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Mild solutions of local non-Lipschitz neutral stochastic partial functional differential equations

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Abstract.

This paper studies the existence and uniqueness of a mild solution for a neutral stochastic partial functional differential equation with infinite delays using local no-Lipschitz. An example is provided to illustrate the obtained result.

1 Introduction

In this paper, a neutral stochastic partial functional differential equation is considered in a real separable Hilbert space of the form

$$\begin{aligned} d[X(t)+a(t,X(t))] &= [AX(t)+f(t,X(t-\tau(t)))]dt+g(t,X(t-\delta(t)))dW(t), \quad t\geq 0, \\ X_0(\cdot) &= \xi\in D^b_{\mathcal{F}_0}([m(0),0],H), \end{aligned} \tag{1.1}$$

where $t-\tau(t), t-\delta(t)\to\infty$ with delays $\tau(t),\delta(t)\to\infty \quad t\to\infty$.

The existence and uniqueness and stability with delays has been considered by many authors. Under a global Lipschitz and linear growth condition, Taniguchi [15] and Luo [5] considered the existence and uniqueness of mild solutions to stochastic neutral partial functional differential equations by the well-known Banach fixed point theorem and strong approximating system, respectively, Govindan [9], showed, by stochastic convolution, the existence, uniqueness and almost sure exponential stability of stochastic neutral partial functional differential equations under global Lipschitz and linear growth condition. By the comparison principle, Govindan [10] the existence and uniqueness of mild solution to stochastic evolution equations with variable delays under a less restrictive hypothesis than the Lipschitz condition on the nonlinear terms. For many practical situations, the nonlinear terms do not obey the global Lipschitz and linear growth condition, even the local Lipschitz condition, and the readers can refer to Govindan [10], Rodkina [3], Taniguchi [14], [15], He [5], Yamada [15] and reference cited therein. More recent results e.g., by Ren and Sakthivel [17], David Castillo-Fernndez [3] and Bin Pei Xue Pei [18] have established the existence and uniqueness of mild solution for a class of second-order neutral stochastic evolution equations with infinite delay and Poisson jumps by means of the successive approximation. Nan Ding [7] established the exponential stability in mean square of mild solution, for neutral stochastic partial functional differential equations with impulses.

With the preceding reason, our objective here is to study the existence and uniqueness of equation (1) exploting the theory of a stochastic convolution integral and local no-Lipschitz condition.

The rest of this paper is organized as follows. In Section 2, we introduce some preliminaries. In Section 3, we prove the existence and uniqueness of the mild solution. Finally, in the 4 sixth Section, we give an example to illustrate the theory.

2 Preliminaries

Throughout this paper, we work in the frameworks used in [5]. Let $\{\Omega, \mathfrak{F}_t, P\}$ be a complete probability space equipped with some filtration $\{\mathfrak{F}_t\}_{t\geq 0}$ satisfying the usual conditions, i.e., the filtration is right continuous and $\mathfrak{F}_0$ contains all P-null sets. Let H, K be two real separable Hilbert spaces and denote by $< \cdot, \cdot >_H$, $< \cdot, \cdot >_K$ their inner products and by $\| \cdot \|_H$,$\| \cdot \|_K$ their vector norms, respectively. We denote by $\mathcal{L}(K, H)$ the set of all linear bounded operators from K into H equipped with the usual operator norm $\| \cdot \|$. In this paper, we always use the same symbol $\| \cdot \|$ to denote norms of operators regardless of the space potentially involved when no confusion possibly arises.

Let $\{W(t), t \geq 0\}$ denote a K-valued $\{\mathfrak{F}_t\}_{t\geq 0}$-Wiener process defined on $\{\Omega, \mathfrak{F}_t, P\}$ with covariance operator Q, i.e.,

$$E < W(t), x >_K < W(s), y >_K = (t \wedge s) < Qx, y >_K \quad for \quad all \quad x, y \in K$$

where Q is a positive, self-adjoint, trace class operator on K. In particular, we shall call such $W(t), t \geq 0$, a K-valued Q-Wiener process with respect to $\{\mathfrak{F}_t\}_{t\geq 0}$.
In order to define stochastic integrals with respect to the Q-Wiener process $W(t)$, we introduce the subspace $K_0 = Q^{1/2}(K)$ of K which, endowed the inner product

$$< u, v >_{K_0} = Q^{-1/2}u, Q^{-1/2}v >_K,$$

is a Hilbert space. Let $\mathcal{L}_2^0 = \mathcal{L}_2(K_0, H)$ denote the space of all Hilbert-Schmidt operators from K_0 into H. It turns out to be a separable Hilbert space equipped with the norm

$$\|\Psi\|^2_{\mathcal{L}_2^0} = tr((\Psi Q^{1/2})(\Psi Q^{1/2})^*) \quad for \quad any \quad \Psi \in \mathcal{L}_2^0.$$

