Journal of Progressive Research in Mathematics(JPRM)
ISSN: 2395-0218

SCITECH
RESEARCH ORGANISATION|

Volume 9, Issue 2
Published online: September 12, 2016|

Journal of Progressive Research in Mathematics
www.scitecresearch.com/journals

Integer Linear Programming Problem in Messebo Cement Factory

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ABSTRACT

To solve integer linear programming problem is very difficult than to solve linear programming problem. In this paper we are going to see the formulation of integer linear programming problem and one of its solution techniques called Branch and Bound method. This paper also contains a real world problem of integer linear programming problem on Messebo Cement Factory (Mekelle, Tigray, Ethiopia) solved using the Branch and Bound method.

Keywords: Integer linear programming problem; Branch and Bound method.

1. Introduction

Many linear programming problems require certain variables to have integer values. Such requirement arises naturally when the variables represent entities like packages or peoples that cannot be fractionally divided. These and other requirements give birth to integer linear programming problem (ILPP).

Integer Linear programming problems are linear programming problems with all the decision variables are restricted to integer value.

In Integer Linear Programming Problem fractional values are meaningless. For example: when we asked how many chairs and tables should the profit maximizing carpenter make, it did not make sense to come up with an answer "three and one half''. May be the carpenter know more than enough to make half a chair (using half the resource needed to make the entire chair), but probably he would not be able to sell half a chair for half the price of a whole chair. So, sometimes it makes sense to add to a problem the additional constraint that all of the variable must take an integer value.

Thus, the general form of Integer linear programming problem is,

$$Max/Min \quad Z = C^T X$$

$$\begin{aligned} subject\ to \quad & AX \leq b \\ & X_i \geq 0\ and\ integers \end{aligned}$$

So, since Integer linear programming problem forced the values of the decision variables to come from the integer part, the optimal solution of Integer linear programming problem is an integer. The feasible set of an Integer linear programming problem is the set of integer points in the feasible region (polyhedral) given by Ax ≤ b

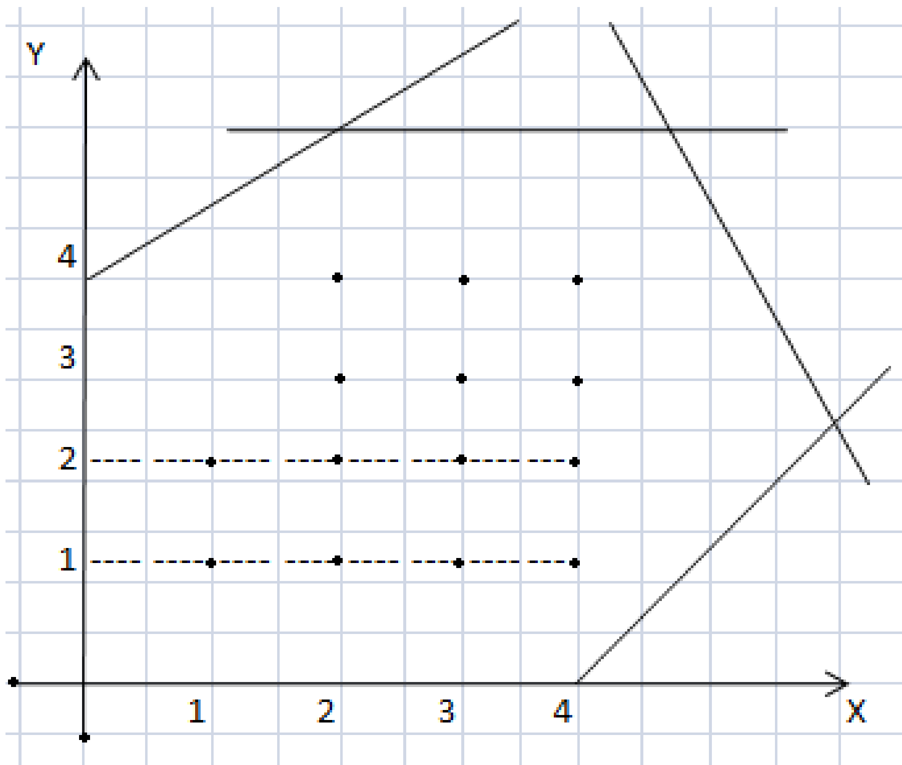


Figure 1: Feasible points of ILPP

Method of solving Integer Linear Programming Problem

There are two general methods for solving integer linear programming problem. These are

(i) Branch and Bound method (BBM)

(ii) Cutting plane method (Gomory's method)

But in this paper we only focus with the Branch and bound method.

