Journal of Progressive Research in Mathematics(JPRM)
ISSN: 2395-0218

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RESEARCH ORGANISATION

Volume 7, Issue 3
Published online: May 08, 2016|

Journal of Progressive Research in Mathematics
www.scitecresearch.com/journals

Collocation Method to Solve Elliptic Equations

Bivariate Poly-Sinc Approximation

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Abstract

The paper proposes a collocation method to solve bivariate elliptic partial differential equations. The method uses Lagrange approximation based on Sinc point collocations. The proposed approximation is collocating on non-equidistant interpolation points generated by conformal maps, called Sinc points. We prove the upper bound of the error for the bivariate Lagrange approximation at these Sinc points. Then we define a collocation algorithm using this approximation to solve elliptic PDEs. We verify the Poly-Sinc technique for different elliptic equations and compare the approximate solutions with exact solutions.

Keywords: Elliptic PDEs; Bivariate approximation; Lagrange interpolation; Sinc points; Conformal maps; Poly-Sinc methods; Collocation method; Dirichlet boundary conditions; Mixed boundary conditions.

AMS Classification: 35J15, 35J25, 65M15, 65D05.

1.Introduction

In this paper we consider the numerical solution of a second order elliptic partial differential equation (PDE) in two dimensions

$$\mathcal{L}(u) = \partial_{x,x}u + \partial_{y,y}u + F(u) = R(x,y) \tag{1}$$
$$\text{in } Q = (a,b) \times (c,d),$$

where $a, b, c, d \in \mathbb{R}$, and u satisfies Dirichlet or Neumann (or mixed) boundary conditions defined on the boundary ∂Q.

Elliptic partial differential equations are common in many areas of physics and engineering [1, 2, 3]. There have been several attempts to develop numerical methods for these equations [4]. The ideal numerical method for solving these problems should be highly accurate, flexible with respect to the geometry, computationally efficient, and easy to implement. Most of the commonly used methods usually fulfill one or two of these criteria, but not all. The most widely used techniques are finite difference methods, spectral methods, finite element methods, wavelet-based methods, and Sinc methods [7, 8, 12, 13, 14].

Finite difference methods can be made highly accurate, but require a structured grid (or a collection of structured grids) [14].

Spectral methods are even more accurate, yet have severe restrictions on the geometry; furthermore, the Fourier methods, also require periodic boundary conditions [8].

Finite element methods are highly flexible, but it is hard to achieve high accuracy, while both coding and mesh generation become increasingly difficult when the number of space dimensions increases [9, 10, 11, 12]. Also, little has been proved regarding the convergence of these schemes.

In contrast, substantial progress has been made recently in proving convergence of wavelet-based adaptive methods for elliptic PDEs. In particular, it has been proved that the adaptive wavelet scheme converges for a variety of elliptic PDEs, as well as for singular integral equations [13].

On the other hand, a very powerful tool are the Sinc methods. Their exponential convergence rate has made them an excellent tool for accurately approximating the solution of partial differential equations. Some of these techniques are based on using separation of variables to find approximate solutions to linear PDEs [7]. Another technique of Sinc methods is a collocation scheme based on Cardinal Sinc approximation [17].

Although Sinc methods possessing an exceptional error formula, in the case of finite interval, they yield unbounded results in the neighborhood of finiteend-points for derivatives. This problem has been resolved by introducing a polynomial approximation based on Sinc points [6]. In [6] we introduced a Lagrange polynomial approximation at some non-equidistant points, known as Sinc points. We proved for the one dimensional case that an exponential decaying rate of the error holds for the function and its derivative over finite and semi-infinite intervals. In the current paper, we introduce the bivariate case of such approximation. We shall establish a bivariate collocation scheme to use Polynomial-Sinc (Poly-Sinc) approximation to solve PDEs

This paper is organized as follows: In section 2, the Poly-Sinc interpolation formula in 1D is introduced. In section 3, we apply a bivariate form of Lagrange approximation to Sinc data and derive the upper bound of the error for this approximation. In section 4, we introduce the Poly-Sinc algorithm based on the approximation proposed in section 3. In section 5, we use the Poly- Sinc algorithm to solve elliptic problems and verify the approximate solution by comparing with exact solutions. Concluding remarks are given in section 6.