Clearly, for any bounded operators $\Psi \in \mathcal{L}(K, H)$, this norm reduces to $\|\Psi\|_{\mathcal{L}_2^0} = tr(\Psi Q \Psi^*)$. For arbitrarily given $T \geq 0$, let $J(t, w), t \in [0, T]$, be an $\mathfrak{F}_t$-adapted, $\mathcal{L}_2^0$-valued process, and we define the following norm for arbitrary $t \in [0, T]$:

$$|J|_t = \{E \int_0^t tr(J(s, w)Q^{1/2})(J(s, w)Q^{1/2})^*)ds\}^{1/2}.$$

In particular, we denote all $\mathcal{L}_2^0$-valued predictable processes J satisfying $|J|_T < \infty$ by $\mathcal{U}^2([0, T]; \mathcal{L}_2^0)$. The stochastic integral

$$\int_0^t j(s, w)dW(s) = L^2 - \lim_{n\to\infty} \sum_{i=1}^{n} \int_0^t \sqrt{\lambda_i} J(s, w)e_i dB_s^i, \quad t \in [0, T],$$

where $W(t) = \sum_{i=1}^{\infty} \sqrt{\lambda_i} B_t^i e_i$. Here $(\lambda_i \geq 0, i \in N)$ are the eigenvalues of Q and $(e_i, i \in N)$ are the corresponding eigenvectors, and $(B_t^i, i \in N)$ are independent standard real-valued Brownian motions. The reader is referred to [3] for a systematic theory concerning stochastic integrals of this kind.
Let $\tau(t), \delta(t) \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfy $t - \tau(t) \to \infty, t - \delta(t) \to \infty \quad as \quad t \to \infty$, and

$$m(0) = \max\{\inf(s - \tau(s), s \geq 0), \inf(s - \delta(s), s \geq 0)\}.$$

We use $D^b_{\mathfrak{F}_0}([m(0),0];H)$ to denote the family of all almost surely bounded, $\mathfrak{F}_0$-measurable, continuous random variables from $[m(0),0]$ to H. Denote the norm $\|\varphi\|_D$ by

$$\|\varphi\|_D = \sup_{m(0)\leq\theta\leq 0} E\|\varphi(\theta)\|_H.$$

A semigroup $\{S(t), t\geq 0\}$ is said to be exponentially stable if there exist positive constants M and a such that $\|S(t)\| \leq Me^{-at}, t\geq 0$. If $M=1$, the semigroup is said to be a contraction. If $\{S(t), t\geq 0\}$ is an analytic semigroup, (see Pazy [1]) with infinitesimal generator A such that $0\in\rho(A)$ (the resolvent set of A), then it is possible to define the fractional $(-A)^\alpha$, for $0\leq\alpha\leq 1$ as a closed linear operator on its domain $D((-A)^\alpha) = H_\alpha$, Furthermore, the subspace $D((-A)^\alpha)$ is dense in H and

$$\|x\|_\alpha = \|(-A)^\alpha x\|_H \quad x\in D((-A)^\alpha)$$

For convenience of the reader, we will state the following lemmas that will be used in the sequel.

Lemma 2.1 *(see [1]) Let A be the infinitesimal generator of an analytic semigroup* $\{S(t), t\geq 0\}$ *If* $0\in\rho(A)$, *then*

(i) $S(t): H\to H_\alpha$ *for every* $t\geq 0, \alpha\geq 0$.

(ii) For every $x\in H_\alpha$, *one has*

$$S(t)(-A)^\alpha x = (-A)^\alpha S(t)x. \tag{2.2}$$

(iii) For every $t\geq 0$ *the operator*

$$\|(-A)^\alpha S(t)\|_H \leq \mu_\alpha t^{-\alpha}e^{-at}, \quad a\geq 0. \tag{2.3}$$

