi(N)$). Since N is ϕ-2-absorbing submodule and $(N : M) \bigcap S = \emptyset$, we have uab $\in$ (N : M) or ax $\in$ N or bx $\in$ N. Therefore ab/sw $\in S^{-1}(N :_R M) \subseteq (S^{-1}N :_{S^{-1}R} S^{-1}M)$ or ax/st $\in S^{-1}N$ or bx/wt $\in S^{-1}N$. Hence $S^{-1}N$ is an $(S^{-1}\phi)$-2-absorbing submodule of $S^{-1}M$. □

Proposition 5.13. *Let $R = R_1 \times R_2 \times ... \times R_n$ and $M = M_1 \times M_2 \times ... \times M_n$ be an R - module, where R_i is a commutative ring and M_i is an R_i - module, for each $i \in \{1, 2, ..., n\}$. Let $N = N_1 \times N_2 \times ... \times N_n$ be a ϕ - 2-absorbing submodule of M, where N_i is a submodule of M_i and let $\psi_i : S(M_i) \longrightarrow S(M_i) \cup \{\emptyset\}$ $\forall$ $i \in \{1,2,...,n\}$ and $\phi(N) = \psi_1(N_1) \times \psi_2(N_2) \times ... \times \psi_n(N_n)$. Then N_j is a ψ_j -2-absorbing submodule of M_j, for each j with $N_j \neq M_j$.*

Proof. Let $x_j \in M_j$ and $a_j, b_j \in R_j$ such that $a_j b_j\ x_j \in N_j$ - $\psi_j(N_j)$. Thus (1,1,...,1,a_j,...,1).(1,1,...,1,b_j,...,1).(0,0,...,0,x_j,...,0)= (0,0,...,0,$a_j b_j x_j$,...,0)$\in N$ - $\phi(N)$, but N is ϕ-2-absorbing submodule. Therefore, (1,1,...,1,a_j ,...,1).(1,1,...,1,b_j,...,1) $\in$ (N : M) or (1,1,...,1,a_j,...,1).(0,0,...,0,x_j,0,...,0) $\in$ N or (1,1,...,1,b_j,...,1)(0,0,...,0,x_j,0,...,0) $\in$ N. So we have $a_j b_j \in (N_j : M_j)$ or $a_j x_j \in N_j$ or $b_j x_j \in N_j$. Thus N_j is ψ_j-2-absorbing submodule for each j. □

Corollary 5.14. *Let $R = R_1 \times R_2 \times ... \times R_n$ and $M = M_1 \times M_2 \times ... \times M_n$ an R-module and $N = N_1 \times N_2 \times ... \times N_n$, where R_i is a commutative ring and M_i is an R_i - module and N_i is a submodule of M_i, for $i \in \{1, 2, ..., n\}$. Let N be a ϕ_n -2-absorbing submodule of M. Then N_j is a ϕ_n -2-absorbing submodule of M_j, for each j with $N_j \neq M_j$ and $n \geq 2$.*

Proof. We have $\phi_n(N)$ = (N:M)$^{n-1}$N = $(N_1$:M$)^{n-1}N_1 \times (N_2$:M$)^{n-1}N_2 \times ... \times (N_n$:M$)^{n-1}N_n$ $= \phi_n(N_1) \times \phi_n(N_2) \times ... \times \phi_n(N_n)$. So the result follows by Proposition 5.13. □

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