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Chatterjea type mappings in metric space and common fixed points theorem

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Abstract

The paper includes theorem giving the sufficient conditions for existence of common point of coincidence and common fixed point for 2n + 1 mappings in metric space under Chatterjea type condition.

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Introduction

A mapping $T : X \to X$, for a metric space (X, d), is called a contraction ([3], [5]), if there exists $L \in]0,1[$ such that $d(Tx, Ty) \leq L \cdot d(x, y)$, for all $x, y \in X$. The following theorem is proved by S. Banach ([3], [5]) and known as the contraction mapping theorem or contraction mapping principle or Banach fixed-point theorem. If (X, d) is a nonempty, complete metric space and $T : X \to X$ is a contraction, then there exists a unique fixed point $x* \in X$, the solution of $Tx* = x*$. Moreover, it provides a constructive method to find those fixed points. The Banach fixed-point theorem has the variety of applications and gives important tool in mathematical analysis. One of the significant application of Banach contraction theorem is the proof of existence and uniqueness of solutions of differential equations. This theorem is generalized and developed in different directions. Various assumptions and contractive conditions are considered by many authors, for example [6], [7], [8], [10]. The paper [9] includes a theorem giving a sufficient condition for existence of common fixed points and points of coincidence for $2n + 1$ $(n \in \mathbb{N})$ mappings in metric space (X, d) under Kannan type contractive contidtion. The present article contains the proof of analogous theorem under Chatterjea type condition.

1 Notations, definitions, lemma

Definition 1.1. ([4], [1]) A mapping $T : X \to X$ - for a metric space (X, d) - is called Chatterjea if there exists $\alpha \in [0, \frac{1}{2})$ such that for all $x, y \in X$

$$d(Tx, Ty) \leq \alpha[d(x, Ty) + d(y, Tx)]$$

Definition 1.2. A point $y \in X$ is called point of coincidence of a family $\{T_j\}_{j\in J}$ of self-mappings on X if there exists a point $x \in X$ such that $y = T_j x$ for all $j \in J$.

Definition 1.3. A pair (F, T) of self-mappings on X is said to be weakly compatible if $FTx = TFx$ whenever $Fx = Tx$.

The lemma presented below is a generalization of the result from the paper [1]. This lemma has been proven in the paper [9], but for convenience of the reader we quote lemma with the proof.

Lemma 1.4. ([9]). *Let $n \in \mathbb{N}$. Let X be a nonempty set and the mappings*

$$F, S_1, ..., S_n, T_1, ..., T_n$$

have a unique point of coincidence v in X. If all pairs (F, T_i), (F, S_i), for $i \in \{1, 2, ..., n\}$ and are weakly compatibles, then v is the unique common fixed point of mappings $F, S_1, ..., S_n, T_1, ..., T_n$.

Proof. Take $u \in X$ such that $Fu = S_1u = ... = S_nu = T_1u = ... = T_nu = v$. By weakly compatibility of all pairs (F, T_i), (F, S_i), for $i \in \{1, 2, ..., n\}$ we have

$$\Big(S_iv = S_iFu = FS_iu = Fv, \text{ and } T_iv = T_iFu = FT_iu = Fv\Big), \text{ for } i \in 1, ..., n.$$

Therefore the point w such that $w = Fv = S_1v = ... = S_nv = T_1v = ... = T_nv$ is a point of coincidence for mappings $F, S_1, ..., S_n, T_1, ..., T_n$, so $w = v$ by uniqueness. From the above v is a unique common fixed point of mappings $F, S_1, ..., S_n, T_1, ..., T_n$. □

The following definition also is quoted from the paper [9].