Branch –and –Bound –method (BBM)

Branch and bound technique is one method of solving ILPP which is based on the concept of divide and conquer. Since the original "large" problem is too difficult to be solved directly, it is divided into smaller sub problems until these sub problems can be conquered. The dividing (branching) is done by partitioning the entire set of feasible solutions into smaller and smaller subsets. The conquering (fathoming) is done partially by bounding how good the best solution in the subset if it's bound indicates that it cannot possibly contain an optimal solution for the original problem. In general, the –branch – and bound approach is based on the principal that the total set of feasible region can be partitioned in to smaller subsets of feasible region. This smaller subset can then be evaluated systematically until the best solution is found.

When the branch and bound approach is applied to an integer linear programming problem, it is used in conjunction with the normal non integer solution approach.

The general formulation of integer linear programming problem is,

$$\begin{aligned} Max \quad & Z = C^T X \\ subject\ to \quad & AX \leq b \\ & X_i \geq 0\ and\ integers. \end{aligned}$$

Basic steps in branch and bound method (BBM)

There are three steps in branch and bound method.

Step1) Initialization: - Relax the integer programming problem (RILPP) and solve the relaxed problem. Relaxed the integer linear programming means the integer linear programming problem without the integrality restriction. In other word it means the corresponding linear programming problem.

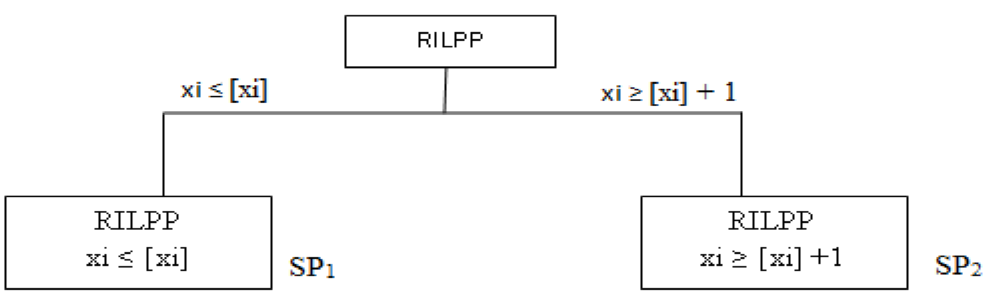
$$\left.\begin{aligned} Max \quad & Z = C^T X \\ subject\ to \quad & AX \leq b \\ & X_i \geq 0\ and\ integers. \end{aligned}\right\} \text{RILPP}$$

Then if the optimal solution to the LPP happens to be integral, then this solution is also the optimal solution to the original ILPP. Otherwise by identifying the upper and lower bounds as follows and go to the next step.

The optimal solution of the Relaxed Integer Linear Programming Problem $= U_b$

The optimal solution obtained by rounding off down = L_b

Step 2) Branching: If the current optimal solution of the Relaxed Integer Linear Programming Problem has fractions, then choose the one with the highest fractional value, and it is the first branching point.



Step3) Fathoming: a sub problem will be fathomed (that is ignore from further branching) for one of the following reasons.

1. If the current optimal solution is in agreement with the integrality Condition.
2. If the current optimal solution is infeasible
3. If the current optimal value is less than Lb

2. FORMULATION

Now let's us solve the real Integer Linear Programming Problem of Messebo Cement Factory using the Branch and Bound Method.

- Messebo Cement Factory is mainly engaged in producing two types of products, Ordinary Portland Cement (OPC) and Portland Pozolana Cement (PPC). To produce one unit OPC, 90% clinker, 5% pozolana and 5% gypsum is needed and similarly to produce one unit of PPC, 70% clinker, 25 % pozolana and 5% gypsum is used and the total available quantities are 106.5 tone/hr clinker, 37 tone/hr pozolana and 6.5 tone/hr gypsum. The prices are 190 birr and 150 birr per unit of OPC and PPC respectively.

