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Maximum Flow Problem in Ethiopian Airlines

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ABSTRACT

Maximum flow problem is a problem which involves a directed network with arcs carrying flow. The problem is to find the maximum flow that can be sent through the arcs of the network from some specified node S, called the source, to a second specified node T, called the sink. In this paper we are going to see what maximum flow problem is and its solution algorithm called the Ford and Fulkerson algorithm. This paper also contains a problem of maximum flow problem in Ethiopian Airlines solved using the Ford and Fulkerson algorithm.

Keywords: Maximum flow problem; augmenting path; and Ford and Fulkerson algorithm.

1. Introduction

A common question about network is "what is the maximum flow rate between a given node and some other node in the network?'' For example, consider a network of pipelines that transports crude oil from wells to refineries. Intermediate booster and pumping stations are installed at appropriate distances to move the crude in the network. Each pipeline has a finite maximum capacity. How can we determine the maximum capacity of the network between wells and the refineries? Another example is traffic engineers may want to know the maximum flow rate vehicles from down town car park to the free way on-ramp because this will influence their decisions on whether to widen the road ways. These and other questions give birth to Maximum flow problem. This problem involves a directed network with arcs carrying flow. The only relevant parameter is the upper bound on arc flow, called *arc capacity*. The problem is to find a flow of maximum value that can be sent through the arcs of the network from some specified node *s*, called the source, to a second specified node *t*, called the sink. Applications of this problem include finding the maximum flow of orders through a job shop, the maximum flow of water through a storm sewer system, and the maximum flow of product through a product distribution system, among others. In particular, the solution is the assignment of flows to arcs. For feasibility, conservation of flow is required at each node except the source and sink, and each arc flow must be less than or equal to its capacity.

The algorithm that we use to solve maximum flow problem is called Ford and Fulkerson algorithm. This algorithm works based on the maximum flow theorem which states below.

Maximum Flow Theorem

A flow has maximum value if and only if it has no augmenting path.

Ford and Fulkerson (FF) Algorithm

In this section we develop the Ford-Fulkerson (FF) algorithm for finding the max-flow in a network. It is simple and practical max-flow algorithm. The main idea of this algorithm is that to find valid flow paths until there is none left, and add them up. Since the 0 –flow is always feasible, Initialize network with null flow and attempt to gradually increase the flow until it gets its maximum value. The Ford-Fulkerson algorithm is simply the following: while there exist an $S \to T$ path P with positive capacity, push the maximum possible flow along P. By the way, these paths P are called augmenting paths, because you use them to augment the existing flow.

Now let's define the steps of Ford and Fulkerson algorithm clearly like below:

Step 1. Find any path from the source node to the destination node that has a strictly positive flow capacity remaining. If there are no more such paths, exit.

Step 2. Determine f, the maximum flow along this path, which will be equal to the smallest flow capacity on any arc in the path (the bottleneck arc).

Step 3. Subtract f from the remaining flow capacity in the forwarded direction for each arc in the path. Add f to the remaining flow capacity in the backwards direction for each arc in the path.

Step 4. Go to step1.

2. FORMULATION (Rerouting Airline Passengers)

Due to certain reason, Ethiopian Airlines had to cancel this evening its flight number 108 which was scheduled to fly from AA (Addis Ababa) to NY (New York) with 125 passengers. Then how can we reroute (transport) as many passengers of flight 108 as possible from AA to NY in this evening and determine the maximum number of these passengers who can get seats (travel) on other flight routes from AA to NY in this evening using the following flight data.

From		**To**						
		Khart.	Cairo	Nairo.	Dakar	Frank.	London	NY
AA	Flight no.	102	103	104	105			~~108~~
	# seats	20	28	35	16			
Khart.	Flight no.		203			206	207	
	# seats		2			15	35	
Cairo	Flight no.				305	306		
	# seats				14	25		
Nairo.	Flight no.		403		405			
	# seats		30		7			
Dakar	Flight no.							508
	# seats							40
Frank.	Flight no.						607	608
	# seats						5	35
London	Flight no.							708
	# seats							22

 # seats = the number of available (unoccupied) seats on the flight.

3. Solution of the problem

First based on the table given above let's figure it out the capacitated digraph of it and it will look like the below figure 1. Where AA stands for Addis Ababa,
Khart. Stands for Khartoum,
Frank. Stands for Frankfurt
Nairo. Stands for Nairobi
NY stands for New York

And here AA is the source node and NY is the sink node all the other nodes are called intermediate nodes. And the question here is that to the maximum how much passengers of the flight 108 can we sent from AA to NY via the other flight numbers. In other word what it asked is the maximum flow between the two special nodes AA and NY.