Definition 1.5. ([9]). Let (X, d) be a metric space. Let $F, S_1, ..., S_n, T_1, ..., T_n : X \to X$ be the mappings such that $S_i(X) \cup T_i(X) \subset F(X)$ for $i = 1, ...n$. We define the sequence $\{x_m\}_{m\geq 0}$ of elements of X as follows. Choose an arbitrary point x_0 in X. Let x_1, x_2 be the point of X such that $Fx_1 = T_1x_0$ and $Fx_2 = S_1x_1$. Continuing, $Fx_3 = T_2x_2$ and $Fx_4 = S_2x_3$,...,$Fx_{2n-1} = T_nx_{2n-2}$, $Fx_{2n} = S_nx_{2n-1}$. Generally, if we have defined x_{kn} for a $k \in \{0, 2, 4, ...\}$, we put

$$Fx_{kn+i} = \begin{cases} T_{\frac{i+1}{2}}x_{kn+i-1}, & \text{for } i \in \{1, 3, ..., 2n-1\}, \\ S_{\frac{i}{2}}x_{kn+i-1}, & \text{for } i \in \{2, 4, ..., 2n\}. \end{cases}$$

2 Main theorem

Theorem 2.1. *Let $n \in \mathbb{N}$. Let (X, d) be a complete metric space and*

$$F, S_1, ..., S_n, T_1, ..., T_n$$

be self mappings of the space (X,d) such that $F(X)$ is the closed subset of X and $\bigcup_{i=1}^{n} S_i(X) \subset F(X)$, $\bigcup_{i=1}^{n} T_i(X) \subset F(X)$. Let us suppose that the following condition is satisfied:

$$d(S_i x, T_j y) \leq A_i d(Fy, S_i x) + B_j d(Fx, T_j y), \quad \text{for } i,j \in \{1,2,...,n\} \tag{2.1}$$

for all $x, y \in X$ where A_i, B_j are non-negative real numbers with $A_i + B_j < 1$, for $i, j \in \{1,2,...,n\}$. Then $F, S_1, ..., S_n, T_1, ..., T_n$ have a unique point of coincidence. If additionally all pairs (F, T_i), (F, S_i), for $i \in \{1,2,...,n\}$ are weakly compatibles, then $F, S_1, ..., S_n, T_1, ..., T_n$ have the unique common fixed point.

Proof. It is easily observe that, if $F, S_1, ..., S_n, T_1, ..., T_n$ have a point of coincidence, then it is unique. Indeed, assume that v, v^* are two distinct point of coincidence for mappings $F, S_1, ..., S_n, T_1, ..., T_n$. Then, there exists u, u^* such that

$$Fu = S_1 u = ... = S_n u = T_1 u = ... = T_n u = v$$

and

$$Fu^* = S_1 u^* = ... = S_n u^* = T_1 u^* = ... = T_n u^* = v^*.$$

By (2.1) we have

$$d(v, v^*) = d(S_1 u, T_1 u^*) \leq A_1 d(Fu^*, S_1 u) + B_1 d(Fu, T_1 u^*),$$

whence

$$d(v, v^*) = d(S_1 u, T_1 u^*) \leq A_1 d(Fu^*, Fu) + B_1 d(Fu, Fu^*).$$

Therefore

$$d(v, v^*) \leq (A_1 + B_1) d(v, v^*),$$

so $v = v^*$. We will prove the existence of a point of coincidence of mappings $F, S_1, ..., S_n$, $T_1, ..., T_n$. Choose an arbitrary point x_0 in X. The sequence $\{x_m\}_{m \geq 0}$ of elements of X is defined by Definition 1.5. Let us consider two cases:

a) there exists $k \in \{0,2,4,...\}$ for which $Fx_{kn} = Fx_{kn+1}$.

b) for every $k \in \{0,2,4,...\}$ $Fx_{kn} \neq Fx_{kn+1}$.