2. Poly-Sinc approximation

In a Lagrange polynomial approximation different types of point sets are used as interpolation points [15]. The most famous set of points are the equidistant points. It is well known that these points deliver sometimes bad results [5, 16].To improve the accuracy of Lagrange approximation other sets of points are used, such as Chebyshev points and modified Chebyshev points [15]. Recently it was shown that it is more effective to use Sinc points as interpolation points[6]. The sequence of points is generated using a conformal map that redistributes the infinite equidistant points on the real line to a finite interval. Such a redistribution by conformal maps locates most of the points near the end-points of the interval. We already proved that using Sinc points as interpolation points we gain a highly accurate approximation. In addition such interpolation points deliver an accuracy similar to the classical Sinc approximation [6].

To define these interpolation points let $\mathbb{Z}$ denote the set of all integers. Let $\mathbb{R}$ be the real line, and $\mathbb{C}$ denote the complex plane. Let h denote a positivestep length on $\mathbb{R}$ and let $k \in \mathbb{Z}, z \in \mathbb{C}$. Let d denote a positive number and let $\mathcal{D} \subset \mathbb{C}$ be a simply connected region defined as:

$$\mathcal{D} = \left\{ z \in \mathbb{C}: \left| arg\left(\frac{z-a}{b-z}\right)\right| < d \right\}, \tag{2}$$

and let $\varphi = \ln\left(\frac{z-a}{b-z}\right)$be a conformal map of $\mathcal{D}$ onto the strip

$$\mathcal{D}_d = \{z \in \mathbb{C}: |Im(z)| < d\}. \quad (3)$$

Let $\Gamma = (a, b) = \varphi^{-1}(\mathbb{R})$be an arc, where $a = \varphi^{-1}(-\infty)$ and $b = \varphi^{-1}(\infty)$ denote the end points of Γ. Then we define the set of Sinc points by

$$x_k = \varphi^{-1}(kh) = \frac{a+be^{kh}}{1+e^{kh}}. \quad (4)$$

Finally, let $\alpha \in (0,1]$ and $\beta \in (0,1]$ denote fixed positive numbers and set $\rho = e^{\varphi(x)}$. Without loss of generality, let us restrict d introduced above to the interval $(0, \pi/2)$. Then $\mathcal{L}_{\alpha,\beta}$ denote the family of all functions f that are analytic in $\mathcal{D}$ such that for all $z \in \mathcal{D}$ we have

$$|f(z)| \leq c_1 \frac{|\rho(z)|^{\alpha}}{[1 + |\rho(z)|]^{\alpha+\beta}}.$$

The space of functions $M_{\alpha,\beta}(\mathcal{D})$ denotes the set of all functions q defined on $\mathcal{D}$ that have finite limits $q(a) = \lim_{z\to a} q(z)$ and $q(b) = \lim_{z\to b} q(z)$, where thelimits are taken from within $\mathcal{D}$, and such that $f \in \mathcal{L}_{\alpha,\beta}(\mathcal{D})$, where,

$$f = q - \frac{q(a) + \rho\, q(b)}{1 + \rho}.$$

Now we define a family of polynomial-like approximation that interpolate given Sinc data of the form $\{x_k, f(x_k)\}_{k=-M}^{N}$where the x_k are Sinc points. This novel family of Lagrange polynomials was recently derived in [6]. The approximation is accurate, provided that the function f with $f_k = f(x_k)$ belongs to the space of analytic functions $\mathcal{L}_{\alpha,\beta}(\mathcal{D})$.

In general a Lagrange polynomial approximation over the interval (a, b) is defined in the following way. Given a set of $n = M + N + 1$ Sinc points $\{x_k, f(x_k)\}_{k=-M}^{N}$,there exists a unique polynomial $P_{M,N}(x)$ of degree at most $(n - 1)$satisfying the interpolation condition,

$$P_{M,N}(x_k) = f(x_k), \;\; k = -M, \dots, N.$$

In this case$P_{M,N}(x)$can be expressed as:

$$P_{M,N}(x) = \sum_{k=-M}^{N} b_k(x) f_k, \quad (5)$$

with,

$$b_k(x) = \frac{v(x)}{(x - x_k)v'(x_k)} \quad (6)$$

where,

$$v(x) = \prod_{j=-M}^{N} (x - x_j).$$

This approximation, like regular Sinc approximation, yields an exceptional accuracy in approximating the function that is known at Sinc points [7]. Unlike Sinc approximation, it gives an exponential convergence rate when differentiating the interpolation formula (5), [6].

In the following, we shall assume that $M = N$ andso$P_{M,N}(x) = P_N(x)$.

Theorem 1. *Let $h = \pi/\sqrt{N}$, and let $\{x_k\}_{k=-M}^{N}$ denote the Sinc points as definedin (4). Let f be in $M_{\alpha,\beta}(\mathcal{D})$ and let $P_N(x)$ be defined as in (5). Then there exist two constants $A > 0$ and $B > 0$, independent of N, such that*

$$|f(x) - P_N(x)| \leq A\ \frac{\sqrt{N}}{B^{2N}}\ exp\left(\frac{-\pi^2 N^{\frac{1}{2}}}{2}\right). \tag{7}$$

Proof. For the proof of (7), see [6].

