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Families of Iterative Methods with Higher Convergence Orders for Solving Nonlinear Equations

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Abstract.

Recently, there has been progress in developing Newton-type methods with higher convergence to solve nonlinear equations. This paper develops new classes of iterative methods with higher convergence orders (more than four) by a construction of two iterative methods of integer-order of convergence, p, q and $p > q$. The construction of such methods provides new classes of methods of order $2p + q$ of convergence. Numerical examples are provided, as well as the use of other existing methods to demonstrate the performance of the presented methods.

Keywords: Nonlinear Equations; Iterative Method; Newton's Method; Order of Convergence.

1. Introduction

The solution of nonlinear equations has been one of the most widely investigated topics in applied mathematics, which has produced a vast amount of literature; see, for example, the references and references therein.

Let $f: D \subset R \rightarrow R$ be a scalar function on an open interval D. If $f(\alpha) = 0$, then α is said to be a zero of $f(x)$. In other words, α is a root of the equation

$$f(x) = 0. \tag{1}$$

It is customary to say that α is a root or zero of an algebraic polynomial $f(x)$, but only a zero if $f(x)$ is not a polynomial.

According to the Kung and Traub's conjecture, an optimal iterative method without memory based on $n + 1$ functional evaluations could achieve the optimal convergence order of 2^n (Li et al. 2011). The famous example of optimal quadratic convergence is the Newton's method for finding the root α based on the follwing iterative scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{2}$$

starting from some initial value x_0.

The following theorem considers a generalization of Newton's method and describes a means for constructing multipoint methods.

Theorem 1:*(Traub 1964) Let α be a simple root of a function $f(x)$ and let $u_n(x)$ define the iterative method of the order p. Next, the composite iterative function x_{n+1} that is introduced by Newton's method* (2)

$$x_{n+1} = u_n - \frac{f(u_n)}{f'(x_n)} \tag{3}$$

defines the iterative method of the order $p+1$.

Using double the Newton method and approximating $f'(g_n)$ by $\frac{f'(x_n)}{w(t_n)}, t_n = \frac{f(g_n)}{f(x_n)}$, to reduce the number of functions evaluations, then the two-step scheme will be (Chun 2008)

$$\begin{aligned} g_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= g_n - w(t_n)\frac{f(g_n)}{f'(x_n)}, t_n = \frac{f(g_n)}{f(x_n)}, \end{aligned} \tag{4}$$

where the multiplicative function or the multiplier $w(t_n)$ in (4) must be determined such that $w(0) = 1, w'(0) = 2$ and $|w''(0)| < \infty$. As a result, the two-point method (4) is the optimal fourth-order of convergence (Petkovic′ 2009). This result is a consequence of the generalization of Traub's theorem (Traub 1964), as presented in the following theorem.

Theorem 2: *Let $u_n(x)$ and $v_n(x)$ be iterative functions of the orders p and q, respectively. Next, the composition of the iterative functions*

$$x_{n+1} = u_n(v_n(x)),$$

defines the iterative method of the order pq.

Recently, a new Newton-type iterative method to solve nonlinear equations with any integer order of convergence was considered by Li et al. (2011); this method be presented in next theorem.

Theorem 3: *Suppose $f(x) \in C^{p+q}(D)$ and there is a zero α of the nonlinear equation $f(x) = 0$. For two iterative functions $u_n(x)$ and $v_n(x)$ of convergence order p and q respectively (p, q are integers and $p > q$). If their results of the $n+1$th iteration are u_{n+1} and v_{n+1}, then the following iterative formula*

$$\mathrm{x}_{n+1} = \mathrm{u}_{n+1} - \frac{\mathrm{f}(\mathrm{u}_{n+1})}{\mathrm{f}'(\mathrm{v}_{n+1})}, \tag{5}$$

has order $p+q$.

Method (5) is a three-step iterative method, and has two extra functional evaluations more than the optimal requirement.

This paper develops new families of Newton-type methods of any integer order of convergence (more than four) to find a zero of a nonlinear function based on any two iterative methods of integer orders p and q of convergence and $p > q$. In Section 2, the methods and proving the order of convergence of the proposed method will be considered. In Section 3, several computational aspects of the proposed methods are given. Section 4 provides the short conclusions.