Here what our objective is to maximize the profit. In other word to maximize our revenue so that our objective function will become $Max\ Z = 190\ x_1 + 150x_2$, where the decision variables $x_1\ and\ x_2$represents the number of unit products OPC and PPC to be produced respectively. And 1 unit represents 100kg.

Resource/constraints	Product type		Total available
	A (OPC)	B (PPC)	
Clinker	90%	70%	106.5
Pozolana	5%	25%	37
Gypsum	5%	5%	6.5
Cost of product	190	150	

Then as you can see it from the table, the constraints are the following

$$0.9\ x_1 + 0.7\ x_2 \leq 106.5$$

$$0.05x_1 + 0.25\ x_2 \leq 37$$

$$0.05x_1 + 0.05\ x_2 \quad \leq 6.5$$

Finally, the ILPP (integer linear programming problem) model or the mathematical expression for the above verbal problem of Messebo Cement Factory will become

$$Max\ Z = 190\ x_1 + 150\ x_2$$

$$s.t \quad 0.9\ x_1 + 0.7\ x_2\ \leq 106.5$$

$$0.05x_1 + 0.25\ x_2 \leq 37$$

$$0.05\,x_1 + 0.05\,x_2 \ \leq 6.5$$

$$x_1, x_2 \geq 0 \; and\; integers$$

Now let's first solve the RILPP (relaxed integer linear programming problem). And since the RILPP is its corresponding LPP (linear programming problem) that is obtained by omitting the integrality constraint, the RILPP of the above ILPP will become

$$\left.\begin{aligned} Max\,Z &= 190\,x_1 + 150\,x_2 \\ s.t \quad & 0.9\,x_1 + 0.7\,x_2 \ \leq 106.5 \\ & 0.05x_1 + 0.25\,x2 \leq 37 \\ & 0.05\,x_1 + 0.05\,x_2 \ \leq 6.5 \end{aligned}\right\} \text{RILPP}$$

$$x_1, x_2 \geq 0$$

And since it is a function of two variables namely x_1 and x_2 we can solve it using the graphical method of solving LPP.

As you can examine it from the graph below (Figure1) the feasible region of the ILPP are the dotted or the integer points inside the feasible region of the LPP (RLPP). Then since the optimal solution of the LPP is found at a corner point(s), we have four candidate points where the optimal solution of the RILPP or LPP can be attained

3. Solution of the problem

Solution of the RILPP

Vertex	Value of the Objective function
$A\,(0,0)$	$Z = 190(0) + 150(0) = 0$
$B\,(118.33,0)$	$Z = 190(118.33) + 150(0) = 22{,}382.7$
$C\,(77.5,52.5)$	$Z = 190(72.5) + 150(52.5) = 22{,}600 \leftarrow max$
$D(0,130)$	$Z = 190(0) + 150(130) = 19{,}500$

From this we can see that the optimal solution of the RILPP is at $({x_1}^*, {x_2}^*) = (77.5,52.5)$ and the optimal value will be 22,600. But since the optimal solution$({x_1}^*, {x_2}^*) = (77.5,52.5)$ does not satisfy the integrality constraint that is ${x_1}^* \sim \in Z(integers) and {x_2}^* \sim \in Z(integers)$ the solution of the RILPP cannot be the solution of the ILPP so we go to the next step.

$$\begin{cases} U_b\text{: Z=22,600 obtained at the solution of the RILPP, } ({x_1}^*, {x_2}^*) = (77.5,52.5). \\ L_b\text{: Z=22,430 obtained by lowered off } ({x_1}^*, {x_2}^*) = (77.5,52.5)\text{at}(x_1, x_2,) = (77,52). \end{cases}$$

W.O.L.G let take x_1- is the branching variable.