(iv) Let $0<\alpha\leq 1$ *and* $x\in H_\alpha$, *Then*

$$\|S(t)x - x\|_H \leq \gamma_\alpha t^\alpha \|(-A)^\alpha x\|_H \tag{2.4}$$

Lemma 2.2 *(see [13]) Let-A be the infinitesimal generator of an analytic semigroup of bounded linear operators $\{S(t), t \geq 0\}$ in K. Then, Then, for any stochastic process $F : [0, \infty) \to H$ which is strongly measurable with $\int_0^T E\|(-A)^\alpha F(t)\|_H^p dt < \infty, p \geq 2$, $0 < T \leq \infty$, the following inequality holds for $0 < t \leq T$:*

$$E\| \int_0^t (-A)S(t-s)F(s)ds\|_H^p \leq k(p,a,\alpha) \int_0^t E\|(-A)^\alpha F(s)\|_H^p ds, \tag{2.5}$$

provided $1/p < \alpha < 1$, where

$$k(p,a,\alpha) = M_{1-\alpha}^p \frac{(p-1)^{p\alpha-1}[\Gamma((p\alpha - 1/(p-1)))]^{p-1}}{(pa)^{p\alpha-1}} \tag{2.6}$$

and $\Gamma(\cdot)$ is the Gamma function.

3 Existence and uniqueness

In this section, we establish the existence and uniqueness of a mild solution of (1.1), under Caratheodory conditions. Let $-A : D(A) \subseteq H \to H$ be the infinitesimal generator of an analytic semigroup of bounded linear operators $\{S(t), t \geq 0\}$ defined on H. Let the functions $f(t,u)$, $a(t,u)$, and $g(t,u)$ be defined as follows:
$f : \mathbb{R}_+ \times H \to H$, $a : \mathbb{R}_+ \times H_\alpha \to \mathcal{L}(K,H)$, $g : \mathbb{R}_+ \times H \to \mathcal{L}(K,H)$ are Borel measurable.

Let the following assumptions hold a.s.:

(H1) $-A$ is the infinitesimal generator of an analytic semigroup of bounded linear operators $\{S(t), t \geq 0\}$ in H and the semigroup is a contraction;

(H2) There exists a function $H(t,r) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $H(t,r)$ is locally integrable in $t \geq 0$ for any fixed $r \geq 0$, and is continuous, monotone nondecreasing, and concave in r for any fixed $t \in [0,T]$. Moreover, for any fixed $t \in [0,T]$ and $\xi \in H$,

$$\|f(t,\xi)\|_H^2 + \|g(t,\xi)\|_{L_0^2} \leq H(t, E\|\xi\|_D^2), \quad t \in [0,T];$$

and any $K(T) > 0$, the differential equation

$$\frac{du}{dt} = K(T)H(t,u), \quad t \in [0,T],$$

has a global solution for any initial value u_0.

(H3) There exists a function $G(t,r) : \mathbb{R}_+ \to \mathbb{R}_+$ such that $G(t,r)$ is locally integrable in $t \leq 0$ for any fixed $r \leq 0$, and is continuous, monotone nondecreasing, and concave in r for any fixed $t \in [0,T]$, *and* $G(t,0) = 0$ for any fixed $t \in [0,T]$, Moreover, for any fixed $t \in [0,T]$ and $\xi, \eta \in H$,

$$\|f(t,\xi) - f(t,\eta)\|_H^2 + \|g(t,\xi) - g(t,\eta)\|_{\mathfrak{L}_0^2}^2 \leq G(t, E\|\xi - \eta\|_D^2), \quad t \in [0,T].$$

For any constant $\bar{K}(T) > 0$, if a nonnegative function $z(t)$ satisfies

$$z(t) \leq \bar{K}(T) \int_0^t G(s, z(s))ds, \quad t \in [0,T],$$

then $z(t) = 0$ for any $t \in [0,T]$.

(H4) The mapping $a(t,x)$ satisfies that exists a number $\alpha \in [0,1]$ and a positive K_0 such that, for any $\xi, \eta \in H$ and $t \geq 0, a(t,x) \in \mathcal{D}((-A^\alpha)$ and

$$\|(-A)^\alpha a(t,\xi) - (-A)^\alpha a(t,\eta)\|_H \leq K_0\|\xi - \eta\|_D.$$

Moreover, we assume that $a(t,0) = 0$.

(H5) The function $a(t,u)$ is continuous and that there exists a positive constant $C = C(T)$ such that

$$\|a(t,u) - a(t,v)\|_{H_\alpha} \leq C_1\|u - v\|_D$$

$$\|a(t,u\|_{H_\alpha} \leq C_2(1 + \|u\|_D).$$

for all $t \in [0,T]$ and $u, v \in D$

We now introduce the concept of a mild solution of (1.1).