2. Development of the Methods

Using the concept of (5) in the iterative method (4), we have

$$x_{n+1} = g_{n+1} - \frac{f(g_{n+1})}{f'(h_{n+1})} - w(t_{n+1})\frac{f\left(g_{n+1} - \frac{f(g_{n+1})}{f'(h_{n+1})}\right)}{f'(h_{n+1})}, \tag{6}$$

where g_{n+1} and h_{n+1} are two iterative methods of order p and q, respectively and $p > q$, $t_{n+1} =$

$\frac{f(g_{n+1}-\frac{f(g_{n+1})}{f'(h_{n+1})})}{f(g_{n+1})}$, and $w(t_{n+1})$ is a function of t_{n+1}.

Theorem 4: *Suppose that $f \in C^{2p+q}(D)$, where the integers $p > q$, and $\alpha \in D$ is a simple root of the nonlinear equation $f(x) = 0$. Let the $n+1$th iteration of two iterative functions be g_{n+1}, h_{n+1} of orders p and q, respectively. Next, the convergence order of (6) is $2p+q$, where $t_{n+1} = \frac{f(g_{n+1}-\frac{f(g_{n+1})}{f'(h_{n+1})})}{f(g_{n+1})}$, and $w(t_{n+1})$ is a function satisfying $w(0) = w'(0) = 1$ and $|w''(0)| < \infty$.*

Proof: Let α be a simple root of $f(x) = 0$, that is, $f(\alpha) = 0$, and the error is expressed as

$$x_n = \alpha + e_n. \tag{7}$$

Using Taylor expansion, we have

$$g_{n+1} = \alpha + c_p e^p + O(e^{p+1}), \tag{8}$$

$$h_{n+1} = \alpha + c_q e^q + O(e^{q+1}), \tag{9}$$

where $c_p, c_q \neq 0$, and $c_k = \frac{f^{(k)}(\alpha)}{k!}, k = 1,2, ...$

Expanding $f(g_{n+1})$ about α, we have

$$\begin{aligned} f(g_{n+1}) &= f(\alpha) + f'(\alpha)(g_{n+1}-\alpha) + \frac{f''(\alpha)}{2}(g_{n+1}-\alpha)^2 + O(g_{n+1}-\alpha)^3 \\ &= f'(\alpha)[(g_{n+1}-\alpha) + \tfrac{A}{2}(g_{n+1}-\alpha)^2 + O((g_{n+1}-\alpha)^3)], \\ &= f'(\alpha)[c_p e^p + \cdots + (c_{2p} + \tfrac{A}{2}c_p^2)e^{2p} + O(e^{2p+1})], \end{aligned} \tag{10}$$

where $A = \frac{f''(\alpha)}{f'(\alpha)}$. Further, we have

$$\begin{aligned} f'(h_{n+1}) &= f'(\alpha) + f''(\alpha)(h_{n+1}-\alpha) + O(h_{n+1}-\alpha)^2 \\ &= f'(\alpha)[1 + Ac_q e^q + O(e^{q+1})]. \end{aligned} \tag{11}$$

Dividing (10) by (11), we obtain

$$\frac{f(g_{n+1})}{f'(h_{n+1})} = c_p e^p + \cdots + (c_{p+q} - c_p c_q A)e^{p+q} + O(e^{p+q+1}). \tag{12}$$

Using (8) and (12), we obtain

$$g_{n+1} - \frac{f(g_{n+1})}{f'(h_{n+1})} = \alpha + c_p c_q A e^{p+q} + O(e^{p+q+1}).$$

Expanding $f(g_{n+1} - \frac{f(g_{n+1})}{f'(h_{n+1})})$ about α, we have

$$f(g_{n+1} - \frac{f(g_{n+1})}{f'(h_{n+1})}) = f'(\alpha)[c_p c_q A e^{p+q} + O(e^{p+q+1})]. \tag{13}$$

Dividing (13) by (11), we have

$$\frac{f(g_{n+1}-\frac{f(g_{n+1})}{f'(h_{n+1})})}{f'(h_{n+1})} = c_p c_q A e^{p+q} + O(e^{p+q+1}). \tag{14}$$

