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On Pseudo Fuzzy Length Space and Quotient of Fuzzy Length Space

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Abstract

In this paper we recall the definition of fuzzy length space on a fuzzy set after that we recall basic definitions and properties of this space. Then we introduce the notion pseudo fuzzy length space on a fuzzy set to prove that the fuzzy completion of pseudo fuzzy length is a fuzzy length space. Finally we defined the quotient of a fuzzy length space then we defined the fuzzy length to the quotient space.

Keywords: Fuzzy length space on fuzzy set; Pseudo fuzzy length space on a fuzzy set; Fuzzy continuous operator.

1. Introduction

The theory of fuzzy set was introduced by Zadeh in 1965[1]. In 1984[2], Katsaras is the first one who introduced the notion of fuzzy norm on a linear space during his studying the notion fuzzy topological vector spaces. In 1984 Kaleva and Seikkala [3] introduced a fuzzy metric space. In 1992 Felbin [4] introduced the notion of fuzzy norm on a linear space so that the corresponding fuzzy metric is of Kaleva and Seikkala type Kramosil and Michalek introduced another idea of fuzzy metric space [5]. In 1994 Cheng and Mordeson [6] introduced the notion fuzzy norm on a linear space so that the corresponding fuzzy metric is of Kramosil and Michalek type. Bag and Samanta [7] in 2003 studied finite dimensional fuzzy normed linear spaces. In 2005 Saadati and Vaezpour [8] studied some results on fuzzy complete fuzzy normed spaces. In 2005 Bag and Samanta [9] studied fuzzy bounded linear proved the fixed point theorems on fuzzy normed linear spaces. In 2009 Sadeqi and Kia [10] studied fuzzy normed linear space and its topological structure. In 2010 Si, Cao and Yang [11] studied the continuity in an intuitionistic fuzzy normed space. In 2015 Nadaban [12] studied properties of fuzzy continuous mapping on a fuzzy normed linear spaces. The concept of fuzzy norm has been used in developing the fuzzy functional analysis and its applications and a large number of papers by different authors have been published for reference please see [13,14,15,16,17,18,19,20,21,22].

In the present paper we recall the definition of fuzzy length on a fuzzy set. The structure of this paper is as follows: In section two we recall basic properties of fuzzy length space on a fuzzy set that's will be needed later. In section three we introduced the notion of pseudo fuzzy length space and we proved some properties of this space. Finally in the last section we defined the quotient of fuzzy length space and we defined the fuzzy length of the quotient space.

2. Basic Concept about fuzzy set

Definition 2.1:[1]

Let U be a classical set of object, called the universal set, whose generic elements are denoted by x.

The membership in a classical subject A of U is often viewed as a characteristic function μ_A from U onto {0, 1} such that μ_A=1 if x $\in$ A and μ_A=0 if x $\notin$ A, {0, 1} is called a valuation set. If a valuation set is allowed to be real interval [0, 1] then A is called a fuzzy set which is denoted in this case by $\widetilde{A}$ and $\mu_{\widetilde{A}}$ is the grade of membership of x in $\widetilde{A}$. Also, it is remarkable that the closer the value of $\mu_{\widetilde{A}}(x)$ to 1, the more belong to $\widetilde{A}$. Clearly, $\widetilde{A}$ is a subset of U that has no sharp boundary. The fuzzy set $\widetilde{A}$ is completely characterized by the set of pairs: $\widetilde{A}$ = { (x, $\mu_{\widetilde{A}}(x)$): x $\in$ U, 0$\leq$ $\mu_{\widetilde{A}}(x)$ $\leq$1}.

Definition 2.2:[11]

Suppose that $\widetilde{D}$ and $\widetilde{B}$ be two fuzzy sets in V$\neq \emptyset$ and W$\neq \emptyset$ respectively then $\widetilde{D} \times \widetilde{B}$ is a fuzzy set whose membership is defined by: $\mu_{\widetilde{D}\times\widetilde{B}}$(d, b) = $\mu_{\widetilde{D}}$(d) $\wedge$ $\mu_{\widetilde{B}}$(b) $\forall$ (d, b) $\in$ V$\times$W.

Definition 2.3:[18]

A fuzzy point p in U is a fuzzy set with single element and is denoted by x_α or (x,α).

Two fuzzy points x_α and y_β are said to be different if and only if x $\neq$ y.

Definition 2.4:[7]

Suppose that d_β is a fuzzy point and $\widetilde{D}$ is a fuzzy set in U. then d_β is said to belong to $\widetilde{D}$ which is written by $d_\beta \in \widetilde{D} \Leftrightarrow$ $\mu_{\widetilde{D}}$(x) > β.

Proposition 2.5:[22]

Suppose that h: V$\rightarrow$ W is a function. Then the image of the fuzzy point d_β in V, is the fuzzy point h(d_β) in W with h(d_β)=(h(d), β).

Definition 2.6:[23]

A binary operation $*$: $[0,1]^2 \rightarrow [0,1]$ is said to be t-norm (or continuous triangular norm) if $\forall$ p, q, t, r $\in$ [0, 1] the conditions are satisfied : (i) p$*$q = q$*$p (ii) p$*$1 = p (iii) (p$*$q) $*$ t = p$*$(q$*$t)

(iv) If p $\leq$ q and t $\leq$ r then p$*$t $\leq$ q$*$r.

Examples 2.7:[23]

When p$*$q = $p.q$ and p$*$q=p$\wedge$q $\forall$p, q $\in$[0, 1] then $*$ is a continuous t-norm.

Remark 2.8:[23]

$\forall$ p>q, there is t such that p$*$t $\geq$ q and for every r, there is e such that r$*$ r $\geq$ e , where p, q, t, r and e belongs to [0, 1].

First we recall the main definition in this paper

Definition 2.9:[24]

Let U be a linear space over field $\mathbb{F}$ and let $\widetilde{A}$ be a fuzzy set in X. let $*$ be a t-norm and $\widetilde{F}$ be a fuzzy set from $\widetilde{A}$ to [0,1] such that:

(FL$_1$) $\widetilde{F}(x_\alpha) > 0$ for all $x_\alpha \in \widetilde{A}$.

(FL$_2$) $\widetilde{F}(x_\alpha) = 1$ if and only if $x_\alpha = 0$.

(FL$_3$) $\widetilde{F}(cx, \alpha) = \widetilde{F}(x, \frac{\alpha}{|c|})$, where $0 \neq c \in \mathbb{F}$.