Definition 3.1 *A stochastic process* $\{X(t), t \in [0,T]\}, 0 \leq T < \infty$*, is called a mild solution of (1.1) if:*

(i) $X(t)$ *is adapted to* $\mathfrak{F}_t, t \geq 0$*;*

(ii) $X(t) \in H$ *has cadlag paths on* $t \in [0,T]$ *almost surely, and for arbitrary* $0 \leq t \leq T$,

$$\begin{aligned} X(t) &= S(t)[\xi(0) + a(0,\xi)] - a(t, X_t) - \int_0^t AS(t-s)a(s, X_s)ds \\ &= + \int_0^t S(t-s)f(s, X(s-\tau(s)))ds + \int_0^t S(t-s)g(s, X(s-\delta(s)))dW(s), \end{aligned}$$

and

$$X_0 = \xi \in D^b_{\mathfrak{F}_0}([m(0), 0], H).$$

Theorem 3.2 *Suppose that the assumptions* $(H1)$*-*$(H5)$ *are satisfied. Then, there exists a unique mild solution to Eq.(1.1).*

Proof. *Denote by* $\mathcal{S}$ *the space of all* $\mathfrak{F}_0$*-adapted processes* $\phi(t,w) : [m(0), \infty) \times \Omega \to \mathbb{R}$*, which is a.s. continuous in* t *for fixed* $w \in \Omega$*. Moreover,* $\phi(s,w) = \xi(s)$ *for* $s \in [m(0), 0]$ *and* $E\|\phi(t,w)\|_H^2 \to 0$ *as* $t \to \infty$*. It is then routine to check that* $\mathcal{S}$ *is a Banach space when it is equipped with a norm defined by*

$$\|\phi\|_{\mathcal{S}} = \sup_{t \geq 0} E\|\phi(t)\|_H^2 \qquad for \quad each \quad \phi \in \mathcal{S}.$$

Define an operator $\Theta : \mathcal{S} \to \mathcal{S}$ *by* $\Theta(x)(t) = \varphi(t)$ *for* $t \in [m(0), 0]$ *and for* $t \geq 0$,

$$\begin{aligned} \Theta(x)(t) &= S(t)[\xi(0) + a(0,\xi)] - a(t, X(t)) - \int_0^t AS(t-s)a(X(s))ds \\ &\quad + \int_0^t S(t-s)f(s, X(s-\tau(s)))ds + \int_0^t S(t-s)g(s, X(s-\delta(s)))dW(s) \\ &:= \sum_{i=1}^{6} I_i(t). \end{aligned} \tag{3.7}$$

We first verify the mean square continuity of Θ *on* $[0,\infty)$. *Let* $x \in \mathcal{S}, t_1 \geq 0$, *and* $|r|$ *be sufficiently small; then*

$$E\|\Theta(x)(t_1+r) - \Theta((x)(t_1)\|_H^2 \leq 6\sum_{i=1}^{6} E\|I_i(t_1+r) - Ii(t_1)\|_H^2.$$

By virtue of closed ness of $(-A)^\alpha$ *and the fact that* $S(t)$ *commutes with* $(-A)^\alpha$ *on* H_α, *we have by Lemma 2.1 and the assumption* $(H4)$ *that*

$$\begin{aligned} E\|I_1(t_1+r) - I_1(t_1)\|_H^2 &= \|S(t_1+r) - S(t_1)\xi(0)\|_H^2 \\ &= E\|(S(r)-I)S(t_1)\xi(0)\|_H^2 \\ &\leq \gamma_\alpha^2\mu_\alpha^2 t_1^{-2\alpha} h^{2\alpha} e^{-2at_1} E\|\xi\|_D^2, \end{aligned}$$

$$\begin{aligned} E\|I_2(t_1+r) - I_2(t_1)\|_H^2 &= E\|(S(r)-I)S(t_1)(-A)^\alpha(-A)^\alpha a(0,\xi)\|_H^2 \\ &\leq \gamma_\alpha^2\mu_\alpha^2 t_1^{-2\alpha} h^{2\alpha} e^{-2at_1} 2C^2C_1\|(-A)^{-\alpha}\|_H^2(1+E\|\xi\|_D^2), \end{aligned}$$

and

$$E\|I_3(t_1+r) - I_3(t_1)\|_H^2 \leq \|(-A)^{-\alpha}\|_H^2 E\|(-A)^{-\alpha}a(t_1+r, X_{t_1+r}) - (-A)^\alpha a(t_1, X_{t_1})\|_H^2.$$