Further, dividing (13) by (10), we have

$$t_{n+1} = \frac{f(g_{n+1} - \frac{f(g_{n+1})}{f'(h_{n+1})})}{f(g_{n+1})}$$
$$= c_q A e^q + O(e^{q+1}). \quad (15)$$

Expanding $w(t_{n+1})$ by the Taylor's polynomial of the second order at the point $t_{n+1} = 0$,

$$w(t_{n+1}) = w(0) + w'(0)t_{n+1} + \frac{w''(0)}{2} t_{n+1}^2$$
$$= 1 + c_q A e^q + O(e^{q+1}). \quad (16)$$

Substituting (13), (14), and (16) into the last step of (6), and takeing $w(0) = w'(0) = 1$ and $|w''(0)| < \infty$, we obtain

$$x_{n+1} = \alpha + c_p^2 c_q \frac{A^2}{2} e^{2p+q} + O(e^{2p+q+1}). \quad (17)$$

This result indicates that the order of convergence of the new classes of of the iterative methods (6) is $2p + q$ and also ends the proof.

Some special methods of (6) are considered depending upon the function $w(t_{n+1})$. In addition, $w(t_{n+1})$ satisfies the last conditions of Theorem 4 in a limiting case when $t \to 0$.

Method 1. Choosing

$$w(t_{n+1}) = -\frac{1}{t_{n+1} - 1}, \ t_{n+1} = \frac{f(g_{n+1} - \frac{f(g_{n+1})}{f'(h_{n+1})})}{f(g_{n+1})},$$

gives new classes of methods defined by the iterative function

$$x_{n+1} = g_{n+1} + \frac{f^2(g_{n+1})}{f'(h_{n+1})} \frac{1}{f(g_{n+1} + \frac{f(g_{n+1})}{f'(h_{n+1})}) - f(g_{n+1})}. \quad (18)$$

Method 2. The function

$$w(t_{n+1}) = 1 + t_{n+1} + \beta t_{n+1}^2.$$

where β is a real parameter and which produces new classification of methods

$$x_{n+1} = g_{n+1} - \frac{f(g_{n+1})}{f'(h_{n+1})} - \frac{f(g_{n+1} - \frac{f(g_{n+1})}{f'(h_{n+1})})}{f'(h_{n+1})} [1 + \frac{f(g_{n+1} - \frac{f(g_{n+1})}{f'(h_{n+1})})}{f(g_{n+1})} + \frac{\beta}{f^2(g_{n+1})} f^2(g_{n+1} - \frac{f(g_{n+1})}{f'(h_{n+1})})].$$
(19)

Method 3. The choice

$$w(t_{n+1}) = \frac{1}{1 - t_{n+1} + \beta t_{n+1}^2},$$

provides a new families defined by the iterative function

$$x_{n+1} = g_{n+1} - \frac{f^2(g_{n+1})}{f'(h_{n+1})} - \frac{f(g_{n+1} - \frac{f(g_{n+1})}{f'(h_{n+1})})}{f'(h_{n+1})}$$

$$\times \frac{f^2(g_{n+1})}{f^2(g_{n+1})-f(g_{n+1})f(g_{n+1}-\frac{f(g_{n+1})}{f'(h_{n+1})})+\beta f^2(g_{n+1}-\frac{f(g_{n+1})}{f'(h_{n+1})})}. \qquad (20)$$

Method 4. For $w(t_{n+1})$ given by

$$w(t_{n+1}) = 1 + \frac{t_{n+1}}{1+\beta t_{n+1}},$$

we obtain a new class of methods

$$x_{n+1} = g_{n+1} - \frac{f(g_{n+1})}{f'(h_{n+1})} - \frac{f(g_{n+1}-\frac{f(g_{n+1})}{f'(h_{n+1})})}{f'(h_{n+1})} \times \frac{f(g_{n+1})+(\beta+1)f(g_{n+1}-\frac{f(g_{n+1})}{f'(h_{n+1})})}{f(g_{n+1})+\beta f(g_{n+1}-\frac{f(g_{n+1})}{f'(h_{n+1})})}. \qquad (21)$$