(FL$_4$) $\widetilde{F}(x_\alpha + y_\beta) \geq \widetilde{F}(x_\alpha) * \widetilde{F}(y_\beta)$.

(FL$_5$) $\widetilde{F}$ is a continuous fuzzy set for all $x_\alpha, y_\beta \in \widetilde{A}$ and α, β ∈ [0,1].

Then the triple $(\widetilde{A}, \widetilde{F}, *)$ is called a fuzzy length space on the fuzzy *set* $\widetilde{A}$.

Definition 2.10:[24]

Suppose that $(\widetilde{D}, \widetilde{F}, *)$ is a fuzzy length space on the fuzzy set $\widetilde{D}$ then $\widetilde{F}$ is continuous fuzzy set if whenever $\{(x_n, \alpha_n)\} \to x_\alpha$ in $\widetilde{D}$ then $\widetilde{F}\{(x_n, \alpha_n)\} \to \widetilde{F}(x_\alpha)$ that is $\lim_{n\to\infty} \widetilde{F}[(x_n, \alpha_n)] = \widetilde{F}(x_\alpha)$.

Proposition 2.11:[24]

Let $(U, \|.\|)$ be a normed space, suppose that $\widetilde{D}$ is a fuzzy set in U. Put $\|x_\alpha\| = \|x\|$. Then $(\widetilde{D}, \|.\|)$ is a normed space.

Example 2.12:[24]

Suppose that $(U, \|.\|)$ is a normed space and assume that $\widetilde{D}$ is a fuzzy set in U. Put $p * q = p.q$ for all p, $q \in [0,1]$. Define $\widetilde{F}_{\|.\|}(x_\alpha) = \frac{\alpha}{\alpha + \|x\|}$. Then $(\widetilde{D}, \widetilde{F}_{\|.\|}, *)$ is a fuzzy length space on the fuzzy set $\widetilde{D}$, is called the fuzzy length Induced by $\|.\|$

Definition 2.13:[24]

Let $\widetilde{A}$ be a fuzzy set in U, and assume that $(\widetilde{A}, \widetilde{F}, *)$ is a fuzzy length space on the fuzzy *set* $\widetilde{A}$.

Let $\widetilde{B}(x_\alpha, r) = \{y_\beta \in \widetilde{A} : \widetilde{F}(y_\beta - x_\alpha) > (1-p)\}$.So $\widetilde{B}(x_\alpha, p)$ is said to be afuzzy open fuzzy ball of center $x_\alpha \in \widetilde{A}$ and radius r.

Definition 2.14:[24]

The sequence $\{(x_n, \alpha_n)\}$ in a fuzzy length space $(\widetilde{A}, \widetilde{F}, *)$ on the fuzzy *set* $\widetilde{A}$ is fuzzy converges to a fuzzy point $x_\alpha \in \widetilde{A}$ if for a given ε, 0 < ε < 1, then there exists a positive number K such that $\widetilde{F}[(x_n, \alpha_n) - x_\alpha] > (1 - \varepsilon)$ $\forall$ n ≥ K.

Definition 2.15:[24]

The sequence $\{(x_n, \alpha_n)\}$ in a fuzzy length space $(\widetilde{A}, \widetilde{F}, *)$ on the fuzzy *set* $\widetilde{A}$ is fuzzy converges to a fuzzy point $x_\alpha \in \widetilde{A}$ if $\lim_{n\to\infty} \widetilde{F}[(x_n, \alpha_n) - x_\alpha] = 1$.

Theorem 2.16:[24]

The two Definitions 1.14 and 1.15 are equivalent.

Lemma 2.17:[24]

Suppose that $(\widetilde{A},\widetilde{F},*)$ is a fuzzy length space on the fuzzy set $\widetilde{A}$. Then $\widetilde{F}(x_\alpha - y_\beta) = \widetilde{F}(y_\beta - x_\alpha)$, for any $x_\alpha, y_\beta \in \widetilde{A}$.

Definition 2.18:[24]

Suppose that $(\widetilde{A},\widetilde{F},*)$ is a fuzzy length space and $\widetilde{D} \subseteq \widetilde{A}$ then $\widetilde{D}$ is called fuzzy open if for every $y_\beta \in \widetilde{D}$ there is $\widetilde{B}(y_\beta,q) \subseteq \widetilde{D}$. A subset $\widetilde{E} \subseteq \widetilde{A}$ is called fuzzy closed if $\widetilde{E}^c = \widetilde{A} - \widetilde{E}$ is fuzzy open.

Theorem 2.19:[24]

Any $\widetilde{B}(y_\beta,q)$ in a fuzzy length space $(\widetilde{A},\widetilde{F},*)$ is a fuzzy open.

Definition 2.20: [24]

Suppose that $(\widetilde{A},\widetilde{F},*)$ is a fuzzy length space, and assume that $\widetilde{D} \subseteq \widetilde{A}$.Then the fuzzy closure of $\widetilde{D}$ is denoted by $\overline{\widetilde{D}}$ or FC($\widetilde{D}$) and is defined by $\overline{\widetilde{D}}$ is the smallest fuzzy closed fuzzy set that contains $\widetilde{D}$.

Definition 2.21:[24]

Suppose that $(\widetilde{A},\widetilde{F},*)$ is a fuzzy length space, and assume that $\widetilde{D} \subseteq \widetilde{A}$.Then $\widetilde{D}$ is said to be fuzzy dense in $\widetilde{A}$ if $\overline{\widetilde{D}} = \widetilde{A}$ or $\mathrm{FC}(\widetilde{D}) = \widetilde{A}$.

Lemma 2.22:[24]

Suppose that $(\widetilde{A},\widetilde{F},*)$ is a fuzzy length space, and assume that $\widetilde{D} \subseteq \widetilde{A}$, Then $d_\alpha \in \overline{\widetilde{D}}$ if and only if we can find $\{(d_n,\propto_n)\}$ in $\widetilde{D}$ such that $(d_n,\propto_n) \to d_\alpha$.

Theorem 2.23:[24]

Suppose that $(\widetilde{A},\widetilde{F},*)$ is a fuzzy length space and let that $\widetilde{D} \subseteq \widetilde{A}$, then $\widetilde{D}$ is fuzzy dense in $\widetilde{A}$ if and only if for any $a_\alpha \in \widetilde{A}$ we can find $d_\beta \in \widetilde{D}$ with $\widetilde{F}[a_\alpha - d_\beta] > (1-\varepsilon)$ for some $0 < \varepsilon < 1$.