Next we extend these results to the case of $2D$ Lagrange interpolation at Sinc points.

3. Bivariate Poly-Sinc Approximation

In this section we define a bivariate form of Lagrange approximation at Sinc data. We will show that this approximation is accurate, provided that the function $f(x,y)$ belongs to a certain space of analytic functions $\mathcal{S}_{\alpha,d}$.

3.1 Definition of Spaces

Let us first define some notations for a function $f(x,y)$and the required space.

Let $Q = [-1,1] \times [-1,1]$ and let $\mathcal{D}$ and $\mathcal{D}_d$as defined in (2) and (3) with$a = -1, b = 1$ and,$= \pi/2$, Define $\overline{\mathcal{D}}$ to be the closure of $\mathcal{D}$. Define Ω to be

$$\Omega = \{[-1,1] \times \overline{\mathcal{D}}\} \cup \{\overline{\mathcal{D}} \times [-1,1]\}.$$

Let $f(x,y) \in \mathbb{C}$ where, $f: \Omega \to \mathbb{C}$. Define the following two functions,

$$f^x: \overline{\mathcal{D}} \to \mathbb{C}\text{as, } f^x(y) = f(x,y), \text{ for } x \in [-1,1],$$
$$f^y: \overline{\mathcal{D}} \to \mathbb{C}\text{as, } f^y(x) = f(x,y), \text{ for } y \in [-1,1].$$

Define the following space of functions $\mathcal{S}_{\alpha,d}$ to be the space of all functions$f(x,y)$ such that,

1. for all $x \in [-1,1], f^x \in Hol(\mathcal{D})$,
2. for all $y \in [-1,1], f^y \in Hol(\mathcal{D})$,
3. there exist$\alpha \in (0,1)$such that,
    - for all$x \in [-1,1], f^x \in Lip_\alpha(\overline{\mathcal{D}})$,
    - for all$y \in [-1,1], f^y \in Lip_\alpha(\overline{\mathcal{D}})$,

where, $Hol(\mathcal{D})$ is the family of all functions f that are analytic in a domain $\mathcal{D}$. A function f is said to be in a class Lip_α on a closed interval $[a,b]$ if there exista constant C such that

$$|f(x_1) - f(x_2)| \leq C|x_1 - x_2|^\alpha,$$

for all points x_1 and x_2 on the interval $[a,b]$.

Define the following two Lagrange approximations,

$$(L_1 f)(x) = \sum_{j=-M}^{N} f^y(x_j) b_j(x) \tag{8}$$

and

$$(L_1 f)(y) = \sum_{k=-M}^{N} f^x(y_k) b_k(y), \tag{9}$$

Then the bivariate Lagrange approximation for a function $f(x,y) \in \mathcal{S}_{\alpha,d}$ is expressed as:

$$(P_{M,N} f)(x,y) = L_1(L_2 f)(x,y) = \sum_{j=-M}^{N} \sum_{k=-M}^{N} f(x_j, y_k) b_j(x) b_k(y), \tag{10}$$

where b_j and b_k are the basis in x and y as defined in (6), respectively. Next, we prove the upper bound of the error in the bivariate formula of Lagrange approximation.

3.2 Upper Bound of Error

Without loss of generality we will restrict $P_{M,N} f$ to the case where $M = N$ and denote the corresponding polynomial $P_{M,N} f$ by $P_N f$.

Theorem 2. *Let $h = \pi/\sqrt{N}$ and let $f(x,y)$ be in $\mathcal{S}_{\alpha,d}$, and let $(P_N f)$ be defined as in (10). Then there exist three positive constants C_1, C_2 and C_3, independent of N, such that*

$$\text{Sup}_{(x,y)\in Q} |f(x,y) - (P_N f)| \le (C_1 + C_2 \log N) \frac{\sqrt{N}}{C_3^{2N}} \exp\left(\frac{-\pi^2 N^{\frac{1}{2}}}{2}\right). \tag{11}$$

Proof.