Next, using Lemmas 2.1 and (2.2) and assumption (H4), we obtain

$$\begin{aligned} E\|I_4(t_1+r) - I_4(t_1)\|_H^2 &= E\ \| \int_0^{t_1} (-A)S(t_1-s)(S(r)-I)a(s,X_s)ds\ \|_H^2 \\ &\leq 2\frac{M_{1-\alpha}^2\Gamma(2\alpha-1)}{(2a)^{2\alpha-1}}\{\int_0^{t_1} E\|(-A)^\alpha(S(r)-I)a(s,X_s)\|_H^2 ds \\ &\quad + \int_{t_1}^{t_1+r} E\|(-A)^\alpha S(r)a(s,X_s)\|_H^2 ds\} \\ &\leq 2\frac{M_{1-\alpha}^2\Gamma(2\alpha-1)}{(2a)^{2\alpha-1}}\{\gamma_\beta^2 h^{2\beta}\int_0^{t_1} E\|(-A)^{\alpha+\beta}a(s,X_s)\|_H^2 ds \\ &\quad + e^{-2ah}\int_{t_1}^{t_1+r} E\|(-A)^\alpha a(s,X_s)\|_H^2 ds\} \\ &\leq 4C_2\frac{M_{1-\alpha}^2\Gamma(2\alpha-1)}{(2a)^{2\alpha-1}}[\gamma_\beta^2 h^2\beta t + he^{-2ah}](1+E\|X_{t_1}\|_D^2), \end{aligned}$$

and

$$E\|I_5(t_1+r) - I_5(t_1)\|_H^2 = E\|\int_0^{t_1}(S(r)-I)S(t_1-s)f(s,X(s-\tau(s))ds$$

$$
\begin{aligned}
&+ \int_0^{t_1+r} S(t_1+r-s) f(s, X(s-\tau(s))ds\|_H^2 \\
\leq\ & 2\int_0^{t_1} (t-s)^{-2\alpha} e^{-2a(t-s)} E\|f(s, X(s-\tau(s)))\|_H^2 ds \\
&+2\int_{t_1}^{t_1+r} e^{-2a(t_1+r-s)} E\|f(s, X(s-\tau(s))\|_H^2 ds.
\end{aligned}
$$

Next, using (H2),

$$
\begin{aligned}
E\|I_5(t_1+r) - I_5(t_1)\|_H^2 \ \leq\ & 2\int_0^{t_1} (t-s)^{-2\alpha} e^{-2a(t-s)} H(s, E\|X(s-\tau(s)))\|_D^2 ds) \\
&+2\int_{t_1}^{t_1+r} e^{-2a(t_1+r-s)} H(s, E\|X(s-\tau(s))\|_D^2) ds.
\end{aligned}
$$

Hence, using similar arguments as in Ahmed [8], Theorem (6.3.2), one can find constants K_1 *and* $K_2 > 0$ *depending on the parameters* $\mu, \alpha, \gamma, k, h$ *such that*

$$
E\|I_5(t_1+r) - I_5(t_1)\|_H^2 \ \leq\ 2k\gamma_\alpha\mu_\alpha[K_1 h^{2\alpha} + K_2 h](1 + H(t_1, E\|X(t_1-\tau(t_1))\|_D^2)).
$$

Assumption (H2) indicates that there is a solution u_t *that satisfies*

$$
u_t = mE\|\varphi\|_D^2 + K(T)\int_0^t H(r, u_r) dr,
$$

where $m = K(T) = 2k\gamma_\alpha\mu_\alpha[K_1 h^{2\alpha} + K_2 h]$. *We have*

$$
E\|I_5(t_1+r) - I_5(t_1)\|_H^2 \leq u_t < \infty,
$$

Next,

$$
\begin{aligned}
E\|I_6(t_1+r) - I_6(t_1)\|_H^2 \ \leq\ & 2E\|\int_0^{t_1} (S(t_1+r-s) - S(t_1-s)) g(s, X(s-\delta(s))) dW(s)\|_H^2 \\
&+2E\|\int_{t_1}^{t_1+r} S(t_1+r-s) g(s, X(s-\delta(s))) dW(s)\|_H^2,
\end{aligned}
$$

$$
\begin{aligned}
E\|I_6(t_1+r) - I_6(t_1)\|_H^2 \ \leq\ & 2\int_0^{t_1} E\|S(t_1+r-s) - S(t_1-s)) g(s, X(s-\delta(s)))\|_H^2 ds \\
&+2\int_{t_1}^{t_1+r} E\|S(t_1+r-s) g(s, X(s-\delta(s)))\|_H^2 ds,
\end{aligned}
$$