Definition 2.24:[24]

Suppose that $(\widetilde{A},\widetilde{F},*)$ is a fuzzy length space. A sequence of fuzzy points $\{(x_n,\propto_n)\}$ is said to be a fuzzy Cauchy if for any given ε, $0 < \varepsilon < 1$, there is a positive number K such that $\widetilde{F}[(x_n,\propto_n) - (x_m,\propto_m)] > (1-\varepsilon)$ for all n, m $\geq$ K.

Definition 2.25:[24]

Suppose that $(\widetilde{A},\widetilde{F},*)$ is a fuzzy length space and $\widetilde{D} \subseteq \widetilde{A}$. Then $\widetilde{D}$ is said to be fuzzy bounded if we can find q ,$0 < q < 1$ such that, $\widetilde{F}(x_\alpha) > (1-q)$, $\forall\, x_\alpha \in \widetilde{A}$.

Definition 2.26:[24]

Let $(\widetilde{A},\widetilde{F}_{\widetilde{A}},*)$ and $(\widetilde{D},\widetilde{F}_{\widetilde{D}},*)$ be two fuzzy length space on fuzzy set $\widetilde{A}$ and $\widetilde{D}$ respectively, let $\widetilde{E} \subseteq \widetilde{A}$ then The operator T: $\widetilde{E} \to \widetilde{D}$ is said to be fuzzy continuous at $a_\alpha \in \widetilde{E}$, if for every $0 < \varepsilon < 1$, there exist $0 < \delta < 1$, such that $\widetilde{F}_{\widetilde{D}}[T(x_\beta) - T(a_\alpha)] > (1-\varepsilon)$ whenever $x_\beta \in \widetilde{E}$ satisfying $\widetilde{F}_{\widetilde{A}}(x_\beta - a_\alpha) > (1-\delta)$. If T is fuzzy continuous at every fuzzy point of $\widetilde{E}$, then T it is said to be fuzzy continuous on $\widetilde{E}$.

Theorem 2.27:[24]

Let $(\tilde{A}, \tilde{F}_{\tilde{A}}, *)$ and $(\tilde{D}, \tilde{F}_{\tilde{D}}, *)$ be two fuzzy length space, let $\tilde{E} \subseteq \tilde{A}$ The operator T: $\tilde{E} \rightarrow \tilde{D}$ is fuzzy continuous at $a_\alpha \in \tilde{E}$ if and only if whenever a sequence of fuzzy points $\{(x_n, \alpha_n)\}$ in $\tilde{E}$ fuzzy converge to a_α, then the sequence of fuzzy points $\{(T(x_n), \alpha_n)\}$ fuzzy converges to $T(a_\alpha)$.

Theorem 2.28:[24]

An operator $T: \tilde{A} \rightarrow \tilde{D}$ is fuzzy continuous if and only if $T^{-1}(\tilde{G})$ is fuzzy open in $\tilde{A}$ for all fuzzy open $\tilde{G}$ of $\tilde{D}$ where $(\tilde{A}, \tilde{F}_{\tilde{A}}, *)$ and $(\tilde{D}, \tilde{F}_{\tilde{D}}, *)$ are fuzzy length space

Theorem 2.29:[24]

The fuzzy length space $(\tilde{A}, \tilde{F}, *)$ is fuzzy compact if and only if every sequence of fuzzy points in $\tilde{A}$ has a subsequence fuzzy converging to a fuzzy point in $\tilde{A}$.

Theorem 2.30:[24]

A fuzzy subspace $\tilde{D}$ of a fuzzy complete fuzzy length space $(\tilde{A}, \tilde{F}, *)$ is itself fuzzy complete if and only if $\tilde{D}$ is fuzzy closed in $(\tilde{A}, \tilde{F}, *)$.

3. Pseudo Fuzzy Length Space

Let V be a vector space over the field $\mathbb{F}[\mathbb{F} = \mathbb{R} \text{ } or \text{ } \mathbb{F} = \mathbb{C}]$, and let $\tilde{A}$ be fuzzy set in V.

Definition 3.1:

A triple $(\tilde{A}, \tilde{F}, *)$ is said to be a pseudo fuzzy length space where $\tilde{A}$ is a fuzzy set, $*$ is a continuous t-norm and $\tilde{F}$ is a fuzzy set from $\tilde{A}$ to $[0,1]$ satisfying the following conditions:

(PF_1) $\tilde{F}(x_\alpha) > 0$ for each $x_\alpha \in \tilde{A}$

(PF_2) If $x_\alpha = 0$ then $\tilde{F}(x_\alpha) = 1$

(PF_3) $\tilde{F}(cx, \alpha) = \tilde{F}(x, \frac{\alpha}{|c|})$ where $0 \neq c \in \mathbb{F}$

(PF_4) $\tilde{F}(x_\alpha) * \tilde{F}(y_\beta) \leq \tilde{F}[x_\alpha + y_\beta]$ for each $x_\alpha, y_\beta \in \tilde{A}$ and $\alpha, \beta \in [0,1]$.

(PF_5) $\tilde{F}$ is continuous fuzzy set

Remark 3.2:

Clearly, that every fuzzy length space is a pseudo fuzzy length space.

Example 3.3:

Let V=$\mathbb{R}$ and let $\tilde{A} = \{\{(x_n, \alpha_n)\}: \{(x_n, \alpha_n)\}$ is a fuzzy convergent sequence of fuzzy points$\}$. Suppose $a * b = a \cdot b$ for all $a, b \in [0,1]$. Define $\tilde{F}[\{(x_n, \alpha_n)\}] = \frac{\alpha}{\alpha + \lim_{n\to\infty}(x_n)}$ where $\alpha = \min\{\alpha_n : n \in \mathbb{N}\}$.

Then $(\tilde{A}, \tilde{F}, *)$ is a pseudo fuzzy length space put it is not fuzzy length space since $\left\{\left(\frac{1}{n}, \frac{1}{n}\right)\right\} \neq \{(0,0)\}$ but $\tilde{F}\left[\left\{\left(\frac{1}{n}, \frac{1}{n}\right)\right\}\right] = 1$.