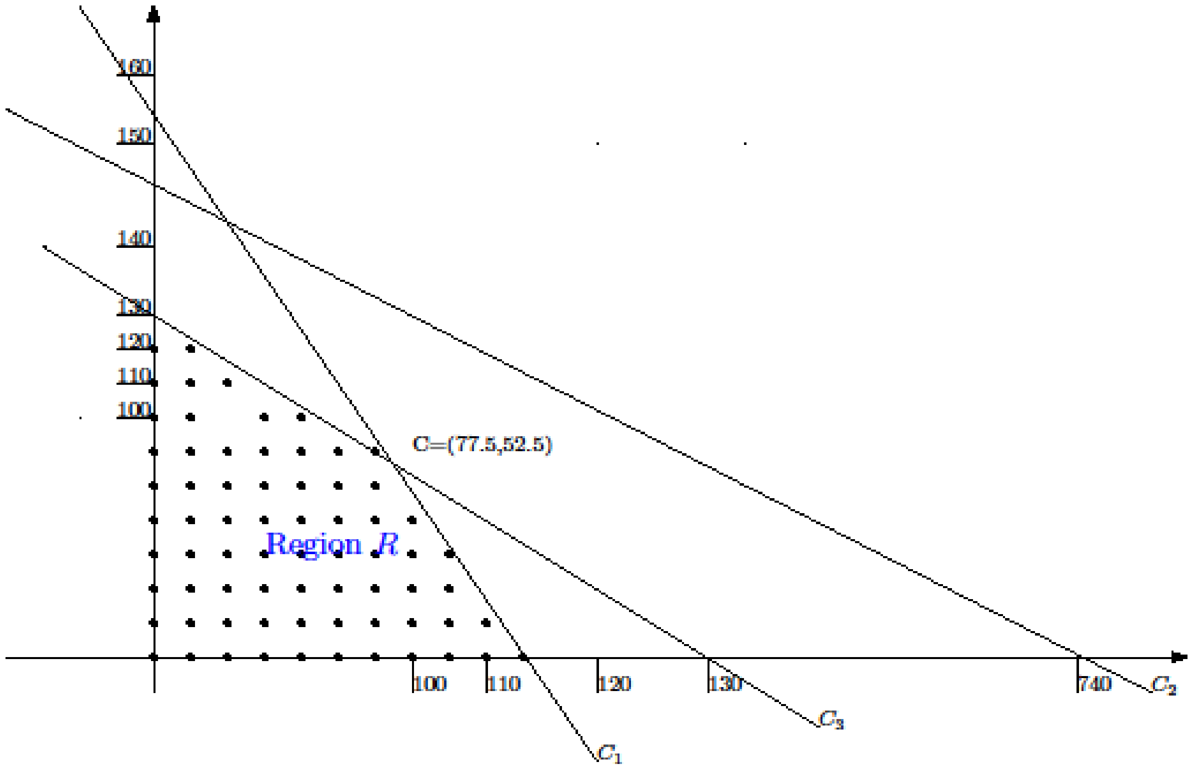
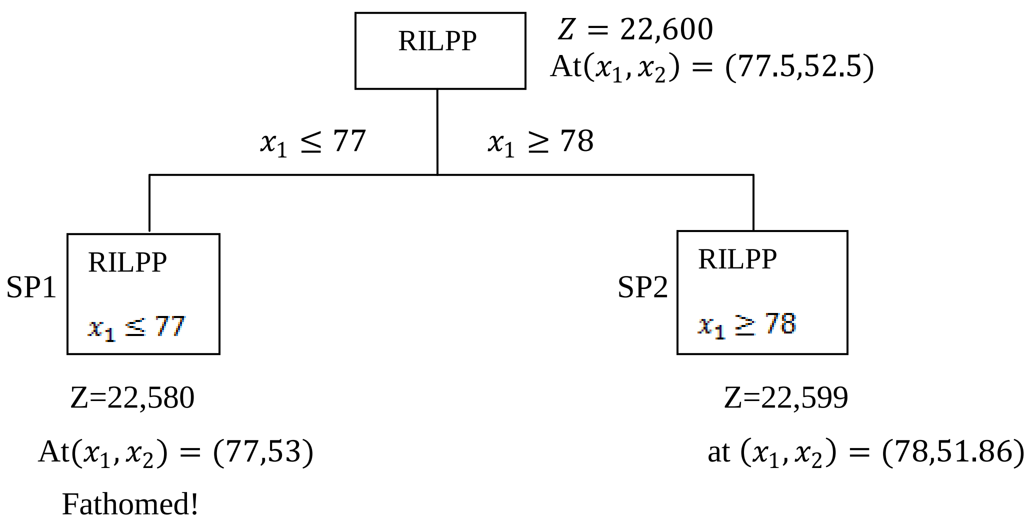


Fig2: Feasible points of ILPP

After this we are in a position to branch the feasible region into two sub regions and then divide the problem into sub problems by introducing a new constraints, $x_1 \leq 77$ and $x_1 \geq 78$ as you see it below



Now, let's solve the sub problems individually.