The difference between the exact function $f(x,y)$ and the approximate polynomial (10) can be written as,

$$\begin{aligned} f(x,y) - (P_N f) &= f - L_1(L_2 f) \\ &= f - L_1(L_2 f) + L_1 f - L_1 f \\ &= (f - L_1 f) + L_1(f - L_2 f). \end{aligned}$$

Then

$$\begin{aligned} \sup_{(x,y)\in Q} |f(x,y) - (P_N f)| &\le \sup_{(x,y)\in Q} |f(x,y) - (L_1 f^y)(x)| + \sup_{(x,y)\in Q} |L_1(f(x,y) - (L_2 f^x)(y))| \\ &\le \sup_{(x,y)\in Q} |f(x,y) - (L_1 f^y)(x)| + \|L_1\| \sup_{(x,y)\in Q} |f(x,y) - (L_2 f^x)(y)|, \end{aligned}$$

where $\|.\|$ is the *Sup* norm. Now using Theorem 1 we have,

$$\text{Sup}_{(x,y)\in Q} |f(x,y) - (L_1 f^y)(x)| \le A_1 \frac{\sqrt{N}}{B_1^{2N}} \, exp\left(\frac{-\pi^2 N^{\frac{1}{2}}}{2}\right), \tag{12}$$

and,

$$\text{Sup}_{(x,y)\in Q} |f(x,y) - (L_2 f^x)(y)| \le A_2 \frac{\sqrt{N}}{B_2^{2N}} \, exp\left(\frac{-\pi^2 N^{\frac{1}{2}}}{2}\right) \tag{13}$$

also,

$$\|L_1\| = \operatorname*{Sup}_{x\in[-1,1]} \sum_{-N}^{N} |b_j(x)|$$
$$\leq \frac{1}{\pi}\log(2N+1) + 1.07618. \qquad (14)$$

The upper bound in (14) is the upper bound of Lebesgue constant for Lagrange approximation at Sinc data. Recently, the Lebesgue constant using Sinc data was discussed in [18]. The result for the Lebesgue constant in (14) has been derived recently and will be published in a forthcoming paper [19]. Now combining the results of (12), (13), and (14) we get (11) which ends the proof.

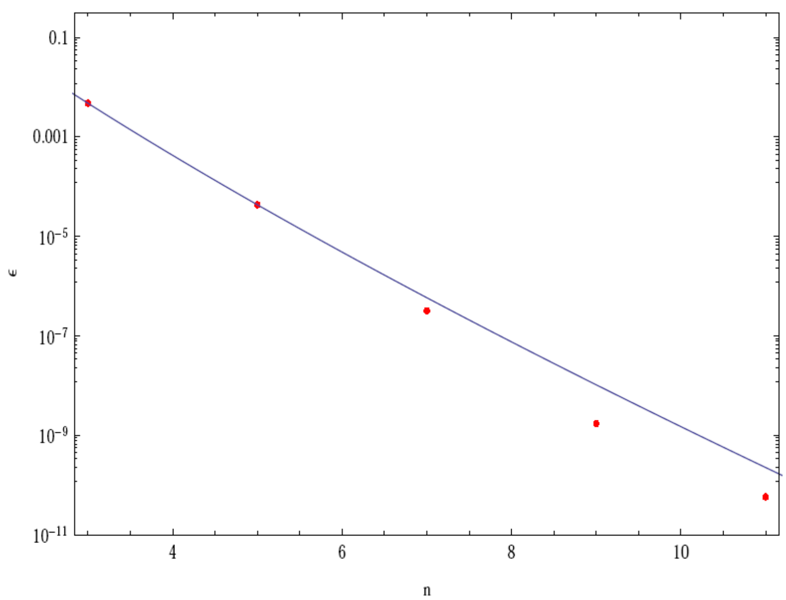

Figure 1: The error for different Sinc points $n = 3, 5, 7$ and,11 for the approximation of the function$f(x, y) = \cos(x\, y)$.

To illustrate the exponential convergence rate of the Poly-Sinc algorithm(Theorem 2), we examined the function$f(x, y) = \cos(x\, y)$ defined on $Q = [-1,1] \times [-1,1]$. For this function we find the Poly-Sinc approximation defined in (10) for different numbers of Sinc points $n = 2N + 1$. For each n, we computethe norm error as defined in (25), see Appendix A. As result a table for each n and the corresponding error has been created. We then use this table in aleast square estimation to find the coefficients of the error function as estimated in (11). Specifically, we used the form $\gamma \log N \sqrt{N} e^{-\mu\sqrt{N}}$, where γ and μ are constants. A least square fit to the collected data delivers the constants γ and μ that optimally represents the error list. In Fig. 1 the solid line represents (11) using the constants γ and μ from the least square fit. Dots in Fig. 1 are the discrete errors computed by (25). The graph in Fig. 1 demonstrates that the error of the Poly-Sinc approximation for $f(x, y) = \cos(x\, y)$ follows anexponentially decay relation.