Method 5. Choosing $w(t_{n+1})$ in the form

$$w(t_{n+1}) = (1+\beta t_{n+1})^{\frac{1}{\beta}}, \beta \neq 0,$$

provides a new family of methods defined by the iterative function

$$x_{n+1} = g_{n+1} - \frac{f(g_{n+1})}{f'(h_{n+1})} - \frac{f(g_{n+1}-\frac{f(g_{n+1})}{f'(h_{n+1})})}{f'(h_{n+1})}[1+\beta f(g_{n+1} - \frac{f(g_{n+1})}{f'(h_{n+1})})]^{\frac{1}{\beta}}. \qquad (22)$$

These classes of methods $(18)-(22)$ have convergence of order $2p+q$. The iterative methods with nth-integer order convergence are not unique because two iterative methods with p and q order convergence, $p > q$ can be chosen by various formulas as long as $2p+q=n$. As a result, these iterative formulas $(18)-(22)$ provide us classes of methods for solving nonlinear equations.

3. Numerical Results

We compare several methods for solving the nonlinear equation (1) with the result of the new constructions of two different iterative methods of integer-order of convergence, using the fife classes of methods $(18)-(22)$ considered in the last section. All computations were performed using MATLAB 7.1 using 2000 digit floating point arithmetics (Digits:=2000). The following stopping criteria were used for the computer programs:

$$(i)|x_{n+1}-x_n| < \varepsilon,$$

$$(ii)|f(x_{n+1})| < \varepsilon,$$

as a result, x_{n+1} is taken as the exact root that is computed when the stopping criterion is satisfied. For numerical illustrations, in this section we used the fixed stopping criterion $\varepsilon = 10^{-30}$.

Tables 1 and 2 present the number of iterations to approximate the zero (IT) and the absolute values of $|f(x_{n+1})|$ and $|x_{n+1}-x_n|$, respectively. Tables 3 and 4 present the values of the computational order of convergence (COC), which is approximated using the formula (Weerakoon and Fernando 2000)

$$COC = \frac{\ln|(x_{n+1}-\alpha)/(x_n-\alpha)|}{\ln|(x_n-\alpha)/(x_{n-1}-\alpha)|}.$$

Example 1. Consider an iterative function constructed by combining Newton's method (2) and the secant method, (PM) for more details see (Petkovic′ 2009)

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = y_n - \frac{f(y_n)(y_n - x_n)}{f(y_n) - f(x_n)}. \qquad (23)$$

Combining (23) and the Newton's method (2) (NM) again, we obtain the following family of iterative methods of order eight of convergence,

$$x_{n+1} = g_{n+1} - \frac{f(g_{n+1})}{f'(h_{n+1})} - w(t_{n+1}) \frac{f(g_{n+1} - \frac{f(g_{n+1})}{f'(h_{n+1})})}{f'(h_{n+1})}, \qquad (24)$$

where g_{n+1} is (23), h_{n+1} is (2) and $w(t_{n+1}), t_{n+1} = \frac{f(g_{n+1} - \frac{f(g_{n+1})}{f'(h_{n+1})})}{f(g_{n+1})}$ are defined in Methods $1-5$.

The result of the methods (24) are denoted by $(S1M1 - S1M5)$ in the Tables 1 and 3, where $\beta = -1$ was used in all the methods except in method $S1M5$, where $\beta = 0.5$ was used to avoid the equality with the method $S1M4$.

The following test functions and their exact root α are used for numerical examples.

Test functions	The approximate zero α
$f_1(x) = x^2 - (1-x)^{25}$,	0.143739259299754,
$f_2(x) = e^{x^2+7x-30} - 1$,	3.0,
$f_3(x) = xe^{x^2} - \sin(x)^2 + 3\cos(x) + 5$,	-1.20764782713092,
$f_4(x) = \sin(x)^2 - x^2 + 1$,	1.40449164821534,
$f_5(x) = (x-1)^3 - 1$,	2.0,
$f_6(x) = x^2 - e^x - 3x + 2$,	0.257530285439861,
$f_7(x) = \cos(\frac{\pi x}{2}) + x^2 - \pi$,	2.03472489627913,
$f_8(x) = xe^x + \log(1 + x + x^4)$,	0.0.