Proposition 3.4:

Let $(\tilde{A}, \tilde{F}, *)$ be a pseudo fuzzy length space. Define $\sim$ on $\tilde{A}$ by $x_\alpha \sim y_\beta$ if and only if $\tilde{F}(x_\alpha - y_\beta) = 1$.

Then $\sim$ is a fuzzy equivalence relation on $\tilde{A}$.

Proof:

(1)$\sim$ is reflexive since $\tilde{F}(x_\alpha - x_\alpha) = 1$

(2)$\sim$ Is symmetric because if $x_\alpha \sim y_\beta$ then $y_\beta \sim x_\alpha$ since $\tilde{F}(x_\alpha - y_\beta) = \tilde{F}(y_\beta - x_\alpha)$.

(3)$\sim$ Is transitive, assume that $x_\alpha \sim y_\beta$ and $y_\beta \sim z_\delta$ so $\tilde{F}(x_\alpha - y_\beta) = 1$ and $\tilde{F}(y_\beta - z_\delta) = 1$. Now,

$\tilde{F}(x_\alpha - z_\delta) \geq \tilde{F}(x_\alpha - y_\beta) * \tilde{F}(y_\beta - z_\delta) \geq 1 * 1 = 1$ it follows that $\tilde{F}(x_\alpha - z_\delta) = 1$ and hence $x_\alpha \sim z_\delta$.

Notation 3.5:

Let $\tilde{A}/\sim$ be denoted by $\hat{\tilde{A}}$ and the element of $\hat{\tilde{A}}$ by $[x_\alpha] = \{y_\beta \in \tilde{A}: y_\beta \sim x_\alpha\}$

Lemma 3.6:

Let $(\tilde{A}, \tilde{F}, *)$ be a pseudo fuzzy length space. Defined $\hat{\tilde{F}}$ from $\hat{\tilde{A}}$ to $[0,1]$ by: $\hat{\tilde{F}}([x_\alpha]) = \tilde{F}(x_\alpha)$.

Then $\hat{\tilde{F}}$ does not depend on the representative x_α.

Proof:

Let $x_\alpha \sim y_\beta$ then $\tilde{F}(x_\alpha - y_\beta) = 1$ so $[x_\alpha] = [y_\beta]$ this implies that

$\hat{\tilde{F}}([x_\alpha]) = \hat{\tilde{F}}([y_\beta]) = \tilde{F}(y_\beta) = \tilde{F}(x_\alpha)$.

Theorem 3.7:

Let $(\tilde{A}, \tilde{F}, *)$ be a pseudo fuzzy length space then $(\hat{\tilde{A}}, \hat{\tilde{F}}, *)$ is a fuzzy length space where $\hat{\tilde{F}}([x_\alpha]) = \tilde{F}(x_\alpha)$.

Proof:

It is clear that $(\hat{\tilde{A}}, \hat{\tilde{F}}, *)$ is a pseudo fuzzy length space. Now if $\hat{\tilde{F}}([x_\alpha]) = 1$, then $\hat{\tilde{F}}([x_\alpha]) = \hat{\tilde{F}}([0])$ and $\hat{\tilde{F}}([x_\alpha] - [0]) = 1$ that is $[x_\alpha] = [0]$. Hence $(\hat{\tilde{A}}, \hat{\tilde{F}}, *)$ is a fuzzy length space.

Proposition 3.8:

Define an operator $T: \tilde{A} \rightarrow \hat{\tilde{A}}$ from the pseudo fuzzy length space $(\tilde{A}, \tilde{F}, *)$ to the fuzzy length space $(\hat{\tilde{A}}, \hat{\tilde{F}}, *)$ by $T(x_\alpha) = [x_\alpha]$, then T is a fuzzy isometry.

Proof:

We show that T is well defined if $x_\alpha = y_\beta$, then $\tilde{F}(x_\alpha - y_\beta) = 1$ so $y_\beta \in [x_\alpha]$ and $y_\beta \sim x_\alpha$ implies $y_\beta \in [x_\alpha]$ and $x_\alpha \in [y_\beta]$, it follows that $[x_\alpha] = [y_\beta]$ or $T(x_\alpha) = T(y_\beta)$ since $\tilde{F}(x_\alpha) = \hat{\tilde{F}}([x_\alpha]) = \hat{\tilde{F}}(T(x_\alpha))$, hence T is fuzzy isometry.

Theorem 3.9:

Let $(\tilde{A}, \tilde{F}, *)$ be a pseudo fuzzy length space and let $T: \tilde{A} \rightarrow \hat{\tilde{A}}$ then the collection of all $\hat{\tilde{F}}$-fuzzy open fuzzy balls is a base for the fuzzy topology $\tau_{\hat{\tilde{F}}}$.

Proof:

Consider $x_\alpha \in \tilde{A}$, $0 < r < 1$ and $\tilde{B}(x_\alpha, r)$ first we show that $T\left(\tilde{B}(x_\alpha, r)\right) = \tilde{B}(T(x_\alpha), r)$, let $b_\beta \in T\left(\tilde{B}(x_\alpha, r)\right)$ then $b_\beta \in T(d_\delta)$ where $d_\delta \in \tilde{B}(x_\alpha, r)$. Now $\tilde{F}(d_\delta - x_\alpha) = \hat{\tilde{F}}([d_\delta - x_\alpha]) = \hat{\tilde{F}}(T(d_\delta) - T(x_\alpha)) > (1-r)$ this implies that $T(d_\delta) \in \tilde{B}(T(x_\alpha), r)$ therefore $b_\beta \in \tilde{B}(T(x_\alpha, r))$ thus $T\left(\tilde{B}(x_\alpha, r)\right) \subseteq \tilde{B}(T(x_\alpha), r)$. Similarly we can show that $\tilde{B}(T(x_\alpha), r) \subseteq T\left(\tilde{B}(x_\alpha, r)\right)$ therefore $T\left(\tilde{B}(x_\alpha, r)\right) = \tilde{B}(T(x_\alpha), r)$. Hence $T^{-1}\left(\tilde{B}(T(x_\alpha), r)\right) = T^{-1}T\left(\left(\tilde{B}(x_\alpha, r)\right)\right) = \tilde{B}(x_\alpha, r)$.

Theorem 3.10

If $(\tilde{A}, \tilde{F}, *)$ is a fuzzy complete pseudo fuzzy length space then $(\hat{\tilde{A}}, \hat{\tilde{F}}, *)$ is fuzzy complete fuzzy length space.