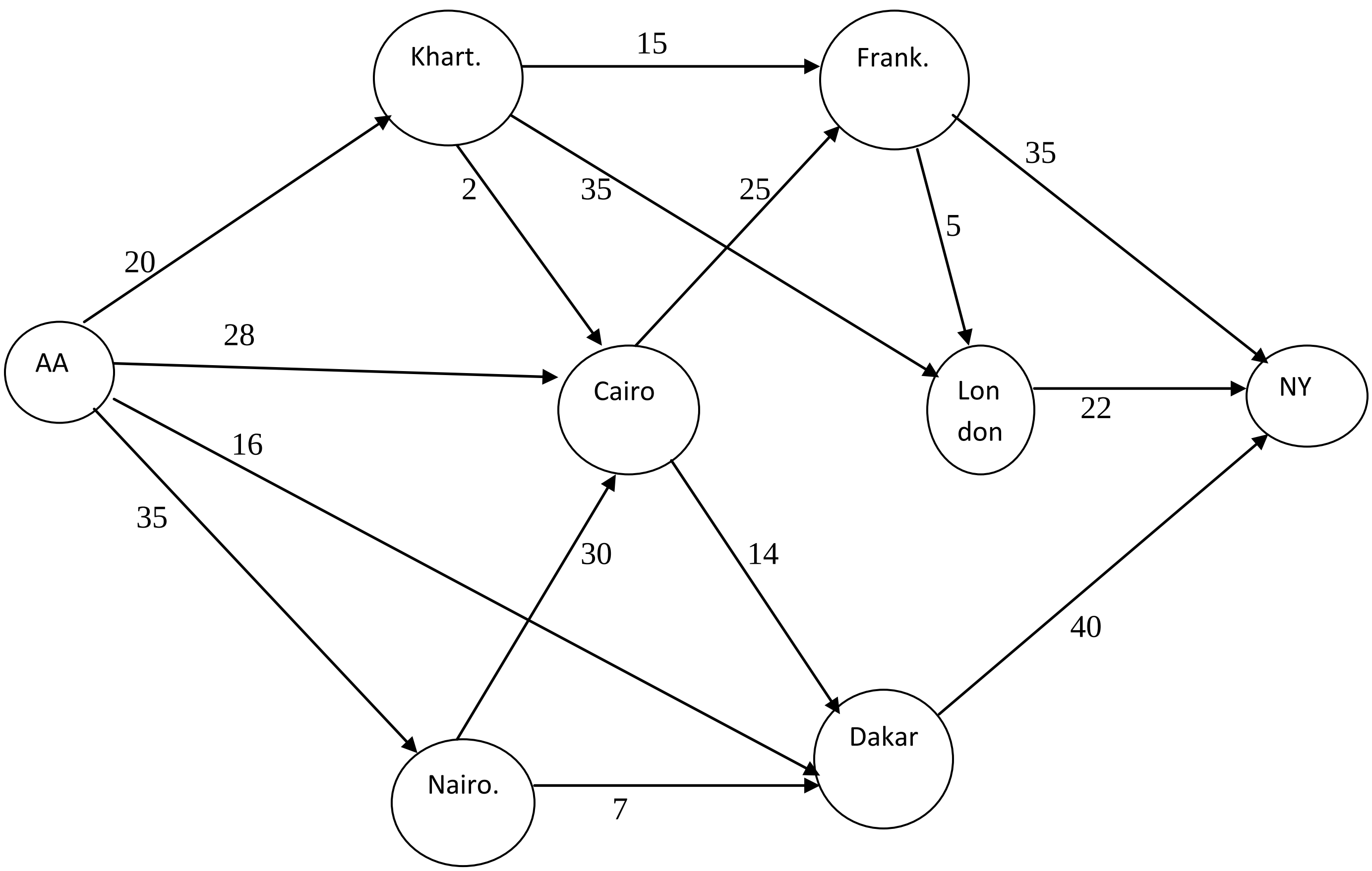


Figure 1: Shows the given problem after changed to network form

Let's do it now using the Ford and Fulkerson algorithm that is let's figure it out the possible augmenting paths that we can have to the maximum and you will get the below

- ❖ P_1: AA $\xrightarrow{20}$ Khart. $\xrightarrow{15}$ Frank. $\xrightarrow{35}$ NY

$$\min\{20,15,35\} = 15$$

- This flow will result the arc from Khartoum to Frankfurt saturated.