$$
\begin{aligned}
E\|I_6(t_1+r) - I_6(t_1)\|_H^2 \ \leq\ & 2\int_0^{t_1} (t-s)^{-2\alpha} e^{-2a(t-s)} E\|g(s, X(s-\tau(s)))\|_H^2 ds \\
&+2\int_{t_1}^{t_1+r} e^{-2a(t_1+r-s)} E\|g(s, X(s-\tau(s))\|_H^2 ds,
\end{aligned}
$$

and

$$\begin{aligned} E\|I_6(t_1+r)-I_6(t_1)\|_H^2 &\leq 2\int_0^{t_1}(t-s)^{-2\alpha}e^{-2a(t-s)}H(s,E\|X(s-\tau(s)))\|_D^2)ds \\ &+2\int_{t_1}^{t_1+r}e^{-2a(t_1+r-s)}H(s,E\|X(s-\tau(s))\|_D^2)ds, \end{aligned}$$

wherein we used Da Prato and Zabczyk [4] (11, Theorem 6.10, page 160) or Lemma 2.4 [1] . Arguing as before, we find constants $K3$ *and* $K4>0$ *such that*

$$E\|I_6(t_1+r)-I_6(t_1)\|_H^2 \leq 2k\gamma_\alpha\mu_\alpha[K_3h^{2\alpha}+K_4h](1+H(s,E\|X(t_1-\tau(t_1))\|_D^2)),$$

Assumption (H2) indicates that there is a solution u_t *that satisfies*

$$u_t = mE\|\varphi\|_D^2 + K(T)\int_0^t H(r,u_r)dr,$$

where $m=K(T)=2k\gamma_\alpha\mu_\alpha[K_3h^{2\alpha}+K_4h]$ *we have*

$$E\|I_6(t_1+r)-I_6(t_1)\|_H^2 \leq u_t < \infty$$

as $h\to 0$*, Thus,* Θ *is indeed continuous in 2-th moment on* $(0,\infty]$*. Next, we show that* $\Theta(\mathcal{S})\subset\mathcal{S}$*. It follows from (3.7) that*

$$\begin{aligned} E\|\Theta(x)(t)\|_H^2 &\leq 6\{E\|S(t)\xi(0)\|_H^2+E\|S(t)a(0,\xi)\|_H^2+E\|a(t,X_t)\|_H^2 \\ &+E\|\int_0^t(-A)S(t-s)a(s,X_s)ds\|_H^2 \\ &+E\|\int_0^t S(t-s)f(s,X(s-\tau(s)))\|_H^2ds \\ &+E\|\int_0^t S(t-s)g(s,X(s-\tau(s)))\|_H^2dWs\} \\ &:= \sum_{i=1}^6 J_i \end{aligned} \tag{3.8}$$

We now estimate each term in (3.8):

$$J_1 \leq 6e^{2ar}e^{-2at}E\|\xi\|_D^2.$$

By Lemma 2.1 and assumption (H4), we have

$$\begin{aligned} J_2 &\leq 6\|(-A)^{-\alpha}\|^2e^{-2at}C_5^2(1+E\|\xi\|_D^2). \\ J_3 &\leq 6\|(-A)^{-\alpha}\|^2C_2^2(1+E\|X(t_1)\|_D^2) \end{aligned}$$

Next, using assumption (H4) and Lemma 2.2, we have

$$\begin{aligned} J_4 &\leq 6\frac{M_{1-\alpha}^2\Gamma(2\alpha-1)}{(2a)^{2\alpha-1}}\int_0^t E\|(-A)^\alpha a(s,X_s)\|_H^2 ds \\ & 12TC_2^2\frac{M_{1-\alpha}^2\Gamma(2\alpha-1)}{(2a)^{2\alpha-1}}(1+E\|X_{t_1}\|_D^2), \end{aligned}$$

and by assumption (H2) and Lemma 2.1, we get

$$\begin{aligned} J_5 &\leq 6E\sup_{-r\leq\theta\leq 0}\int_0^{t+\theta} e^{-2a(t+\theta-s)}H(s,E\|X(s-\tau(s)\|)_H^2)ds \\ &\leq 6TC_1^2(1+H(t_1,E\|X(t_1)\|_D^2)) \end{aligned}$$

Lastly, by [4], Theorem 6.10 and assumption (H2), we have

$$\begin{aligned} J_6 &\leq 6k\int_0^t E\|b(s,X(s-\tau(s))\|_H^2 ds \\ & 6kTC_1^2(1+H((t_1),E\|X(t_1)\|_D^2). \end{aligned}$$