Example 2. A modification for the fourth order method of Kou et. al. (2007), (25) done by Cordero et. al. (2010) to produce the following sixth order method (CM)

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$z_n = x_n - \theta \frac{f(x_n) + f(y_n)}{f'(x_n)} - (1-\theta) \frac{f(x_n)}{f(x_n) - f(y_n)} \frac{f(x_n)}{f'(x_n)}, \qquad (25)$$
$$x_{n+1} = z_n - \frac{f(z_n)}{f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n)}. \qquad (26)$$

where $\theta \in R - \{-1\}$, and $\theta = 0.5$ is taken in the Tables 2 and 4.

The optimal fourth-order iterative method of King (1973) (KM) for solving equation (1) is defined by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = y_n - \frac{f(y_n)}{f'(x_n)} \frac{f(x_n) + \beta f(y_n)}{f(x_n) + (\beta - 2) f(y_n)}, \beta \in R, \qquad (27)$$

Let the iterative method (26) be u_{n+1} and the King's methods, (27), be v_{n+1}, then the new family methods are

$$x_{n+1} = u_{n+1} - \frac{f(u_{n+1})}{f'(v_{n+1})} - w(t_{n+1})\frac{f(u_{n+1} - \frac{f(u_{n+1})}{f'(v_{n+1})})}{f'(v_{n+1})}, \qquad (28)$$

with the order sixteen of convergence. In addition, substituting methods 1-5 from last section in (28) gives $(S2M1 - S2M5)$ in Tables 2 and 4.

4. Conclusions

In this work we presented an approach that can be used for constructing families of $2p + q$th-order iterative methods that use two iterative methods of integer order of convergence p and q, where $p > q$. However, noted that per iteration the considered methods $(S1M1 - S1M5)$ and $(S2M1 - S2M5)$ require three more evaluations of functions than do the requirement of the optimality condition. The presented classes of methods are tested on several nonlinear functions and demonstrate remarkably fast convergence. In this paper, we constructed five such families, and more families could be constructed using the presented theory for solving systems of nonlinear of equations, which will be exploited in detail in a forthcoming paper.

Table 1: Numerical results for the different functions of Example 1

Method	IT	$\lvert f(x_{n+1})\rvert$	$\lvert x_{n+1} - x_n\rvert$	Method	IT	$\lvert f(x_{n+1})\rvert$	$\lvert x_{n+1} - x_n\rvert$
$f_1(x), x_0 = 0.35$				$f_2(x), x_0 = 3.5$			
NM	8	0.14148e-81	0.43566e-41	NM	14	0.11642e-93	0.11669e-47
PM	8	0.46151e-92	0.41981e-31	PM	9	0.10687e-111	0.57495e-38
LM	7	0.11012e-559	0.14451e-112	LM	7	0.59673e-205	0.90685e-42
S1M1	5	0.27897e-280	0.10394e-35	S1M1	6	0.37595e-851	0.40178e-107
S1M2	6	0.4e-1010	0.29522e-175	S1M2	6	0.37071e-483	0.40108e-61
S1M3	6	0.37071e-483	0.40108e-61	S1M3	5	0.28350e-302	0.16356e-38
S1M4	5	0.27897e-280	0.10394e-35	S1M4	6	0.37595e-851	0.40178e-107
S1M5	6	0.4e-1010	0.44022e-217	S1M5	6	0.16040e-622	0.15232e-78
$f_3(x), x_0 = -1.1$				$f_4(x), x_0 = 1.1$			
NM	7	0.39878e-99	0.36156e-50	NM	7	0.34436e-68	0.42075e-34
PM	5	0.23257e-189	0.17186e-63	PM	5	0.26050e-129	0.55495e-43
LM	4	0.15215e-376	0.12976e-75	LM	4	0.33032e-193	0.15457e-38
S1M1	3	0.10278e-319	0.37199e-40	S1M1	4	0.14e-1008	.61976e-167
S1M2	3	0.32768e-357	0.76466e-45	S1M2	4	0.1e-1008	0.57789e-211
S1M3	3	0.36695e-310	0.58159e-39	S1M3	4	0.14e-1008	0.50133e-148
S1M4	3	0.10278e-319	0.37199e-40	S1M4	4	0.14e-1008	0.61976e-167
S1M5	3	0.23951e-329	0.23252e-41	S1M5	4	0.14e-1008	0.19624e-189
$f_5(x), x_0 = 2.5$				$f_6(x), x_0 = 2.0$			
NM	8	0.84331e-111	0.16766e-55	NM	7	0.21174e-110	0.77434e-55