After these 15 passengers sent through the specified path1 (P_1) we have to ammend the capacity of the those arcs in the path that means we have to deduct 15 from each arc capacity on the specified path and will become

$$\begin{cases} \text{AA} \xrightarrow{5} \text{Khart.} \\ \text{Khart.} \xrightarrow{0} \text{Frank. (Saturated!)} \\ \text{Frank.} \xrightarrow{20} \text{NY} \end{cases}$$

- P_2: AA $\xrightarrow{28}$ Cairo $\xrightarrow{25}$ Frank. $\xrightarrow{20}$ NY

  $\min\{28,25,20\} = 20$

- This flow will result the arc from Frankfurt to NY saturated.

Then after these 20 passengers sent through this path2 (P_2) the arc capacity of those arcs in the specified path will become

$$\begin{cases} \text{AA} \xrightarrow{8} \text{Cairo} \\ \text{Cairo} \xrightarrow{5} \text{Frank.} \\ \text{Frank.} \xrightarrow{0} \text{NY (Saturated!)} \end{cases}$$

- P_3: AA $\xrightarrow{35}$ Nairobi $\xrightarrow{7}$ Dakar $\xrightarrow{40}$ NY

  $\min\{35,7,40\} = 7$

- This flow will result the arc from Nairobi to Dakar saturated

After these 7 passengers sent via path3 (P_3) the arc capacity of those arcs in the specified path will become

$$\begin{cases} \text{AA} \xrightarrow{28} \text{Khart.} \\ \text{Nairobi} \xrightarrow{0} \text{Dakar (Saturated!)} \\ \text{Dakar} \xrightarrow{33} \text{NY} \end{cases}$$

- P_4: AA $\xrightarrow{5}$ Khart. $\xrightarrow{35}$ London $\xrightarrow{22}$ NY

  $\min\{5,35,22\} = 5$

- This flow will result the arc from AA to Khartoum saturated

After these 5 passengers sent via path4 (P_4) the arc capacity of those arcs in the specified path will become

$$\begin{cases} \text{AA} \xrightarrow{0} \text{Khart. (Saturated!)} \\ \text{Khart.} \xrightarrow{30} \text{London} \\ \text{London} \xrightarrow{17} \text{NY} \end{cases}$$

- P_5: AA $\xrightarrow{8}$ Cairo $\xrightarrow{5}$ Frank. $\xrightarrow{5}$ London $\xrightarrow{17}$ NY

  $\min\{8,5,5,17\} = 5$

- This flow will result the arc from Cairo to Frankfurt saturated

After these 5 passengers sent via path5 (P_5) the arc capacity of those arcs in the specified path will become

$$\begin{cases} AA \xrightarrow{3} Cairo \\ Cairo \xrightarrow{0} Frank.\ (Saturated!) \\ Frank. \xrightarrow{0} London\ (Saturated!) \\ London \xrightarrow{12} NY \end{cases}$$

- P_6: AA $\xrightarrow{16}$ Dakar $\xrightarrow{33}$ NY

  $\min\{16,33\} = 16$

  - This flow will result the arc from AA to Dakar saturated

After these 16 passengers sent via path 6 (P_6) the arc capacity of those arcs in the specified path will become

$$\begin{cases} AA \xrightarrow{0} Dakar\ (Saturated!) \\ Dakar \xrightarrow{17} NY \end{cases}$$

- P_7: AA $\xrightarrow{3}$ Cairo $\xrightarrow{14}$ Dakar $\xrightarrow{17}$ NY

  $\min\{3,14,17\} = 3$

  - This flow will make the arc from AA to Cairo saturated

After these 3 passengers sent via this path 7 (P_7) the arc capacity of those arcs in the specified path will become

$$\begin{cases} AA \xrightarrow{0} Cairo\ \ (Saturated!) \\ Cairo \xrightarrow{11} Dakar \\ Dakar \xrightarrow{14} NY \end{cases}$$