4. Poly-Sinc Algorithm

In this section we set up the collocation method based on the use of Lagrange interpolation at Sinc points defined in (10).

The idea of the Poly-Sinc algorithm is to transform the bivariate equation (1) and its corresponding boundary conditions to an algebraic system of equations which is solved afterwards to acquire the approximate solution.

The algorithm can be described by the following six steps:

1. Replace $u(x,y)$ in equation (1) and in the boundary conditions by the

   Lagrange polynomial defined in (5),

$$u(x,y) \approx \sum_{j=-M}^{N} \sum_{k=-M}^{N} u_{jk} b_j(x) b_k(y). \tag{15}$$

2. Collocate the equations by replacing x by Sinc points

$$x_k = \varphi_x^{-1}(kh) = (a + be^{kh})/(1 + e^{kh}), k = -N, \ldots, N.$$

3. Collocate the equations by replacing y by Sinc points

$$y_j = \varphi_y^{-1}(kh) = (c + de^{jh})/(1 + e^{jh}), j = -N, \ldots, N.$$

4. The differential equation has been transformed to a system of $(2N+1)^2$ algebraic equations.

   The unknowns are $u_{jk}, j = -N, \ldots, N$ and $k = -N, \ldots, N$.

5. Next use Newton's root finding method for the case where$F(u)$is linearto find the solution of the algebraic system.
6. Finally insert the coefficients u_{jk} into the Lagrange polynomial (15) to getthe approximate solution.

5. Numerical Results

In this section we verify the Poly-Sinc collocation algorithm to solve specific elliptic PDEs. To test the algorithm, we compare the obtained approximate solution with the exact solution of the boundary problem. We examine different types of boundary value problems. Some of them with Dirichlet boundary conditions and the others are with mixed boundary conditions. For each of these examples we derive the approximate solution and estimate the local and norm errors. We also compare the obtained error with the error obtained by using different techniques in literature. These examples will show an improvement in the error if Poly-Sinc technique is used.

Example 5.1.

First let us examine the following Poisson equation taken from [22].

$$\partial_{x,x} u + \partial_{y,y} u = f(x,y) \text{ in } Q, \quad (16)$$

with the homogeneous Dirichlet boundary conditions,

$$u = 0, on\ \partial Q, \tag{17}$$

Where $Q = (-1,4) \times (0,1)$ and$f(x,y)$ is compatible with the exact solution $u_{ex} = (x+1)(x-4)(1-y^2)y^2/3.1596$. Lybeck solved this problem by domain decomposition using Sinc approximation [22]. With 21×21 basis functionsalong the x and y directions, a norm error $\epsilon_n = 10^{-2}$ has been obtained [22]. Our aim is to improve this error and verify the exact solution using the Poly-Sinc algorithm for (16) and (17).

Here we choose $\varphi(x) = \ln\left(\frac{x+1}{4-x}\right)$and $\varphi(y) = \ln(y/(1-y))$. Then the Sinc points for x and y can be defined by, $x_k = \varphi_x^{-1}(kh) = (-1+4e^{kh})/(1+e^{kh})$, and $y_j = \varphi_y^{-1}(kh) = \frac{e^{jh}}{1+e^{jh}}$, respectively. Apply the Poly-Sinc algorithm with $N = 5$; i.e. we have 11×11 Sinc points along the$x$ and y directions which are halve the numbers of points used in [22]. We solve the system of 121 algebraic equations using Newton's root finding method to get the approximate solution.

In Fig. 2, the left panel represents the approximate solution of equation (16) with Dirichlet boundary conditions (17). The right panel in Fig. 2 represents the local absolute error $E_n = |u_{ex} - u|$,where u is the approximate solution. Calculating the norm error using (25), we find $\epsilon_n = 4.27725 \times 10^{-6}$ which is 3 orders of magnitude less than in [22].

The following example shows that the Poly-Sinc algorithm can handle, without any modifications, the inhomogeneous Dirichlet boundary conditions.

Example 5.2.

Given the following Poisson equation taken from [7]

$$\partial_{x,x}u + \partial_{y,y}u = f(x,y) \text{ in } Q, \tag{18}$$

with the inhomogeneous Dirichlet boundary conditions,

$$u(x,y) = \frac{1}{(1+r^2)^2} \text{ on } \partial Q, \tag{19}$$

where $r^2 = x^2 + y^2, Q = (0,1)^2$, and $f(x,y)$ is compatible with the exact solution $u_{ex}(x,y) = 1/(1+r^2)^2$. The inhomogeneous boundary conditions require extra collocation step for each boundary condition. This collocation step will transform the boundary condition to an algebraic system of equations. Adding the resulting algebraic equations of the boundary conditions to the algebraic system of equations resulting from the collocated PDE (18) will give a system of equations to be solved to determine the coefficient u_{jk}. The resulting system is solved using Newton's method to get the approximate solution.