Consequently, $E\|(\Theta(X)(T)\|_D^2 < \infty$*, implying that* Θ *maps* $\mathcal{S}$ *into itself. Thirdly, we will show that* Θ *is contractive. For* $x, y \in \mathcal{S}$*, proceeding as we did previously, we can obtain*

$$\begin{aligned} &\sup_{s\in[0,T]} E\|\Theta(x)(t)-\Theta(y)(t)\|_H^2 \\ &\leq 4E\sup_{s\in[0,T]}\|a(t,X(t))-a(t,Y(t))\|_H^2 \\ &\quad 4E\sup_{s\in[0,T]}\|\int_0^t(-A)S(t-s)[a(t,X(t))-a(t,Y(t))]ds\|_H^2 \\ &\quad 4E\sup_{s\in[0,T]}\|\int_0^t S(t-s)(f(s,X(s-\tau(s)))-f(s,Y(s-\tau(s))))ds\|_H^2 \\ &\quad 4E\sup_{s\in[0,T]}\|\int_0^t S(t-s)(g(s,X(s-\delta(s)))-g(s,Y(s,\delta(s))))dW(s)\|_H^2 \\ &\leq 4C_2^2\|-A^{-\alpha}\|^2\sup_{s\in[0,T]}E\|X(t)-Y(t)\|_D^2 \\ &\quad 4\frac{M_\alpha^2\Gamma(2\alpha-1)}{(2a)^{2\alpha-1}}TC_4^2\sup_{s\in[0,T]}E\|X(t)-Y(t)\|_D^2 \\ &\quad T4\int_0^t H(r,E(\sup_{s\in[0,T]}\|X(t)-Y(t)\|_D^2)dr \\ &\quad 4C_1\int_0^t H(r,E(\sup_{s\in[0,T]}\|X(t)-Y(t)\|_D^2)dr \end{aligned}$$

Now choosing $T > 0$ sufficiently small, we can find a positive number $L(T) \in [0,1]$ such that

$$\|\Theta(X) - \Theta(Y)\|_{\mathcal{S}} \leq L(T)[\|X(t) - Y(t)\|_D + H(t, X(t) - Y(t))].$$

For any $X, Y \in \mathcal{S}$.Hence, by the Banach fixed point theorem, G has a unique fixed point $X \in \mathcal{S}$ and this fixed point is the unique mild solution of (1.1) on $[0,T]$. Next, we continue the solution for $t \geq T$, see Ahmed and Govindan [8],[12] . for notational convenience, set $T = t_1$. For $t \in [t_1, t_2]$, where $t_1 < t_2$, we say that a function $l(t)$ is a continuation of $l(t)$ to the interval $[t_1, t_2]$if

(a) $l(t) \in C([-\infty, t_2], L(\Omega, H)), and$

(b)

$$\begin{aligned} l(t) &= S(t-t_1)[\xi(t_1) + a(t_1, \xi)] - a(t, l_t) - \int_0^t AS(t-s)a(s, l_s)ds \\ &\quad + \int_0^t S(t-s)f(s, l(s-\tau(s)))ds + \int_0^t S(t-s)g(s, l(s-\delta(s)))dW(s), \end{aligned}$$

The terminology mild continuation applied to $l(t)$ is justified by the observation that if we define a new function $n(t)$ on $[0, t_2]$ by setting

$$n(t) = \begin{cases} X(T) & if \quad 0 \leq t \leq t_1, \quad ; \\ l(t) & if \quad t_1 \leq t \leq t_2, \quad . \end{cases}$$

and $l(t) = \xi(t), t \in [-\infty, 0]$, then $l(t)$ is a mild solution of (1.1) on $[0, t_2]$. The existence and uniqueness of the mold continuation $l(t)$ is demonstrated exactly as above with only some minor changes. The details are therefore omitted. Repeating this procedure, one continues the solution till the time $t_m = t_{max}$ where $[0, t_m]$ is the maximum interval of the existence and uniqueness of a solution. For t_m finite, $limE|X(t)|^2 = \infty$ as $t \to t_m$. If not, then there exists a sequence $\{\tau_n\}$ converging to t_m and a finite positive number δ such that $E|X(\tau_n)|^2 \leq \delta$ for all n. Taking n sufficiently large so that τ_n is infinitesimally close to t_m, one can use the previous arguments to extend the solution beyond t_m, which is a contradiction. Next assume that $t + \theta \leq 0$. In that case,

$$E||(\Theta X)_t - (\Theta Y)_t||_C^2 = 0.$$

This complete the proof. positive numbers δ $\square$

4 Example

Consider the neutral stochastic partial functional differential equation with finite delays $\tau_1(t), \tau_2(t), \tau_3(t) \to \infty, t \to \infty$*:*