Proof:

Let $\{(\hat{x}_n, \hat{\alpha}_n)\}$ be a fuzzy Cauchy sequence of fuzzy points in $(\hat{\tilde{A}}, \hat{\tilde{F}}, *)$, where $(\hat{x}_i, \hat{\alpha}_i) = [(x_i, \alpha_i)]$ for each i then $\{(x_n, \alpha_n)\}$ is a sequence of fuzzy points in $\tilde{A}$ and for given $0 < \varepsilon < 1$, there is a positive integer K such that $\hat{\tilde{F}}((\hat{x}_m, \hat{\alpha}_m) - (\hat{x}_n, \hat{\alpha}_n)) > (1 - \varepsilon)$. But $\hat{\tilde{F}}((\hat{x}_m, \hat{\alpha}_m) - (\hat{x}_n, \hat{\alpha}_n)) = \tilde{F}((x_m, \alpha_m) - (x_n, \alpha_n))$. Hence $\tilde{F}((x_m, \alpha_m) - (x_n, \alpha_n)) > (1 - \varepsilon)$ for all $n, m \geq K$.Therefore $\{(x_n, \alpha_n)\}$ is fuzzy Cauchy sequence of fuzzy points in $\tilde{A}$ but $\tilde{A}$ is fuzzy complete hence there is $x_\alpha \in \tilde{A}$ such that $\lim_{n\to\infty} \tilde{F}((x_n, \alpha_n) - x_\alpha) = 1$. Now put $\hat{x}_\alpha = [x_\alpha]$ then $\hat{x}_\alpha \in \hat{\tilde{A}}$ and $\lim_{n\to\infty} \hat{\tilde{F}}((\hat{x}_n, \hat{\alpha}_n) - \hat{x}_\alpha) = 1$. It follows that $\{(\hat{x}_n, \hat{\alpha}_n)\}$ fuzzy converges to $\hat{x}_\alpha \in \hat{\tilde{A}}$. Hence $(\hat{\tilde{A}}, \hat{\tilde{F}}, *)$ is fuzzy complete.

Definition 3.11:

Two pseudo fuzzy length spaces $\tilde{F}$ and $\tilde{F}'$ on same fuzzy set $\tilde{A}$ are said to be fuzzy equivalent, if for a sequence of fuzzy points $\{(x_n, \alpha_n)\}$ and fuzzy point x_α are in $\tilde{A}$ then $\lim_{n\to\infty} \tilde{F}((x_n, \alpha_n) - x_\alpha) = 1$ if and only if $\lim_{n\to\infty} \tilde{F}'((x_n, \alpha_n) - x_\alpha) = 1$

Theorem 3.12:

If $T: \tilde{A} \to \tilde{A}$ is a fuzzy continuous operator where $(\tilde{A}, \tilde{F}, *)$ is a pseudo fuzzy length space, then there exists a pseudo fuzzy length space $\tilde{F}'$ such that $\tilde{F}$ is fuzzy equivalent to $\tilde{F}'$.

Proof:

For all $x_\alpha\, in\, \tilde{A}$ defined $\tilde{F}'(x_\alpha) = \tilde{F}(x_\alpha) * \tilde{F}(T(x_\alpha))$. We now prove that $\tilde{F}'$ is a fuzzy length space on $\tilde{A}$.

(P$\tilde{F}_1$) $\tilde{F}'(x_\alpha) > 0$ since $\tilde{F}(x_\alpha) > 0$ and $\tilde{F}(T(x_\alpha)) > 0$

(P$\tilde{F}_2$) If $x_\alpha = 0$ then $\tilde{F}(x_\alpha) = 1$ and $\tilde{F}(T(x_\alpha)) = 1$ implies that $\tilde{F}(x_\alpha) = 1$

(P$\tilde{F}_3$) $\tilde{F}'(c\, x, \alpha) = \tilde{F}(c\, x, \alpha) * \tilde{F}(T(c\, x, \alpha)) = \tilde{F}(x, \frac{\alpha}{|c|}) * \tilde{F}(T(x, \frac{\alpha}{|c|})) = \tilde{F}'(x, \frac{\alpha}{|c|})$

(P$\tilde{F}_4$) let $x_\alpha, y_\beta \in \tilde{A}$, we have $\tilde{F}'(x_\alpha + y_\beta) \geq \tilde{F}'(x_\alpha) * \tilde{F}'(y_\beta)$

$\tilde{F}'(x_\alpha + y_\beta) = \tilde{F}(x_\alpha + y_\beta) * \tilde{F}(T(x_\alpha) + T(y_\beta))$

$\geq \tilde{F}(x_\alpha) * \tilde{F}(y_\beta) * \tilde{F}(T(x_\alpha)) * \tilde{F}(T(y_\beta))$

$\geq \tilde{F}(x_\alpha) * \tilde{F}(T(x_\alpha)) * \tilde{F}(y_\beta) * \tilde{F}(T(y_\beta))$

$\geq \tilde{F}'(x_\alpha) * \tilde{F}'(y_\beta)$

(P$\tilde{F}_5$) $\tilde{F}'$ is continuous since $\tilde{F}$ is continuous. Hence $(\tilde{A}, \tilde{F}', *)$ is pseudo fuzzy length space. To prove that $\tilde{F}'$ is fuzzy equivalent to $\tilde{F}$. Let $\{(x_n, \alpha_n)\}$ be a sequence of fuzzy points in $\tilde{A}$ and x_α is fuzzy point in $\tilde{A}$, suppose that $\lim_{n\to\infty} \tilde{F}((x_n, \alpha_n) - x_\alpha) = 1$ then $\lim_{n\to\infty} \tilde{F}(T(x_n, \alpha_n) - T(x_\alpha)) = 1$ since T is fuzzy continuous. Hence $\lim_{n\to\infty} \tilde{F}'((x_n, \alpha_n) - x_\alpha) = \lim_{n\to\infty} \tilde{F}((x_n, \alpha_n) - x_\alpha) * \lim_{n\to\infty} \tilde{F}(T(x_n, \alpha_n) - T(x_\alpha)) = 1 * 1 = 1$. It follows that $\lim_{n\to\infty} \tilde{F}'((x_n, \alpha_n) - x_\alpha) = 1$.