In [7], this problem has been solved by a Sinc convolution using Green's function. But here we use a simpler collocation technique to handle such kind of inhomogeneous Dirichlet boundary conditions and get a norm error $\epsilon_n = 5.37184 \times 10^{-4}$. The approximate solution and the absolute error E_n are givenin Fig. 3. In the calculations we used $N = 5$.

The next example tests the algorithm to solve the modified Helmholtz equation.

Example 5.3.

Consider the following Helmholtz problem, [20]

$$\partial_{x,x}u + \partial_{y,y}u - ku = 4 - k(x^2 + y^2), \text{in } Q. \tag{20}$$

with the Dirichlet boundary conditions,

$$u(x,y) = x^2 + y^2 \text{ on } \partial Q, \tag{21}$$

Where $Q = (0,1)^2$. The exact solution of this boundary value problem is $u_{ex}(x,y) = x^2 + y^2$. In [20] this problem has been solved, for different values of k, by a collocation technique called the method of fundamental solutions. Here, we solve the problem for $k = 1, 9, 25.$ For each k we calculate the norm error ϵ_n, see table 1. A comparison between the Poly-Sinc results and the results in [20] is collected in table 1. Also, for $k = 1$ we present, graphically, theapproximate solution and the absolute error E_n, see Fig. 4.

	$k = 1$	$k = 9$	$k = 25$
Poly-Sinc	10^{-8}	10^{-8}	10^{-8}
[20]	10^{-4}	10^{-4}	10^{-2}

Table 1: Errors in example 5.3 using Poly-Sinc approximation compared with [20].

The next example demonstrates the application of the algorithm to mixed boundary conditions,

Example 5.4.

Given the following Laplace equation, [21]

$$\partial_{x,x} u + \partial_{y,y} u = 0, \text{in } Q = (0,1)^2 \tag{22}$$

with the mixed Neumann boundary conditions,

$$\begin{aligned} &u(0, y) = -y^3, \\ &u(1, y) = -1 - y^3 + 3y^2 + 3y, \\ &u_x(x, 0) = 3x^2, \\ &u_y(x, 1) = 3x^2 + 6x - 3. \end{aligned} \tag{23}$$

The exact solution of the problem is $u_{ex}(x, y) = -x^3 - y^3 + 3xy^2 + 3x^2y$. Applying the Poly-Sinc with $N = 5$ means with 11×11 points along the x and y directions, we get a norm error $\epsilon_n = 2.70464 \times 10^{-8}$. In [21], this problem has been solved using a collocation method based on reproducing kernel approximations. A distribution of 33×33 points along the x and y directions has been used to get $\epsilon_n = 10^{-5}$. The plot of the approximate solution and the absolute error E_n are given in Fig. 5. The left panel represents the approximate solution and the right panel shows the absolute error, E_n, between the exact solution u_{ex} and the approximate solution u.