Solution of SP1 ($RILPP$ & $x_1 \leq 77$)

Vertex	**z-value**
$A(0,0)$	$Z = 190(0) + 150(0) = 0$
$B(77,0)$	$Z = 190(77) + 150(0) = 14{,}630$
$C(77,53)$	$Z = 190(77) + 150(53) = 22{,}580 \leftarrow max$
$D(0,130)$	$Z = 190(0) + 150(130) = 19{,}500$

- And the solution of SP1 $(x_1, x_2) = (77,53)$ is agreed with the integrality constraint so it will be fathomed. In other word going through this SP1 will not give us a better solution than the obtained one at this level so we stop here.

Solution of SP2 ($RILPP$ & $x_1 \geq 78$)

Vertex	**z-value**
$A(78,0)$	$Z = 190(78) + 150(0) = 14{,}820$
$B(118.33,0)$	$Z = 190(118.33) + 150(0) = 22{,}482.7$
$C(78,51.86)$	$Z = 190(78) + 150(51.86) = 22{,}599 \leftarrow max$

- But since the solution of SP2 is $(x_1, x_2) = (78,51.86)$ is not in agree with the integrality constraint we go further by branching it into two sub problems namely SP3 and SP4 by adding new constraints $x_2 \leq 51$ and $x_2 \geq 52$. As we see it in Fig2 below.
- And since **SP4** is infeasible it will be fathomed.

Solution of SP3 ($SP2$ & $x_2 \leq 51$)

Vertex	**z-value**
$A(118.33,0)$	$Z = 190(118.33) + 150(0) = 22{,}482.7$
$B(78,0)$	$Z = 190(78) + 150(0) = 14{,}820$
$C(78,51)$	$Z = 190(78) + 150(51) = 22{,}470$
$D(78.67, 51)$	$Z = 190(78.67) + 150(51) = 22{,}597.3 \leftarrow max$

- Since the solution of SP3, $(x_1, x_2) = (78.67,51)$ not satisfying the integrality constraint SP3 will be branched into two sub problems namely SP5 and SP6. See Fig3 below.