In the case a) - by (2.1) - for every $i \in \{1,2,...,n\}$ - we have

$$\begin{aligned} d(S_i x_{kn}, Fx_{kn}) &= d(S_i x_{kn}, Fx_{kn+1}) = d(S_i x_{kn}, T_1 x_{kn}) \\ &\leq A_i d(Fx_{kn}, S_i x_{kn}) + B_1 d(Fx_{kn}, T_1 x_{kn}) \\ &= A_i d(Fx_{kn}, S_i x_{kn}) + B_1 d(Fx_{kn}, Fx_{kn+1}) \\ &= A_i d(Fx_{kn}, S_i x_{kn}) \end{aligned}$$

and

$$\begin{aligned} d(Fx_{kn}, T_i x_{kn}) &= d(S_n x_{kn}, T_i x_{kn}) \\ &\leq A_n d(Fx_{kn}, S_n x_{kn}) + B_i d(Fx_{kn}, T_i x_{kn}) \\ &= B_i d(Fx_{kn}, T_i x_{kn}). \end{aligned}$$

This yields that the point y defined as

$$y := Fx_{kn} = S_1x_{kn} = S_2x_{kn} = ... = S_nx_{kn} = T_1x_{kn} = T_2x_{kn} = ... = T_nx_{kn}$$

is the required unique point of coincidence for mappings $F, S_1, ..., S_n, T_1, ..., T_n$.

In the case b) the reasoning is as follows. Let $k \in \{0, 2, 4, ...\}$. We have $Fx_{kn} \neq Fx_{kn+1}$. By (2.1) we get

$$\begin{aligned} d(Fx_{kn}, Fx_{kn+1}) &= d(S_nx_{kn-1}, T_1x_{kn}) \\ &\leq A_nd(Fx_{kn}, S_nx_{kn-1}) + B_1d(Fx_{kn-1}, T_1x_{kn}) \\ &= B_1d(Fx_{kn-1}, Fx_{kn+1})) \\ &\leq B_1[d(Fx_{kn-1}, Fx_{kn}) + d(Fx_{kn}, Fx_{kn+1})]. \end{aligned}$$

Hence

$$\begin{aligned} d(Fx_{kn}, Fx_{kn+1}) &\leq \frac{B_1}{1-B_1}d(Fx_{kn-1}, Fx_{kn}) \\ &\leq \lambda d(Fx_{kn-1}, Fx_{kn}), \end{aligned}$$

where

$$0 < \lambda := max\left\{\frac{A_i}{1-A_i}, \frac{B_i}{1-B_i} : \ i = 1, 2, ...n\right\} < 1.$$

Moreover

$$\begin{aligned} d(Fx_{kn}, Fx_{kn-1}) &= d(S_nx_{kn-1}, T_nx_{kn-2}) \\ &\leq A_nd(Fx_{kn-2}, S_nx_{kn-1}) + B_nd(Fx_{kn-1}, T_nx_{kn-2}) \\ &= A_nd(Fx_{kn-2}, Fx_{kn}) + B_nd(Fx_{kn-1}, Fx_{kn-1}) \\ &= A_nd(Fx_{kn-2}, Fx_{kn}) \\ &\leq A_n[d(Fx_{kn-2}, Fx_{kn-1}) + d(Fx_{kn-1}, Fx_{kn})]. \end{aligned}$$

Therefore

$$\begin{aligned} d(Fx_{kn}, Fx_{kn-1}) &\leq \frac{A_n}{1-A_n}d(Fx_{kn-1}, Fx_{kn-2}) \\ &\leq \lambda d(Fx_{kn-1}, Fx_{kn-2}). \end{aligned}$$

From the above we get easily for any $m \in \mathbb{N}$

$$d(Fx_m, Fx_{m+1}) \leq \lambda^m d(Fx_0, Fx_1)$$

and for any $m_2 > m_1$

$$\begin{aligned} d(Fx_{m_1}, Fx_{m_2}) &\leq d(Fx_{m_1}, Fx_{m_1+1}) + d(Fx_{m_1+1}, Fx_{m_1+2}) + ... + d(Fx_{m_2-1}, Fx_{m_2}) \\ &\leq [\lambda^{m_1} + \lambda^{m_1+1} + ... + \lambda^{m_2}]d(Fx_0, Fx_1) \\ &\leq \left[\frac{\lambda^{m_1}}{1-\lambda}\right]d(Fx_0, Fx_1), \end{aligned}$$