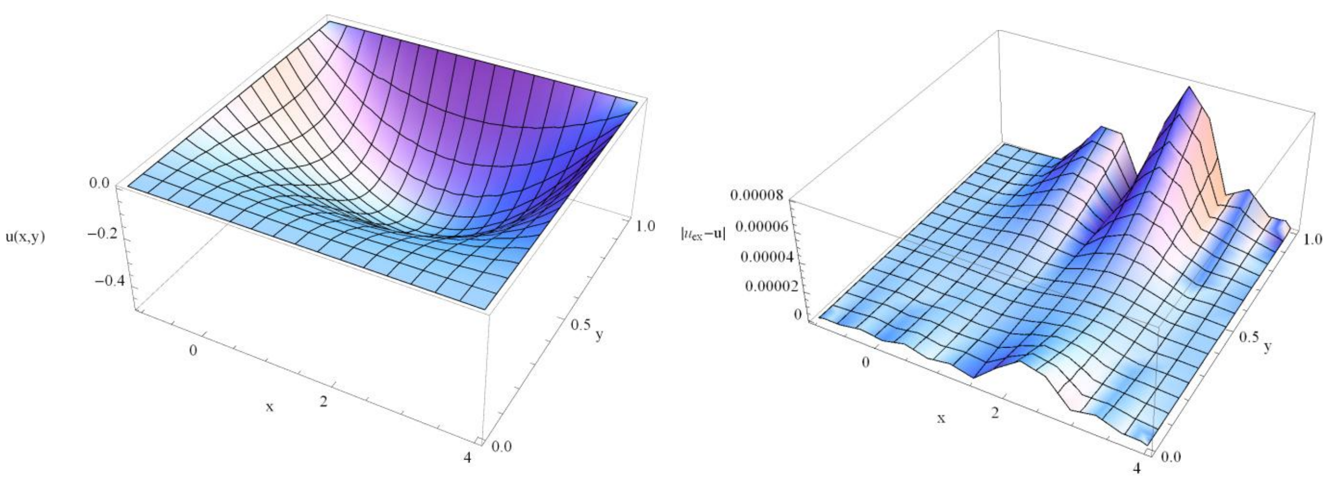

Figure 2: Approximate solution of equation (16) and boundary conditions (17) with $N = 5$, left panel. Right panel shows the local absolute error with respect to the exact solution $u_{ex} = (x + 1)(x - 4)(1 - y^2)y^2/3.1596$.

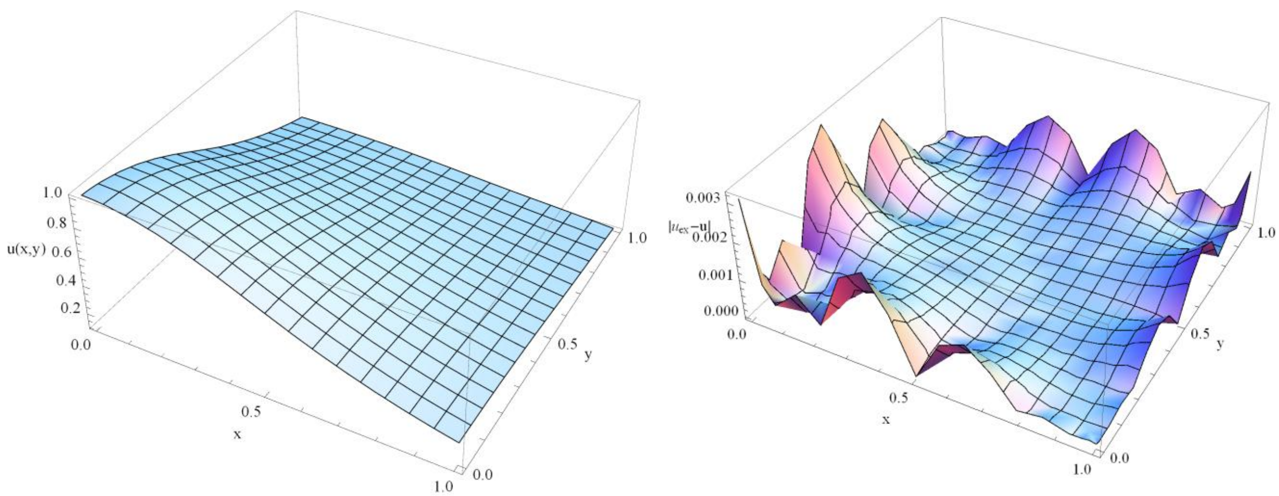

Figure 3: Approximate solution of equation (18) and boundary conditions (19) with $N = 5$, left panel. Right panel shows the local absolute error with respect to the exact solution $u_{ex}(x, y) = 1/(1 + r^2)^2$.