Conversely, suppose that $\lim_{n\to\infty} \tilde{F}'((x_n, \alpha_n) - x_\alpha) = 1$. This means that

$\lim_{n\to\infty} \tilde{F}'((x_n, \alpha_n) - x_\alpha) = \lim_{n\to\infty} \tilde{F}((x_n, \alpha_n) - x_\alpha) *$

$\lim_{n\to\infty} \tilde{F}(T(x_n, \alpha_n) - T(x_\alpha)) = 1$. It follows that $\lim_{n\to\infty} \tilde{F}((x_n, \alpha_n) - x_\alpha) = 1$

Lemma 3.13:

In Theorem (3.12), if we take $* = \wedge$ then $T: (\tilde{A}, \tilde{F}', *) \rightarrow (\tilde{A}, \tilde{F}, *)$ is uniformly fuzzy continuous.

Proof:

Clearly that $\tilde{F}\left(T(x_\alpha) - T(y_\beta)\right) \geq \tilde{F}'(x_\alpha - y_\beta)$ now, given $0 < r < 1$ such that $\tilde{F}'(x_\alpha - y_\beta) > 1 - r$.

Let $\delta = r$, then $\tilde{F}\left(T(x_\alpha) - T(y_\beta)\right) > (1 - \delta)$ therefore: $T: (\tilde{A}, \tilde{F}', *) \rightarrow (\tilde{A}, \tilde{F}, *)$ is uniformly fuzzy continuous.

Theorem 3.14:

If $(\tilde{A}, \tilde{F}, *)$ is a fuzzy compact pseudo fuzzy length space then $(\hat{\tilde{A}}, \hat{\tilde{F}}, *)$ is fuzzy compact fuzzy length space.

Proof:

Let $\{(\hat{x}_n, \hat{\alpha}_n)\}$ be a sequence of fuzzy points in $(\hat{\tilde{A}}, \hat{\tilde{F}}, *)$ where $(\hat{x}_i, \hat{\alpha}_i) = [(x_i, \alpha_i)]$ for each i, then $\{(x_n, \alpha_n)\}$ is a sequence of fuzzy points in $\tilde{A}$, but $\tilde{A}$ is fuzzy compact hence $\{(x_n, \alpha_n)\}$ contains a subsequence $\left\{\left(x_{n_j}, \alpha_{n_j}\right)\right\}$ which fuzzy converges to some $x_\alpha \in \tilde{A}$; that is given, $0 < r < 1$ there is positive integer K such that $\tilde{F}\left(\left(x_{n_j}, \alpha_{n_j}\right) - x_\alpha\right) > (1 - r)$ for all $n_j \geq K$.

Hence $\tilde{\hat{F}}\left(\left(\hat{x}_{n_j},\hat{\alpha}_{n_j}\right)-\hat{x}_\alpha\right)=\tilde{F}\left(\left(x_{n_j},\alpha_{n_j}\right)-x_\alpha\right)>(1-r)$. For all $n_j\geq K$, it follows that $\{(\hat{x}_n,\hat{\alpha}_n)\}$ contains a subsequence $\left\{\left(\hat{x}_{n_j},\hat{\alpha}_{n_j}\right)\right\}$ which fuzzy converges to $\hat{x}_\alpha$ where $\left(\hat{x}_{n_j},\hat{\alpha}_{n_j}\right)=\left[\left(x_{n_j},\alpha_{n_j}\right)\right]$ and $\hat{x}_\alpha=[x_\alpha]$. Hence $(\hat{\tilde{A}},\tilde{\hat{F}},*)$ is fuzzy compact.

4. Quotient of Fuzzy Length Space

Definition 4.1:

Suppose that $(\tilde{A},\tilde{F},*)$ is a fuzzy length space and let $\tilde{E}$ be a subset of $\tilde{A}$. Let $Q:\tilde{A}\to\frac{\tilde{A}}{\tilde{E}}$ be the natural operator, $Q(x_\alpha)=x_\alpha+\tilde{E}$.We define $\tilde{F}_1[x_\alpha+\tilde{E}]=\sup\{\tilde{F}[x_\alpha+y_\beta]:y_\beta\in\tilde{E}\}$.

Theorem 4.2:

Let $(\tilde{A},\tilde{F},*)$ be a fuzzy length space if $\tilde{E}$ is fuzzy closed subset of $\tilde{A}$ then $(\frac{\tilde{A}}{\tilde{E}},\tilde{F}_1,*)$ is a fuzzy length space.

Proof:

(FL_1) It is clear that $\tilde{F}_1[x_\alpha+\tilde{E}]\geq 0$ for all $x_\alpha+\tilde{E}\in\frac{\tilde{A}}{\tilde{E}}$.

(FL_2) Let $\tilde{F}_1[x_\alpha+\tilde{E}]=1$ then there is a sequence of fuzzy points $\{(x_n,\alpha_n)\}$ in $\tilde{E}$ such that $\tilde{F}_1[x_\alpha+(x_n,\alpha_n)]\to 1$ so $x_\alpha+(x_n,\alpha_n)\to 0$ or $(x_n,\alpha_n)\to -x_\alpha$ but $\tilde{E}$ is fuzzy closed so $x_\alpha\in\tilde{E}$ which implies that $x_\alpha+\tilde{E}=\tilde{E}$ where $\tilde{E}$ is the zero element of $\frac{\tilde{A}}{\tilde{E}}$.

Conversely, if $x_\alpha+\tilde{E}=\tilde{E}$ then $x_\alpha\in\tilde{E}$ since $\tilde{E}$ is fuzzy closed then there is $\{(x_n,\alpha_n)\}\in\tilde{E}$ such that $(x_n,\alpha_n)\to -x_\alpha$ that is $\tilde{F}[x_\alpha+(x_n,\alpha_n)]\to 1$. Hence $\tilde{F}_1[x_\alpha+\tilde{E}]=1$. (FL_3) Let $r\neq 0$, r is scalar

$$\tilde{F}_1[r(x_\alpha+\tilde{E})]=\tilde{F}_1[rx_\alpha+\tilde{E}]$$
$$=\sup\{\tilde{F}[r\,x+r\,y,\lambda]:y_\beta\in\tilde{E},\lambda=\alpha\wedge\beta\}=\sup\{\tilde{F}\left[x+\ y,\frac{\lambda}{|r|}\right]:y_\beta\in\tilde{E},\lambda=\alpha\wedge\beta\}=$$
$$\tilde{F}_1\left[x_\alpha+\tilde{E},\frac{\lambda}{|r|}\right]$$