$$\begin{aligned} d[u(t,x) + \frac{l_3(t)}{\|(-A)^{3/4}\|}\int_{-\tau_3(t)}^{0} u(t+w,x)dw] &= [\frac{\partial^2}{\partial x^2}u(t,x) + l_1(t)\int_{-\tau_1(t)}^{0} u(t+w,x)dw]dt \\ &\quad +l_2(t)u(t-\tau_2(t),x)d\beta(t), \quad t>0, \quad (4.9) \end{aligned}$$

$$l_i : \mathbb{R}^+ \to \mathbb{R}^+, \quad i = 1,2,3; \quad u(t,0) = u(t,\pi) = 0 \quad t > 0,$$

$$u(s,x) = \xi$$

where $\beta(t)$ *is a standard one dimensional Wiener process,* $l_i(t), i = 1,2,3$ *are continuous functions and* $E\|\xi\|^2 < \infty$*. Take* $H = L^2[0,\pi], K = \mathbb{R}^1$*. Define* $-A : H \to H$ *by* $-A = \frac{\partial^2}{\partial x^2}$ *with domain* $D(-A) = [z \in H : z, \partial z/\partial x$ *are absolutely continuous,* $\partial^2 z/\partial x^2 \in H, z(0) = z(\pi) = 0]$*. Then*

$$-Az = \sum_{n=1}^{\infty} n^2(z, z_n)z_n, \quad z \in D(-A), \tag{4.10}$$

where $z_n(x) = \sqrt{2/\pi}sin \quad nx, n = 1,2,3...$ *is the orthonormal set of eigenvectors of* $-A$*. It is well known that* $-A$ *is the infinitesimal of an analytic semigroup* $S(t), t \geq 0$ *in* H *and is given by*

$$S(t)z = \sum_{n=1}^{\infty} e^{-n^2 t}(z, z_n)z_n, \quad z \in H, \tag{4.11}$$

that satisfies $\| S(t) \| \leq e^{-\pi^2 t}, t \geq 0,$*and hence is a contraction semigroup.*
Define

$$\begin{aligned} a(t,w_t) &= \frac{l_3(t)}{\| (-A)^{3/4} \|}\int_{-\tau_3(t)}^{0} u(t+w,x)dw, \\ f(t,w_t) &= l_1(t)\int_{-\tau_3(t)}^{0} u(t+w,x)dw, \\ g(t,w_t) &= l_2 u(t-\tau_2(t),x). \end{aligned} \tag{4.12}$$

Then

$$\begin{aligned}\| a(t, w_t) \|_{3/4} &= \frac{l_3(t)}{\| (-A)^{3/4} \|} \mid (-A)^{3/4} \int_{-\tau_3(t)}^{0} u(t+w, x) dw \mid \\ &\leq l_3(T)\tau_3(t) \| u \| \quad a.s. \end{aligned} \tag{4.13}$$

Let $u(t,v) = \theta(t)\bar{u}(v), t \in [0,T]$ *where* $\theta(t) \geq 0$ *is locally integrable and* $\bar{u}(v)$ *is a concave nondecreasing function from* $\mathbb{R}^+$ *to* $\mathbb{R}^+$ *such that* $\bar{u}(0) = 0, \bar{u}(v) > 0$ *for* $u > 0$ *and* $\int_{0+} \frac{1}{\bar{u}(v)} dv = \infty$. *The comparison theorem of ordinary differential equation shows that assumption* (H_2). *Now let us give some concrete examples of the function* $\bar{u}$. *Let* $K > 0$ *and let* $\delta \in (0,1)$ *be sufficiently small. Define*

$$\bar{u}_1(v) = Kv, \quad v \geq 0.$$

$$\bar{u}_2(v) = \begin{cases} vlog(v^{-1}), & 0 \leq v \leq \delta, \\ \delta log(\delta^{-1}) + \bar{u}_2'(\delta-)(v-\delta), & v > \delta \quad . \end{cases} \tag{4.14}$$

$$\bar{u}_3(v) = \begin{cases} vlog(v^{-1}loglog(v^{-1}), 0 \leq \delta & , \\ \delta log(\delta^{-1})loglog(\delta^{-1}) + \bar{u}_3'(\delta-)(v-\delta), & v > \delta \quad . \end{cases} \tag{4.15}$$

where $\bar{u}'$ *denotes the derivative of function* $\bar{u}$. *They are all concave nondecreasing functions satisfying* $\int_0 \frac{dv}{\bar{u}(v)} = +\infty \quad (i = 1, 2, 3)$.
Hence, there exists a unique mild solution by Theorem (3.2)

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