- P_8: AA $\xrightarrow{28}$ Nairobi $\xrightarrow{30}$ Cairo $\xrightarrow{11}$ Dakar $\xrightarrow{14}$ NY

  $\min\{28,30,11,14\} = 11.$

  - This flow will result the arc from Cairo to Dakar saturated

After these 11 passengers sent via this path8 (P_8) the arc capacity of those arcs in the specified path will become

$$\begin{cases} AA \xrightarrow{17} Nairobi \\ Nairobi \xrightarrow{19} Cairo \\ Cairo \xrightarrow{0} Dakar\ (Saturated!) \\ Dakar \xrightarrow{3} NY \end{cases}$$

Then since there is no more flow augmenting path the algorithm will terminate here and the maximum flow will be the sum of 15+20+7+5+5+16+3+11 which is sent through the augmenting paths and it's 82. This means to the maximum Ethiopian Air lines can send 82 passengers out of the 125 to NY using the other Flights. And after performing all the 8 augmenting path list above the remaining arc capacity of those arcs in the original network will become like below

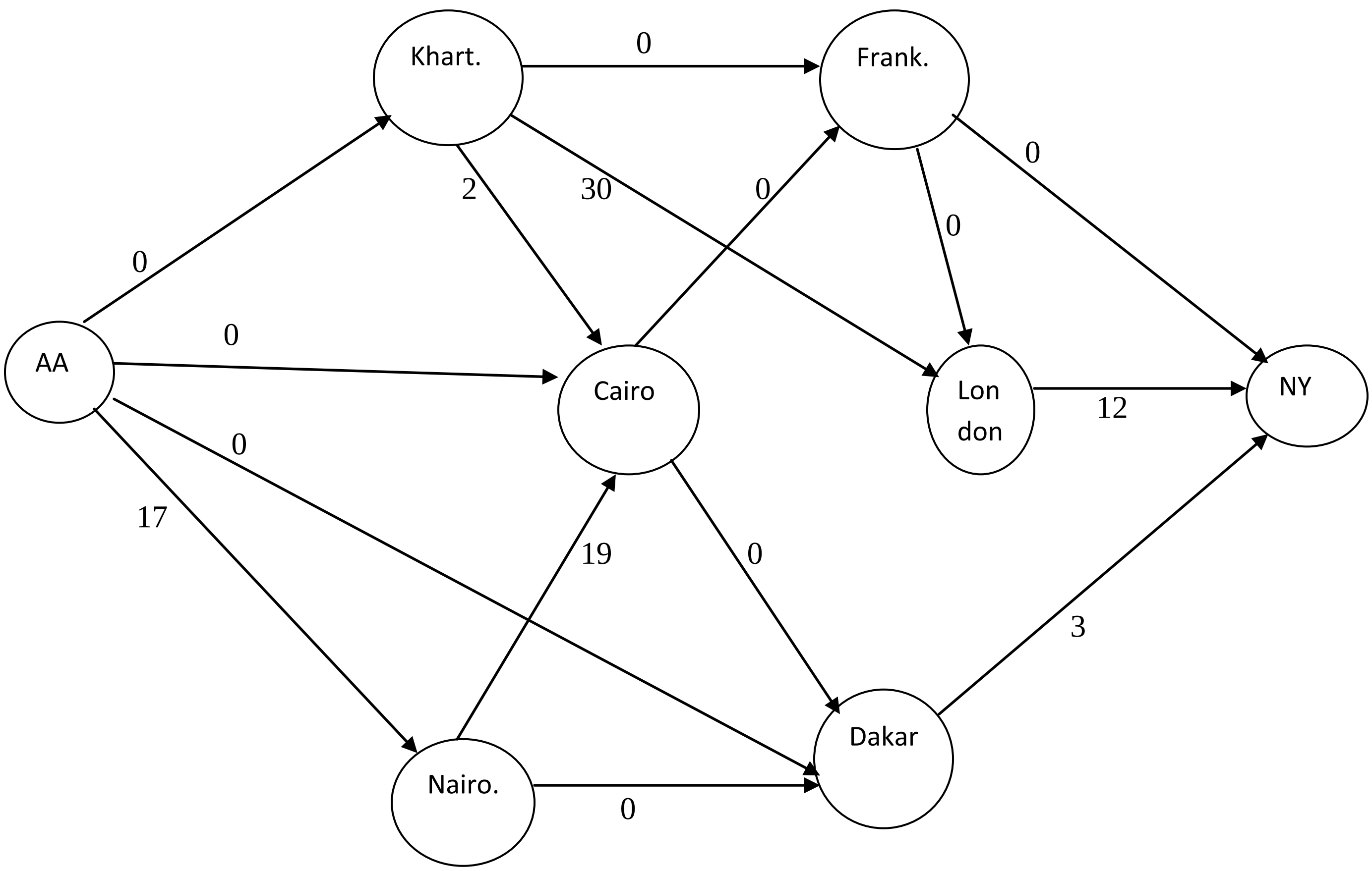


Figure 2: the remaining arc capacities of the original network after the above 8 paths are performed

4. CONCLUSION

In ford and Fulkerson algorithm after a flow has been sent to the augmenting path at least one arc in that augmenting path will be saturated. And we might get different augmenting paths other than the listed above for the Ethiopian Air line problem even in terms of the number of augmenting paths but we get the same maximum flow value. This means even though we can have different augmenting paths, different number of augmenting paths the maximum flow value is unique. For example we can solve the Ethiopian Air lines problem above like below using 7 augmenting paths unlike the above use 8 augmenting paths.

- P_1: AA $\xrightarrow{20}$ Khart. $\xrightarrow{15}$ Frank. $\xrightarrow{35}$ NY

  $\min\{20,15,35\} = 15$

After these 15 passengers sent via path1 (P_1) the arc capacities in that specified path will become
$$\begin{cases} AA \xrightarrow{5} Khart \\ Khart. \xrightarrow{0} Frank. \text{ (Saturated!)} \\ Frank. \xrightarrow{20} NY \end{cases}$$

- P_2: $AA \xrightarrow{28} Cairo. \xrightarrow{25} Frank. \xrightarrow{20} NY$