(FL_4) $\tilde{F}_1\left[(x_\alpha+\tilde{E})+(y_\beta+\tilde{E})\right]=\tilde{F}_1[(x_\alpha+y_\beta)+\tilde{E})]$
$$=\sup\{\tilde{F}[(x_\alpha+y_\beta)+z_\sigma]:z_\sigma\in\tilde{E}\}\geq\sup\{\tilde{F}[x_\alpha+z_\sigma]:z_\sigma\in\tilde{E}]\}*\sup\{\tilde{F}[y_\beta+z_\sigma]:z_\sigma\in\tilde{E}]\}=$$
$$\tilde{F}_1[x_\alpha+\tilde{E}]*\tilde{F}_1[y_\beta+\tilde{E}]$$

(FL_5) Let $\{(x_n,\alpha_n)+\tilde{E}\}$ be a sequence of fuzzy points in $\frac{\tilde{A}}{\tilde{E}}$ fuzzy converges to $x_\alpha+\tilde{E}$. Now

$$\lim_{n\to\infty}\tilde{F}_1[(x_n,\alpha_n)+\tilde{E}]=\lim_{n\to\infty}sup\{\tilde{F}[(x_n,\alpha_n)+y_\beta]:y_\beta\in\tilde{E}\}=\sup\{\lim_{n\to\infty}\tilde{F}[(x_n,\alpha_n)+y_\beta]:y_\beta\in\tilde{E}\}=sup\{\tilde{F}[x_\alpha+y_\beta]:y_\beta\in\tilde{E}\}$$
$$=\tilde{F}_1[x_\alpha+\tilde{E}]$$
. Hence $\tilde{F}_1$ is continuous.

Lemma 4.3:

Let $(\tilde{A}, \tilde{F}, *)$ be a fuzzy length space and let $\tilde{E}$ be a fuzzy closed subset of $\tilde{A}$. Then $\tilde{F}_1[Q(x_\alpha)] \geq \tilde{F}[x_\alpha]$ for all $x_\alpha \in \tilde{A}$.

Proof:

$\tilde{F}_1[Q(x_\alpha)] = \tilde{F}_1[x_\alpha + \tilde{E}] = sup\{\tilde{F}[x_\alpha + y_\beta]: y_\beta \in \tilde{E}\} \geq \tilde{F}[x_\alpha]$.

Lemma 4.4:

Let $(\tilde{A}, \tilde{F}, *)$ be a fuzzy length space and let $\tilde{E}$ be fuzzy closed subset of $\tilde{A}$. Suppose that $(\frac{\tilde{A}}{\tilde{E}}, \tilde{F}_1, *)$ be the quotient fuzzy length space. Then the operator Q is fuzzy continuous.

Proof:

Let $\{(x_n, \alpha_n)\}$ be a sequence of fuzzy points in $\tilde{A}$ such that $(x_n, \alpha_n) \to x_\alpha \epsilon \tilde{A}$ that is $\lim_{n\to\infty} \tilde{F}[x_n - x, \lambda] = 1$ where $\lambda = \min\{\alpha, \alpha_n : n \in \mathbb{N}\}$. Now $Q[(x_n, \alpha_n)] = (x_n, \alpha_n) + \tilde{E}$, hence $\lim_{n\to\infty} \tilde{F}_1[(x_n - x, \lambda) + \tilde{E}] \geq \lim_{n\to\infty} \tilde{F}[(x_n - x, \lambda)] = 1$. This implies that $\lim_{n\to\infty} \tilde{F}_1[(x_n - x, \lambda) + \tilde{E}] = 1$. This means that $\{(x_n - \alpha_n) + \tilde{E}\} \to x_\alpha + \tilde{E}$ or $Q[(x_n, \alpha_n)] \to Q[x_\alpha]$. Hence Q is fuzzy continuous.

Lemma 4.5:[20]

If $J \subset [0,1]$ and $\sup J = \beta$ then for each $\varepsilon \in [0, \beta]$ there exists $\alpha \in J$ such that $\varepsilon * \beta \leq \alpha$.

Theorem 4.6:

Let $(\tilde{A}, \tilde{F}, *)$ be a fuzzy length space and $\tilde{E}$ be a fuzzy closed subset of $\tilde{A}$. If $\tilde{A}$ is fuzzy complete then $\frac{\tilde{A}}{\tilde{E}}$ is a fuzzy complete.

Proof:

Let $\{(x_n, \alpha_n) + \tilde{E}\}$ be a fuzzy Cauchy sequence in $\frac{\tilde{A}}{\tilde{E}}$. Then for any given $r, 0 < r < 1$ there is a positive number K suck that $\tilde{F}_1[(x_n - x_m, \lambda) + \tilde{E}] \geq (1 - r)$ for all $n, m \geq K$. Let $[(x_n, \alpha_n) + (y_n, \beta_n)] \in \{(x_n, \alpha_n) + \tilde{E}\}$ for all $n \in \mathbb{N}$. Then by (lemma 4.5) there is $\varepsilon \in [0, \tilde{F}_1]$, $\tilde{F}[(x_n - x_m, \lambda) + (y_n - y_m, \gamma)] > (1 - r) * \varepsilon$ for all $n, m \geq K$ where $\lambda = \min\{\alpha_n : n \in \mathbb{N}\}$ and $\gamma = \min\{\beta_n : n \in \mathbb{N}\}$. Hence $\{(x_n + y_n, \delta)\}$ is a fuzzy Cauchy sequence in $\tilde{A}$ where $\delta = \gamma \wedge \lambda$. But $\tilde{A}$ is fuzzy complete so $\{(x_n + y_n, \delta)\} \to x_\alpha + y_\beta$ so $[(x_n + y_n, \delta)] = (x_n, \alpha_n) + \tilde{E} \to Q[(x_\alpha + y_\beta)] = x_\alpha + \tilde{E}$. Since Q is fuzzy continuous. Therefore every fuzzy Cauchy in $\frac{\tilde{A}}{\tilde{E}}$ is fuzzy converges. Hence $\frac{\tilde{A}}{\tilde{E}}$ is fuzzy complete.

Theorem 4.7:

Let $(\tilde{A}, \tilde{F}, *)$ be a fuzzy length space and let $\tilde{E}$ be a fuzzy closed subset of $\tilde{A}$. Then $\tilde{A}$ is fuzzy complete if and only if $\tilde{E}$ and $\frac{\tilde{A}}{\tilde{E}}$ are fuzzy complete.