PM	5	0.45099e-105	0.53172e-35	PM	5	0.19671e-160	0.84158e-53
LM	4	0.11642e-215	0.39570e-43	LM	4	0.72206e-400	0.36254e-79
S1M1	4	0.0	0.31786e-186	S1M1	3	0.55746e-314	0.27161e-38
S1M2	4	0.0	0.19147e-170	S1M2	3	0.51742e-315	0.20179e-38
S1M3	4	0.0	0.49706e-207	S1M3	3	0.17170e-313	0.31262e-38
S1M4	4	0.95358e-186	0.37574e-23	S1M4	4	0.0	0.14753e-314
S1M5	4	0.0	0.29881e-178	S1M5	3	0.23418e-314	0.24370e-38
$f_7(x), x_0 = 1.87$				$f_8(x), x_0 = 0.25$			
NM	6	0.98787e-64	0.66530e-32	NM	6	0.76186e-68	0.12344e-33
PM	5	0.37519e-245	0.14627e-81	PM	5	0.60304e-260	0.36404e-86
LM	4	0.18716e-524	0.10623e-104	LM	4	0.33321e-581	0.10129e-115
S1M1	3	0.34782e-440	0.97417e-55	S1M1	3	0.34642e-485	0.49401e-60
S1M2	3	0.29714e-454	0.16986e-56	S1M2	3	0.19984e-478	0.34584e-59
S1M3	3	0.53057e-435	0.43307e-54	S1M3	3	0.21555e-489	0.14722e-60
S1M4	3	0.34782e-440	0.97417e-55	S1M4	3	0.34642e-485	0.49401e-60
S1M5	3	0.94106e-445	0.26162e-55	S1M5	3	0.19928e-482	0.10933e-59

Table 2: Numerical results for different functions of Example 2

Method	IT	$\lvert f(x_{n+1})\rvert$	$\lvert x_{n+1} - x_n\rvert$	Method	IT	$\lvert f(x_{n+1})\rvert$	$\lvert x_{n+1} - x_n\rvert$
$f_1(x), x_0 = 0.35$				$f_2(x), x_0 = 3.5$			
KM	5	0.18288e-378	0.43263e-95	KM	div		
CM	4	0.34080e-425	0.20156e-71	CM	6	0.28397e-723	0.44547e-121
LM	4	0.79247e-620	0.16347e-62	LM	5	0.85104e-846	0.41765e-85
S2M1	4	0.3e-2000	0.36152e-319	S1M1	4	0.44788e-752	0.18568e-47
S2M2	4	0.3e-1550	0.78853e-251	S1M2	5	0.0	0.82521e-216
S2M3	5	0.3e-1550	0.11647e-144	S1M3	5	0.0	0.29486e-219
S2M4	4	0.3e-2000	0.36152e-319	S1M4	4	0.44788e-752	0.18568e-47
S2M5	4	0.6e-1550	0.53129e-353	S1M5	5	0.0	0.73116e-224
$f_3(x), x_0 = -1.1$				$f_4(x), x_0 = 2.5$			
KM	4	0.24634e-202	0.77346e-51	KM	4	0.20712e-136	0.64532e-34
CM	3	0.54768e-180	0.45226e-30	CM	4	0.56433e-562	0.18662e-93
LM	3	0.40769e-827	.98107e-83	LM	3	0.10389e-490	0.80218e-49
S2M1	3	0.7e-1548	0.79815e-224	S1M1	3	0.1e-1549	0.38770e-133
S2M2	3	0.7e-1548	0.19173e-212	S1M2	3	0.1e-1549	0.66521e-122
S2M3	3	0.7e-1548	0.43533e-214	S1M3	3	0.1e-1549	0.14096e-133