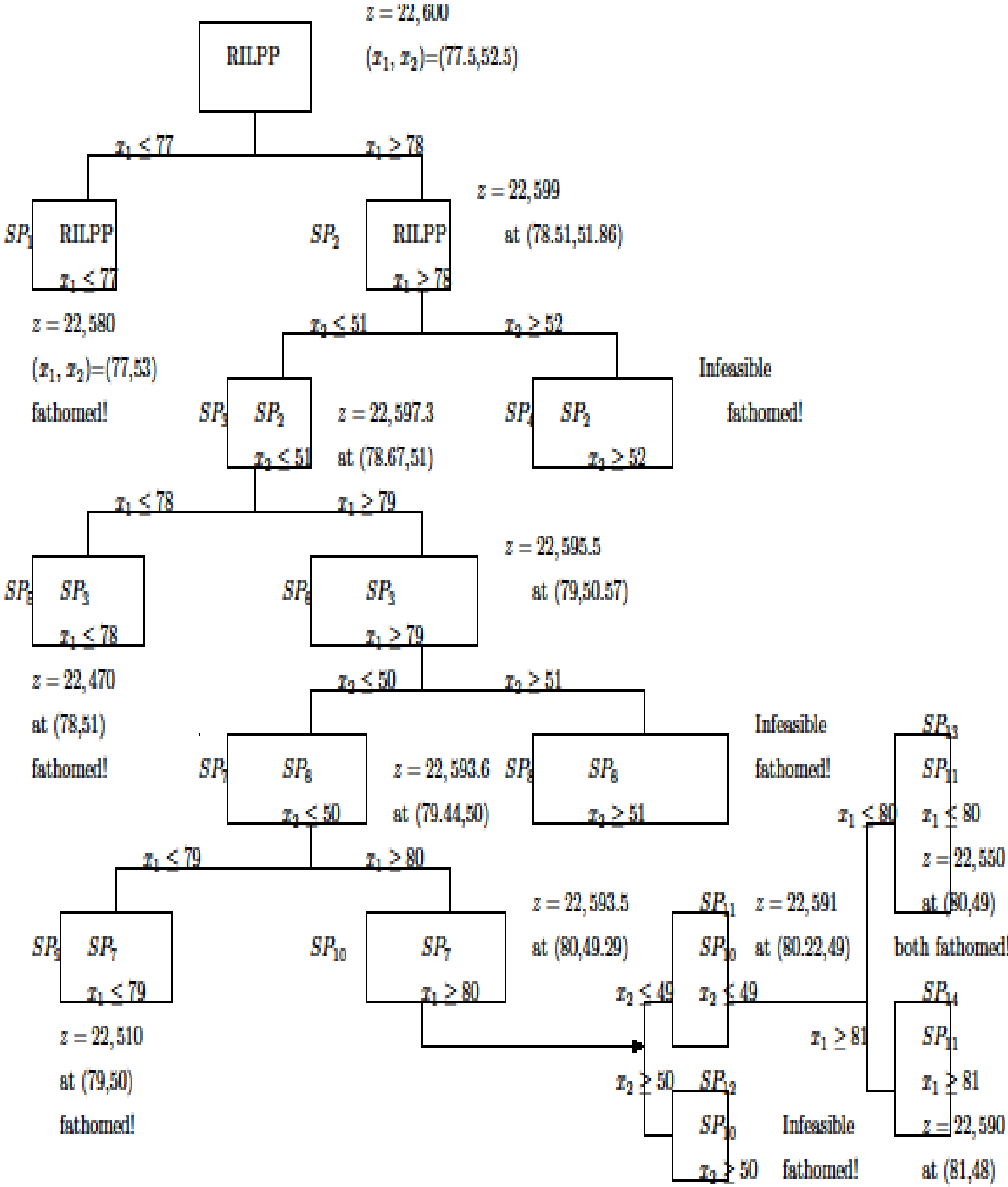


Fig3: The tree that shows the branching of the sub problems

Solution of SP5 ($\boldsymbol{SP3}$ & $\boldsymbol{x_1} \leq 78$)

Vertex	**z-value**
$A(78,0)$	$Z = 190(78) + 150(0) = 14{,}820$
$B(78,51)$	$Z = 190(78) + 150(51) = 22{,}470 \leftarrow max$

- Since the solution of SP5 $(x_1, x_2) = (78,51)$ satisfies the integrality constraint it will be Fathomed.

Solution of SP6 ($\boldsymbol{SP3}$ & $\boldsymbol{x_1} \geq 79$)

Vertex	**z-value**
$A(79,0)$	$Z = 190(79) + 150(0) = 15{,}010$
$B(118.33,0)$	$Z = 190(118.33) + 150(0) = 22{,}482.7$
$C(79,50.57)$	$Z = 190(79) + 150(50.57) = 22{,}595.5 \leftarrow max$

- The solution of SP6 $(x_1, x_2) = (79,50.57)$ is not in agreed with the integrality constraint so it will be branched into two sub problems namely SP7 and SP8.
- **SP8** is infeasible so it will be Fathomed.

Solution of SP7 ($\boldsymbol{SP6}$ & $\boldsymbol{x_2} \leq 50$)

Vertex	**z-value**
$A(79,0)$	$Z = 190(79) + 150(0) = 15{,}010$
$B(118.33,0)$	$Z = 190(118.33) + 150(0) = 22{,}482.7$
$C(79,50)$	$Z = 190(79) + 150(50) = 22{,}510$
$D(79.44,50)$	$Z = 190(79.44) + 150(50) = 22{,}593.6 \leftarrow max$

- The solution of SP7 $(x_1, x_2) = (79.44,50)$ is not in agreed with the integrality constraint so it will be branched into two sub problems namely SP9 and SP10.

Solution of SP9 ($\boldsymbol{SP7}$ & $\boldsymbol{x_1} \leq 79$)

Vertex	**z-value**
$A(79,0)$	$Z = 190(79) + 150(0) = 15{,}010$
$B(79,50)$	$Z = 190(79) + 150(50) = 22{,}510 \leftarrow max$

- Since the solution $(x_1, x_2) = (79,50)$ agree with the integrality constraint so, SP9 is Fathomed.