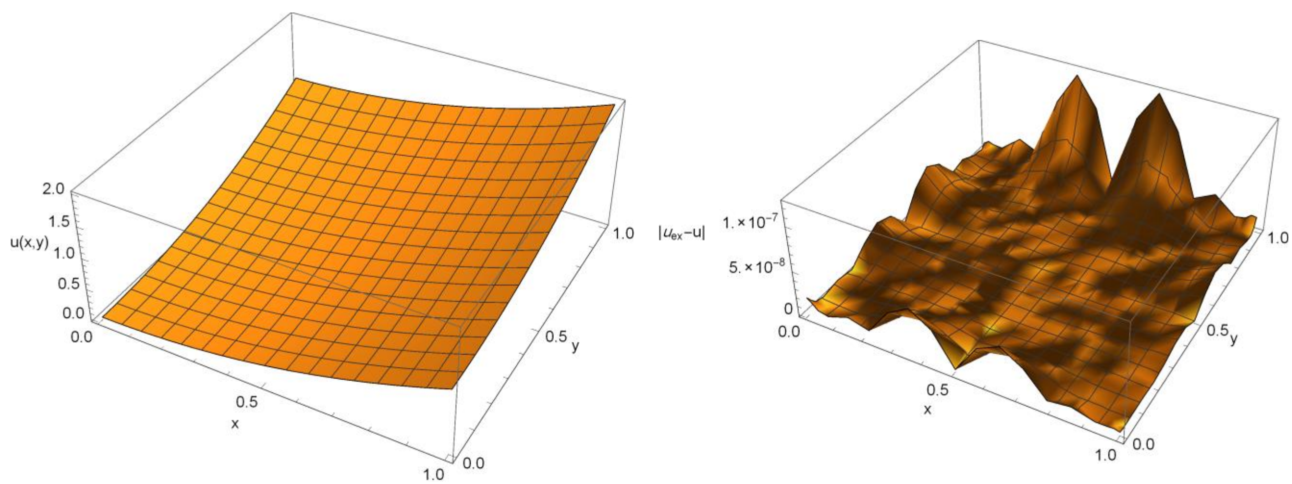

Figure 4: Approximate solution of equation (20) and boundary conditions (21) with $N = 5$, left panel. Right panel shows the local absolute error with respect to the exact solution $u_{ex}(x, y) = x^2 + y^2$.

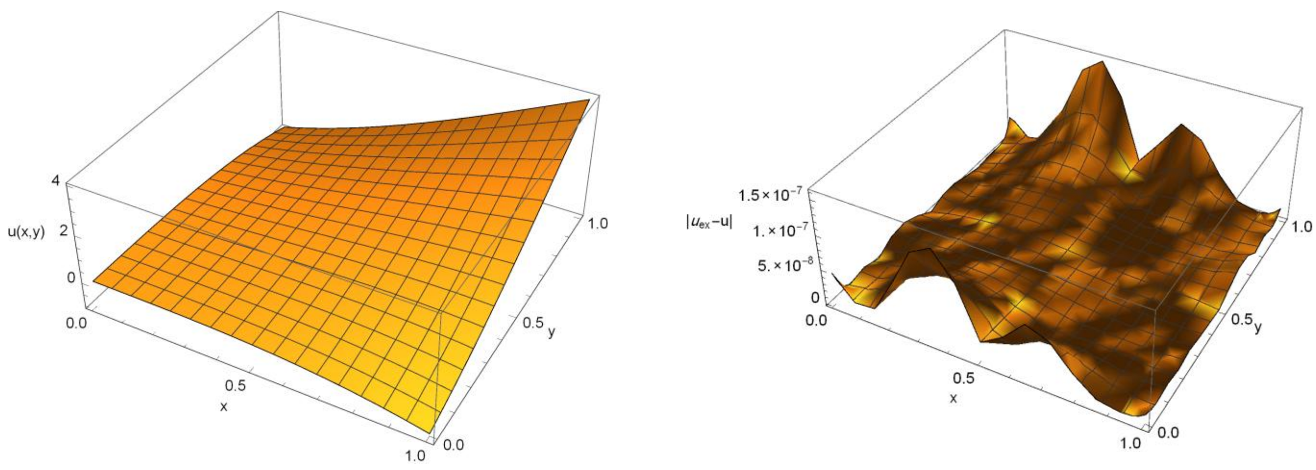

Figure 5: Approximate solution of equation (22) and boundary conditions (23) with $N = 5$, left panel. Right panel shows the local absolute error with respect to the exact solution $u_{ex}(x,y) = -x^3 - y^3 + 3xy^2 + 3x^2y$.

6.Conclusion

In this work we have proposed a collocation method for the solution of elliptic partial differential equations. This method has the advantages of: easy to compute and implement; treating efficiently different types of boundary conditions; the method is very accurate for the class of bivariate Poisson and Laplace problems. Using a small number of Sinc points we can reach a small error level. Consequently any solution of bivariate elliptic PDEs can be represented to arbitrarily high accuracy using a small number of collocation points.

Appendix A

For practical purposes, we use two forms of error estimations:

- **Absolute Error:** The absolute local error is the absolute difference between the exact solution $u_{ex}(x,y)$ and the approximate solution $u(x,y)$obtained by the Poly-Sinc algorithm, defined as:

$$E_n = |u_{ex}(x,y) - u(x,y)|. \tag{24}$$

  We will mainly use this error in graphing of the local error for a certain problem.

- **Norm Error:** The norm error is given by:

$$\epsilon_n == \left[\int_c^d \int_a^b [u_{ex}(x,y) - u(x,y)]^2 dx\, dy\right]^{\frac{1}{2}} \tag{25}$$

where $u_{ex}(x,y)$ is the exact solution and $u(x,y)$ is the approximation obtained by the Poly-Sinc algorithm.

Acknowledgement

We acknowledge partial financial support by the DAAD/BMBF project No.57128284.

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