so $(F_{x_m})_{m\in\mathbb{N}}$ is a Cauchy sequence. Let us define $v := \lim_{m\to\infty} F_{x_m}$. Since $F(X)$ is a closed subset of X then there exists $u \in X$ such that $F(u) = v$. Let $j \in \{1, 2, ..., n\}$. We have for a $k \in \{0, 2, 4, ...\}$ and $i_0 \in \{2, 4, ..., 2n\}$

$$\begin{aligned} d(Fu, T_j u) &\le d(Fu, Fx_{kn+i_0}) + d(Fx_{kn+i_0}, T_j u) \\ &= d(Fu, Fx_{kn+i_0}) + d(S_{\frac{i_0}{2}} x_{kn+i_0-1}, T_j u) \\ &\le d(Fu, Fx_{kn+i_0}) + A_{\frac{i_0}{2}} d(Fu, S_{\frac{i_0}{2}} x_{kn+i_0-1}) \\ &\quad + B_j d(Fx_{kn+i_0-1}, T_j u) \\ &\le d(Fu, Fx_{kn+i_0}) + A_{\frac{i_0}{2}} d(Fu, S_{\frac{i_0}{2}} x_{kn+i_0-1}) \\ &\quad + B_j [d(Fx_{kn+i_0-1}, Fu) + d(Fu, T_j u)]. \end{aligned}$$

Then

$$\begin{aligned} d(Fu, T_j u) &\le \frac{1}{1-B_j} d(Fu, Fx_{kn+i_0}) \\ &\quad + \frac{A_{\frac{i_0}{2}}}{1-B_j} d(Fu, Fx_{kn+i_0}) \\ &\quad + \frac{B_j}{1-B_j} d(Fx_{kn+i_0-1}, Fu) \end{aligned}$$

so for sufficently large k - the distance $d(Fu, T_j u)$ can be arbitrarily small, then $T_j u = Fu$, for any $j \in \{1, 2, ..., n\}$. Moreover

$$d(Fu, S_j u) = d(S_j u, T_j u) \le A_j d(Fu, S_j u) + B_j d(Fu, T_j u) = A_j d(Fu, S_j u),$$

whence $S_j u = Fu$, for any $j \in \{1, 2, ..., n\}$. We have proved that v is a unique point of coincidence of mappings $F, S_1, ..., S_n, T_1, ..., T_n$. If additionally all pairs (F, T_i), (F, S_i), for $i \in \{1, 2, ..., n\}$ are weakly compatibles, then by Lemma 1.4, $F, S_1, ..., S_n, T_1, ..., T_n$ have the unique common fixed point. □

3 Examples

Example 3.1. Let $X = [\frac{1}{4}, \frac{23}{7}]$ and d be the Euclidean metric on X. Let $n = 2$. We define the mappings $F, S_1, S_2, T_1, T_2 : X \to X$ as follows:

$$F(x) = \begin{cases} \frac{1}{x}, & \text{for } x \in [\frac{1}{4}, 1], \\ 1, & \text{for } x \in [1, 2], \\ \frac{1}{x-1}, & \text{for } x \in [1, \frac{23}{7}], \end{cases}$$

and for $i = 1, 2$

$$S_i(x) = \begin{cases} (1 - \frac{1}{5-i})x + \frac{1}{5-i}, & \text{for } x \in [\frac{1}{4}, 1], \\ 1, & \text{for } x \in [1, \frac{23}{7}[, \end{cases}$$

$$T_i(x) = \begin{cases} (\frac{i}{i+1})x + \frac{i}{i+1}, & \text{for } x \in [\frac{1}{4}, 1], \\ 1, & \text{for } x \in [1, \frac{23}{7}[, \end{cases}$$