  $\min\{28,25,20\} = 20$

After these 20 passengers sent via path2 (P_2) the arc capacities in that specified path will become
$$\begin{cases} AA \xrightarrow{8} Cairo \\ Cairo \xrightarrow{5} Frank. \\ Frank. \xrightarrow{0} NY \text{ (Saturated!)} \end{cases}$$

- P_3: $AA \xrightarrow{35} Nairobi \xrightarrow{7} Dakar \xrightarrow{40} NY$

  $\min\{35,7,40\} = 7$

After these 7 passengers sent via this path3 (P_3) the arc capacities in that specified path will become
$$\begin{cases} AA \xrightarrow{28} Nairobi \\ Nairobi \xrightarrow{0} Dakar \\ Dakar \xrightarrow{33} NY \end{cases}$$

- P_4: $AA \xrightarrow{28} Nairobi \xrightarrow{30} Cairo \xrightarrow{14} Dakar \xrightarrow{33} NY$

  $\min\{28, 30, 14, 33\} = 14$

After these 14 passengers sent via this path4 (P_4) the arc capacities in that specified path will become
$$\begin{cases} AA \xrightarrow{14} Nairobi \\ Nairobi \xrightarrow{16} Cairo \\ Cairo \xrightarrow{0} Dakar \text{ (Saturated!)} \\ Dakar \xrightarrow{19} NY \end{cases}$$

- P_5: $AA \xrightarrow{16} Dakar \xrightarrow{19} NY$

  $\min\{16, 19\} = 16$

After these 16 passengers sent via this path5 (P_5) the arc capacities in that specified path will become
$$\begin{cases} AA \xrightarrow{0} Dakar \text{ (Saturated!)} \\ Dakar \xrightarrow{3} NY \end{cases}$$

- P_6: $AA \xrightarrow{5} Khart. \xrightarrow{35} London \xrightarrow{22} NY$

  $\min\{5, 35, 22\} = 5$

After these 5 passengerssent via this path6 (P_6) the arc capacities in that specified path will
$$\begin{cases} AA \xrightarrow{0} Khart. \text{ (Saturated!)} \\ Khart. \xrightarrow{30} London \end{cases}$$

become London $\xrightarrow{17}$ NY

- ❖ P_7: AA $\xrightarrow{8}$ Cairo $\xrightarrow{5}$ Frank. $\xrightarrow{5}$ London $\xrightarrow{17}$ NY

$$\min\{8, 5, 5, 17\} = 5$$

After these 5 passengers sent via this path7 (P_7) the arc capacities in that specified path will become

$$\begin{cases} \text{AA} \xrightarrow{3} \text{Cairo} \\ \text{Cairo} \xrightarrow{0} \text{Frank. (Saturated!)} \\ \text{Frank.} \xrightarrow{0} \text{London (Saturated!)} \\ \text{London} \xrightarrow{12} \text{NY} \end{cases}$$