Proof:

If $\tilde{A}$ is fuzzy complete then $\frac{\tilde{A}}{\tilde{E}}$ is fuzzy complete by (theorem 4.6). Since $\tilde{E}$ is fuzzy closed subset of $\tilde{A}$ so $\tilde{E}$ is fuzzy complete by (theorem 2.30)

Conversely, assume that $\tilde{E}$ and $\frac{\tilde{A}}{\tilde{E}}$ are fuzzy complete. Let $\{(x_n, \alpha_n)\}$ be a fuzzy Cauchy sequence in $\tilde{A}$. So given $r, 0 < r < 1$ there is a positive number K such that $\tilde{F}[(x_n - x_m, \lambda)] > (1-r)$ for all $n, m \geq K$, where $\lambda = \min\{\alpha_n : n \in \mathbb{N}\}$. Hence $\tilde{F}_1[(x_n - x_m, \lambda) + \tilde{E}] > \tilde{F}[(x_n - x_m, \lambda)] \geq (1-r)$ for all $n, m \geq K$. That is $\{(x_n, \alpha_n) + \tilde{E}\}$ is a fuzzy Cauchy sequence in $\frac{\tilde{A}}{\tilde{E}}$. But $\frac{\tilde{A}}{\tilde{E}}$ is fuzzy complete, hence there is $(x_\alpha + \tilde{E}) \in \frac{\tilde{A}}{\tilde{E}}$ such that $\tilde{F}_1[(x_n - x, \lambda) + \tilde{E}] > (1-r)$ for all $n \geq K$.

Now there is $(1-\varepsilon) \in \left[0, \tilde{F}_1[(x_n - x, \lambda) + \tilde{E}]\right]$ such that $\tilde{F}[(x_n - x, \lambda)] > (1-r) * (1-\varepsilon)$. That is $\lim_{n\to\infty} \tilde{F}[(x_n - x, \lambda)] = 1$, hence $(x_n, \alpha_n) \to x_\alpha$. Therefore $\tilde{A}$ is fuzzy complete.

Conclusion

In the present paper our aim is to define the notion of pseudo fuzzy length space and proved some basic properties of this space. Also we define the quotient of a fuzzy length space to introduce the definition of the fuzzy length of the quotient space in order to prove that the quotient space is fuzzy complete.

References

[1] Zadeh, L(1965). ,**Fuzzy sets**, Inf. Control. Vol. 8.338-353.

[2] Katsaras, A. (1984) , **Fuzzy topological vector spaces II**, Fuzzy sets and Systems, Vol. 12.143-154.

[3] Kaleva,O. and Seikkala, S. (1984), **On fuzzy metric spaces, fuzzy sets and systems** Vol.121.215-229.

[4] Felbin, C(1992)., **Finite dimensional fuzzy normed linear spaces**. Fuzzy sets and Systems, Vol. 48.239-248.

[5] Kramosil, O. and Michalek, J. (1975) ,**Fuzzy metrics and statistical metric spaces**, Kybernetika, Vol. 11.326-334.

[6] Cheng, S. and Mordeson, J, (1994) **Fuzzy linear operators and fuzzy normed linear spaces**, Ball. Cal. Math. Soc. Vol. 86.429 - 436.

[7] Bag, T. and Samanta, S.,(2003) **Finite dimensional fuzzy normed linearspaces,** J. Fuzzy Math.Vol.11(3).687-705.

[8] Saadati, R. and Vaezpour, M. (2005) ,**Some results on fuzzy Banach spaces** J. Appl. Math. and Computing. Vol. 171.475-484.

[9] Bag, T. and Samanta, S., (2005) **Fuzzy bounded linear operators**, Fuzzy sets and Systems, Vol.151(3).513-547.

[10] Sadeqi, I. and Kia, F., (2009) **Fuzzy normed linear space and its topological structure,** Chaos Solitions and Fractals, Vol. 40 (5). 2576-2589.

[11] Si, H. Cao, H. and Yang, P., (2010) **Continuity in an intuitionistic fuzzy normed space** Seventh, I. Conference on fuzzy systems and knowledge Discovery. 144-148.

[12] Nadaban, S. (2015) **Fuzzy continuous mapping in fuzzy normed linear spaces**, I. J. Computer Communications and Control, Vol 10 (6). 834-842.

[13] Amini, A. and Saadati, R., ,(2004) **Some Properties of continuous t-norms and s-norms**, Int. J. Pure Appl.Math.Vol.16,157-164.

[14] Bag, T. and Samanta, S., (2007) **Some fixed point theorems in fuzzy normed linear spaces**, Information sciences Vol.177. 3271-3289.

[15] Bag, T. and Samanta, S., (2006) **Fixed point theorems on Fuzzy normed spaces,** Inf. sci. Vol.176. 2910-2931.

[16] Congxin, W. and Ming, M., (1993) **Continuity and bounded mappings between fuzzy normed spaces**, Fuzzy Math, Vol.1. 13-24.

[17] Goudarzi, M. and Vaezpour, S. (2010) , **Best simultaneous approximation in fuzzy normed spaces,** Iranian J. of Fuzzy Systems, Vol.7.87-69.

[18] Jameel, R., (2014). **On some results of analysis in a standard fuzzy normed spaces**, M.Sc. Thesis, University of Technology, Iraq.

[19] Kider, J. (2012) ,**New fuzzy normed spaces**, J. Baghdad Sci. , Vol 9. 559-564.

[20] Oregan, D. and Saadati, R. (2010)), **L-Random and fuzzy normed spaces and classical theory**, CUBO Mathematical J. Vol. (2 (2010))71-81.

[21] Xiao, J. and Zhu, X. (2004), **Fuzzy normed spaces of operators and it is completeness**, Fuzzy sets and Systems. Vol.133. 437-452.

[22] George, A. and Veeramani, P., (1994) **On some results in fuzzy metric Spaces,** Fuzzy sets and Systems, Vol. 64. 395-399.

[23] Golet, I. , (2010) **On generalized fuzzy normed spaces and coincidence theorems**, Fuzzy sets and Systems, Vol.161. 1138-1144.

[24] Mousa, J. I. 2016. **Properties of fuzzy norm on fuzzy set**, M. Sc. Thesis University of Technology, Iraq.