We have $F(X) = [\frac{7}{16}, 4]$, $S_1(X) \cup S_2(X) = [\frac{7}{16}, 1]$, $T_1(X) \cup T_2(X) = [\frac{5}{8}, 1]$. We put $A_1 = A_2 = \frac{5}{12}$ and $B_1 = B_2 = \frac{1}{2}$. One can easily observe that the condition (2.1) is satisfied. For $x_0 = \frac{1}{4}$ - by Definition 1.5 - we get the sequence $(Fx_m)_{m\in\mathbb{N}}$ as follows: $\frac{1}{4}, \frac{5}{8}, \frac{23}{32}, \frac{87}{96}, \frac{270}{288}, \frac{279}{288} \dots$. So $v = \lim_{m\to\infty} Fx_m = 1$ is a unique point of coincidence of mappings F, S_1, S_2, T_1, T_2. Since all pairs (F, T_i), (F, S_i), for $i \in \{1, 2\}$ are weakly compatibles, then - by Lemma 1.4 - $v = 1$ is the unique common fixed point for mappings F, S_1, S_2, T_1, T_2. Moreover, let us remark that if we change definition of the function F as follows

$$F(x) = \begin{cases} \frac{1}{x}, & \text{for } x \in [\frac{1}{4}, 1], \\ \frac{7}{16}, & \text{for } x = 1, \\ 1, & \text{for } x \in]1, 2], \\ \frac{1}{x-1}, & \text{for } x \in [1, \frac{23}{7}], \end{cases}$$

then there is no common fixed point for mappings F, S_1, S_2, T_1, T_2. Since $1 = F(2) = S_1(2) = S_2(2) = T_1(2) = T_1(2)$, the point 1 is only the point of coincidence of a family F, S_1, S_2, T_1, T_2. In this case the condition (2.1) is still satisfied but all pairs (F, S_i), (F, T_i) for $i \in \{1, 2\}$ are not weakly compatibles.

Example 3.2. Let $X = [0, \infty[$ and d be the Euclidean metric on X. Let $n = 3$. We define the mappings $F, S_1, S_2, S_3, T_1, T_2, T_3 : X \to X$ as follows:

$$F(x) = x, \quad \text{for } x \in X$$

and for $i = 1, 2, 3$

$$S_i(x) = \pi + \frac{1}{2i}\sin x, \quad T_i(x) = \pi + \frac{1}{3i}\sin x.$$

We have $F(X) = X$, $\bigcup_{i=1}^n S_i(X) = [\pi - \frac{1}{2}, \pi + \frac{1}{2}] \subset F(X)$, and $\bigcup_{i=1}^n T_i(X) = [\pi - \frac{1}{3}, \pi + \frac{1}{3}] \subset F(X)$. We put $A_1 = A_2 = A_3 = \frac{6}{13}$ and $B_1 = B_2 = B_3 = \frac{6}{13}$. One can easily observe that the condition (2.1) is satisfied. For $x_0 = \frac{\pi}{2}$ - by Definition 1.5 - we get the sequence $(Fx_m)_{m\in\mathbb{N}}$ as follows:

$$\frac{\pi}{2},$$

$$\pi + \frac{1}{3},$$

$$\pi - \frac{1}{2}\sin\frac{1}{3},$$

$$\pi + \frac{1}{5}\sin\left[\frac{1}{2}\sin\frac{1}{3}\right],$$

$$\pi - \frac{1}{4}\sin\left\{\frac{1}{5}\sin\left[\frac{1}{2}\sin\frac{1}{3}\right]\right\},$$

$$\pi + \frac{1}{7}\sin\left[\frac{1}{4}\sin\left\{\frac{1}{5}\sin\left[\frac{1}{2}\sin\frac{1}{3}\right]\right\}\right],$$
$$\pi - \frac{1}{6}\sin\left\{\frac{1}{7}\sin\left[\frac{1}{4}\sin\left\{\frac{1}{5}\sin\left[\frac{1}{2}\sin\frac{1}{3}\right]\right\}\right]\right\},$$
$$\ldots$$

So $v = \lim_{m\to\infty} Fx_m = \pi$ is a unique point of coincidence of mappings $F, S_1, S_2, S_3, T_1, T_2, T_3$. Since all pairs (F, T_i), (F, S_i), for $i \in \{1, 2, 3\}$ are weakly compatibles, then - by Lemma 1.4 - $v = \pi$ is the unique common fixed point for mappings $F, S_1, S_2, S_3, T_1, T_2, T_3$.

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