Then since there is no more flow augmenting path the algorithm will terminate here and the maximum flow will be the sum of 15+20+7+14+16+5+5 which is again 82. This means to the maximum Ethiopian Air lines can send 82 passengers from AA to NY using the other Flights. Thus, the maximum flow result is always the same (unique). And after these augmenting paths are performed to the original network (figure1) the remaining arc capacities of those arcs will look like below

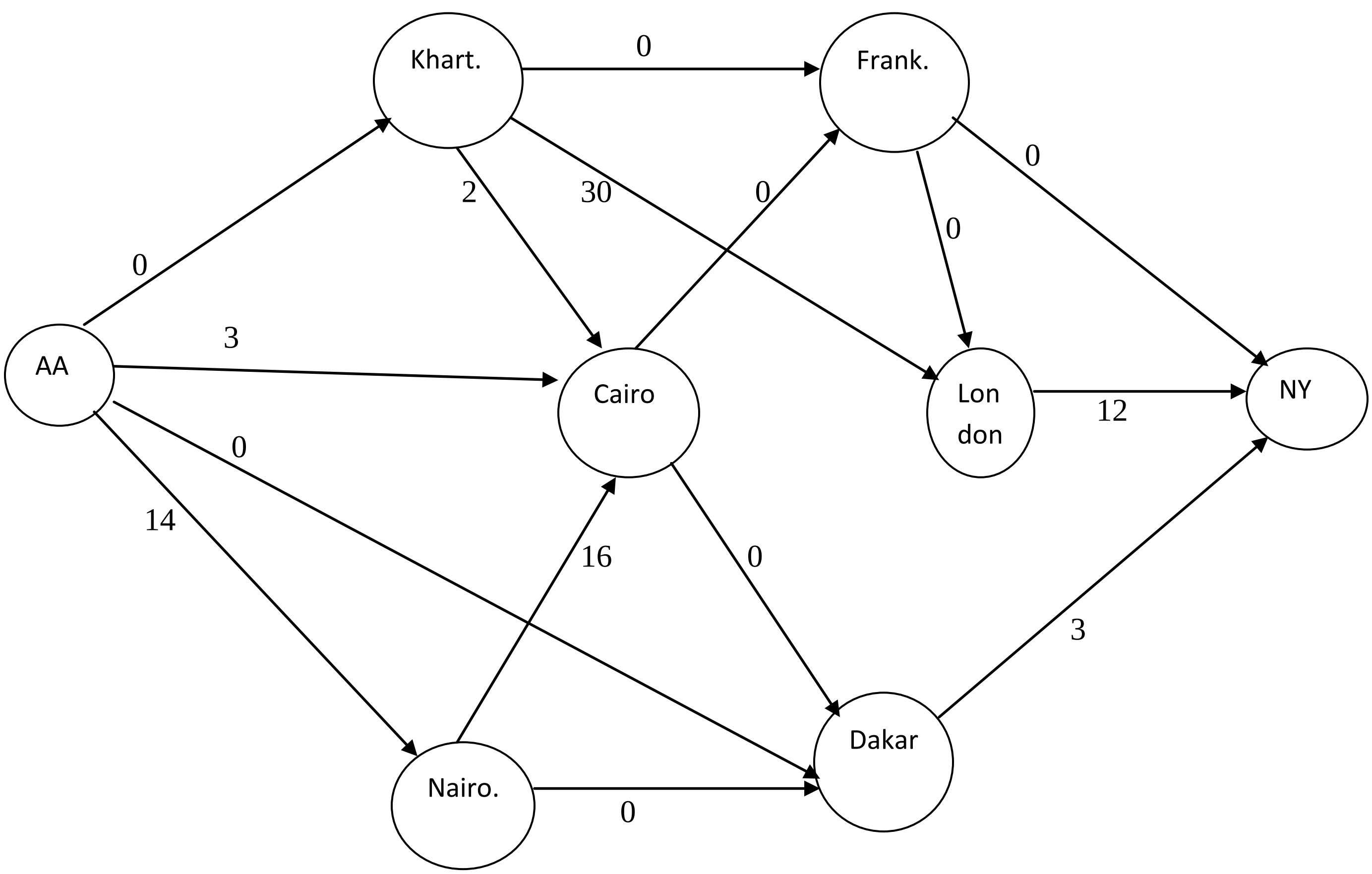


Figure 3: the remaining arc capacities of the original network after the above 7 paths are performed