S2M4	3	0.7e-1548	0.79815e-224	S1M4	3	0.1e-1549	0.38770e-133
S2M5	3	0.7e-1548	0.52484e-222	S1M5	3	0.1e-1549	0.79066e-127
$f_5(x), x_0 = 2.5$				$f_6(x), x_0 = 2.0$			
KM	5	0.48954e-422	0.20099e-105	KM	4	0.86048e-124	0.40869e-30
CM	4	0.50210e-565	0.48730e-94	CM	4	0.12805e-474	0.25485e-78
LM	3	0.38401e-377	0.14902e-37	LM	3	0.13879e-356	0.71758e-35
S2M1	3	0.0	0.15044e-100	S1M1	3	0.2e-1548	0.44264e-111
S2M2	3	0.29122e-1511	0.30128e-94	S1M2	3	0.28366e-1129	0.99320e-70
S2M3	3	0.0	0.11110e-108	S1M3	3	0.13736e-1134	0.46223e-70
S2M4	3	0.0	0.15044e-100	S1M4	3	0.2e-1548	0.44264e-111
S2M5	3	0.0	0.13880e-97	S1M5	3	0.17259e-1210	0.83378e-75
$f_7(x), x_0 = 1.87$				$f_8(x), x_0 = 0.25$			
KM	4	0.24204e-258	.24761e-64	KM	4	0.26221e-262	0.53821e-65
CM	3	0.17932e-199	0.58788e-33	CM	3	0.79314e-182	0.78205e-30
LM	3	0.30008e-934	0.45690e-93	LM	3	0.75470e-915	0.64969e-91
S2M1	3	0.1e-1548	0.12741e-241	S1M1	3	0.21826e-1550	0.96140e-246
S2M2	3	0.1e-1548	0.31293e-241	S1M2	3	0.43136e-1550	0.31731e-239
S2M3	3	0.1e-1548	0.79727e-242	S1M3	3	0.10420e-1550	036604e-256
S2M4	3	0.1e-1548	0.12741e-241	S1M4	3	0.21826e-1550	0.96140e-246
S2M5	3	0.1e-1548	0.17951e-241	S1M5	3	0.11156e-1549	0.14923e-242

Table 3: Comparison of the computational order of convergence (COC) for the various methods of Example 1.

Method	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	$f_5(x)$	$f_6(x)$	$f_7(x)$	$f_8(x)$
NM	2.00000	2.00000	2.00000	2.00000	2.00000	2.00000	2.00000	2.00000
PM	3.00000	3.00000	3.00000	3.00000	3.00000	3.00000	3.00000	3.00000
LM	5.00000	5.00000	5.00000	5.00000	5.00000	5.00000	5.00001	5.00001
S1M1	8.00000	7.99641	8.00000	7.99954	7.99983	7.99999	8.00000	8.00000
S1M2	8.00009	8.00000	8.00000	8.00005	7.99981	8.00743	8.00000	8.00000
S1M3	8.00000	8.00000	8.00000	7.99878	7.99991	8.00932	8.00000	8.00000
S1M4	8.00000	7.99641	8.00000	7.99954	7.99983	7.99999	8.00000	8.00000
S1M5	8.00000	8.00000	8.00000	7.99984	7.99980	7.99999	8.00000	8.00000

Table 4: Comparison of the computational order of convergence (COC) for the various methods of Example 2.

Method	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	$f_5(x)$	$f_6(x)$	$f_7(x)$	$f_8(x)$
KM	4.00000	div	4.00000	4.00000	4.00000	4.00000	4.00000	4.00000

CM	6.00000	5.99996	6.00000	6.00000	6.00000	6.00000	6.00000	6.00000
LM	10.0000	10.0000	10.0000	10.0000	10.0000	9.99997	10.0000	10.0000
S1M1	15.9416	16.0000	15.9782	16.1111	15.6962	15.9831	16.0393	16.0795
S1M2	16.0607	16.0824	16.0342	16.2216	16.0000	16.0000	16.0411	16.1110
S1M3	16.0529	16.0463	16.0257	16.1072	15.6420	16.0000	16.0383	16.0327
S1M4	15.9416	16.0000	15.9782	16.1111	15.6962	15.9831	16.0393	15.8489
S1M5	16.0152	16.0513	15.9867	16.1706	15.7305	16.0000	16.0400	16.0947

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