Solution of SP10 ($\boldsymbol{SP7}$ & $\boldsymbol{x_1} \geq 80$)

Vertex	**z-value**
$A(80,0)$	$Z = 190(80) + 150(0) = 15{,}200$
$B(118.33,0)$	$Z = 190(118.33) + 150(0) = 22{,}482.7$
$C(80,49.29)$	$Z = 190(80) + 150(49.29) = 22{,}593.5 \leftarrow max$

- The solution of SP10 $(x_1, x_2) = (80,49.29)$ is not satisfying the integrality constraint so it will be branched into two sub problems namely SP11 and SP12.
- **SP12** is infeasible so it will be Fathomed.

Solution of SP11 ($\boldsymbol{SP10}$ & $\boldsymbol{x_2} \leq 49$)

Vertex	**z-value**
$A(80,0)$	$Z = 190(80) + 150(0) = 15{,}200$
$B(118.33,0)$	$Z = 190(118.33) + 150(0) = 22{,}482.7$
$C(80,49)$	$Z = 190(80) + 150(49) = 22{,}550$
$D(80.22,49)$	$Z = 190(80.22) + 150(49) = 22{,}591.8 \leftarrow max$

- The solution of SP11 $(x_1, x_2) = (80.22,49)$ is not in agreed with the integrality constraint so it will be branched into two sub problems namely SP13 and SP14.

Solution of SP13 ($\boldsymbol{SP11}$ & $\boldsymbol{x_1} \leq 80$)

Vertex	**z-value**
$A(80,0)$	$Z = 190(80) + 150(0) = 15{,}200$
$B(80,49)$	$Z = 190(80) + 150(49) = 22{,}550 \leftarrow max$

- Since the solution $(x_1, x_2) = (80,49)$ satisfies the integrality constraints SP13 is Fathomed.

Solution of SP14 ($\boldsymbol{SP11}$ & $\boldsymbol{x_1} \geq 81$)

Vertex	**z-value**
$A(81,0)$	$Z = 190(81) + 150(0) = 15{,}390$
$B(118.33,0)$	$Z = 190(118.33) + 150(0) = 22{,}482.7$
$C(81,48)$	$Z = 190(81) + 150(48) = 22{,}590 \leftarrow max$

- Since the solution $(x_1, x_2) = (81,48)$ satisfies the integrality constraints SP14 is Fathomed.

Now since the entire sub problems are Fathomed, the solution of a sup problem which results with highest Z-value among all the sup problems will be the solution of the ILPP. In this case the solution of SP14 which is$(x_1, x_2) = (81,48)$ has the highest Z-value with 22,590 among all the sub problems thus the optimal solution of the ILPP will be$({x^*}_1, {x^*}_2) = (81,48)$and the optimal value will become $Z^* = 22{,}590$. this means, MCF has to produce 8100 kg of OPC and 4800 kg of PPC to maximize its profit.

4. CONCLUSION

In BBM, we add a constraint and the added constraint are used to divide the feasible region in to two sub regions. In integer linear programming problem rounding off is meaningless. For instance, take the ILPP

$$\begin{aligned} \max Z &= 5x_1 + 4x_2 \\ s.t. \quad & x_1 + x_2 \le 5 \\ & 10x_1 + 6x_2 \le 45 \\ & x_1, x_2 \ge 0 \; and\ integers \end{aligned}$$

Then the optimal value of the RILPP is $z^* = 23.75$ at the optimal solution $(x_1, x_2) = (3.75,1.25)$ and if we rounding off down, we get $(x_1, x_2) = (3,1)$ and at $(3,1)$ we get the value $Z = 19$ but, at the feasible point$(x_1, x_2) = (3,2)$ we get a better optimal value Z= 23.so in ILPP rounding off is meaningless.

References:

[1] Dimitris Bertsimas, John N.Tsitsiklis, Introduction to Linear Optimization, Athena Scientific, Belmont, Massachusetts, 1997.
[2] Frederick S.Hillier,Gerald J.Lieberman, introduction to Operations research, seventh edition, McGraw hill, New York, 2000.
[3] Hamdy A.Taha, Operations research: An introduction, eighth edition, Pearson Prentice Hall, New Jersey, 2007.
[4] Singiresu S.RAO, Engineering Optimization theory and practice, fourth edition, John Wiley & Sons, 2009.
[5] Wayne L.winston, Operations research applications and algorithms, Duxbury press, fourth edition, Belmont, 2004.