- ➢ Another thing here is that we might have a flow augmenting path which includes back ward arcs. To elaborate this let's consider the below example by bearing in mind that an FAP (Flow Augmenting Path) is a path from S to T along which we can send more flow than is currently in the network.

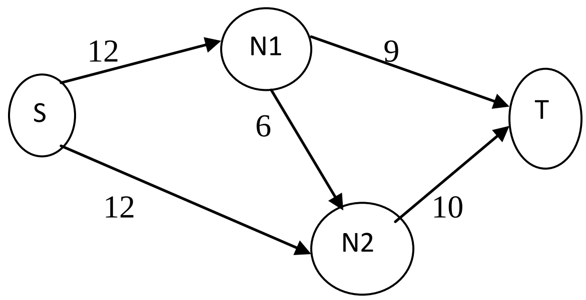
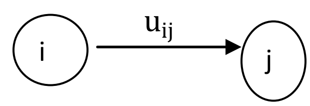


Figure 4:



arc (i,j) is called saturated if the flow (x_{ij}) & its capacity (u_{ij}) are equal. (I.e. $x_{ij} = u_{ij}$)

Now let's use the below augmenting paths to solve the max-flow problem

❖ P_1: $S \xrightarrow{12} N1 \xrightarrow{6} N2 \xrightarrow{10} T$

$$\min\{12, 6, 10\} = 6$$

After these 6 units sent via this path1 (P_1) the arc capacity of those arcs in that specified path will become
$$\begin{cases} S \xrightarrow{6} N1 \\ N1 \xrightarrow{0} N2 \text{ (Saturated!)} \\ N2 \xrightarrow{4} T \end{cases}$$

❖ P_2: $S \xrightarrow{6} N1 \xrightarrow{9} T$

$$\min\{6, 9\} = 6$$

After these 6 units sent via this path2 (P_2) the arc capacity of those arcs in that specified path will become
$$\begin{cases} S \xrightarrow{0} N1 \text{ (Saturated!)} \\ N1 \xrightarrow{3} T \end{cases}$$

❖ P_3: $S \xrightarrow{12} N2 \xrightarrow{4} T$

$$\min\{12, 4\} = 4$$

After these 4 units sent via this path3 (P_3) the arc capacity of those arcs in that specified path will become
$$\begin{cases} S \xrightarrow{8} N2 \\ N2 \xrightarrow{0} T \text{ (Saturated!)} \end{cases}$$

No more forward path from S to T through which we can send more flow!
The value of the current flow is 6+6+4 which is 16. Is this the maximum flow? NO! In fact the maximum flow is 19.

What we have to do here is that to find a way to reroute some units of flow through different path say S-N2-N1-T. Send 3 ($\min\{8,6,3\} = 3$) more units through this path. Thus, FAP is not necessarily a forward path from S to T. we need to have the following situation in order to be able to send more flow along a path from S to T:

i) Each forward arc (i,j) on the path must be not saturated
ii) There must be a non-zero flow on the back ward arc in order to send it back for rerouting.

After all the 4 paths are performed the remaining arc capacity of the original network above of figure 4 will look like below

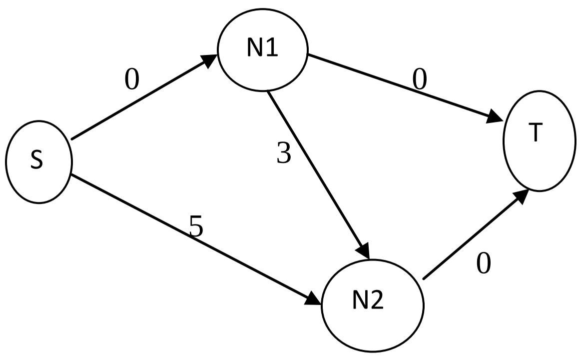


Figure 5

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