Journal of Progressive Research in Mathematics(JPRM)
ISSN: 2395-0218

SCITECH
RESEARCH ORGANISATION

Volume 7, Issue 3
Published online: May 02, 2016|

Journal of Progressive Research in Mathematics
www.scitecresearch.com/journals

On some generalizations of (n, m)-normal powers operators on Hilbert space

Eiman H. Abood and Mustafa A. Al-loz

Department of mathematics, College of science, University of Baghdad, Baghdad, Iraq.

Abstract.

Recall that an operator $T \in B(H)$ is called (n, m)-*normal powers* operator if and only if $T^n(T^m)^* = (T^m)^*T^n$ for some nonnegative integers n and m.Throughout this paper,we introduce some types of generalizations of (n, m)-normal powers operatorsand study some of them properties.

Keywords: Normal operators, (n, m)-normal powers operator, n-power quasi-normaloperator, n-power class (Q).

1. Introduction.

Recall that operator $T \in B(H)$ is said to be normal operator if $TT^* = T^*T$. In [3], Alzuraqi introduced a new class of operators n-normal operators which is defined as follows: $T \in B(H)$ is called an n-normal operator if $T^nT^* = T^*T^n$ for some nonnegative integer n. He gave some basic properties of these operators and described the n-normal operators.

In [1], the authorssuggested the class of (n, m)-normal powers operators and study some properties of such class of operators for different values of the parametersn, m which is defined as follows: $T \in B(H)$is called (n, m)-*normal powers* operator if and only if $T^n(T^m)^* = (T^m)^*T^n$ for some nonnegative integers n and m. Clearly, every bounded normal operator is (1,1)-normal powers operator. Moreover, one can see that every n -normal operator is $(n, 1)$-normal powers . But the converse is not necessary true in general.It is simply seen that, $T = \begin{bmatrix} 2 & 1 \\ 0 & -2 \end{bmatrix}$is (3,2)-normal powers operator but it is not 3-normal .Hence, the class of n-normal operator is subclass of the class of (n, m)-normal powers operators.For more information about (n, m)-normal powers operators, we can refer the reader to [1] and [2].

In this paper, we establish new classes of operators which are generalizations of (n,m)-normal powers operators on a Hilbert.

2. On $(\mathbf{n},\mathbf{m})$-Powers Quasi-Normal Operators.

Recall that **[4]**, an operator $T \in B(H)$ is called *quasi-normal* if $T(T^*T) = (T^*T)T$. In **[**6], the author introduced the*n-power quasi-normal* as follows: T is called n-power quasi-normal if and only if $T^n(T^*T) = (T^*T)T^n$ for some nonnegative integer n. It is clear that every quasi-normal operator is 1-power quasi-normal. In this section we introduce a new class of operator, which is called (n,m)-power quasi-normal is introduced as follows:

Definition 2.1: Let $T \in B(H)$, T is called (n,m)-*power quasi-normal* if and only if $T^n(T^{*m}T) = (T^{*m}T)T^n$ for some nonnegitive integers n,m.

It is clear that every quasi-normal operator is $(1,1)$-power quasi-normal operator. Moreover, one can see that every n-power quasi-normal operator is $(n,1)$-power quasi-normal operator. But the converse is not true in general. It is easily seen that, $T = \begin{bmatrix} 2 & 1 \\ 0 & -2 \end{bmatrix}$ is $(3,2)$-power quasi-normal operator but it is not 3-power quasi-normal.

Theorem 2.2: If T is (n,m)-power quasi-normal operator, then so are:

1) kT for any scalar k.

2) Any $S \in B(H)$ which is unitary equivalent to T.

3) The restriction $T|M$ of T to any closed subspace M of H that reduces T.

Proof: Since T is (n,m)-power quasi-normal operator, then $T^n(T^{*m}T) = (T^{*m}T)T^n$ for someative integers n,m.

1)$(kT)^n((kT)^{*m}kT) = k^nT^n\left(\bar{k}^mT^{*m}kT\right) = k^n\bar{k}^mkT^n(T^{*m}T) = k^n\bar{k}^mk(T^{*m}T)T^n$

$= \left(\bar{k}^mT^{*m}kT\right)k^nT^n = ((kT)^{*m}kT)(kT)^n.$

Therefore, kT is (n,m)-power quasi-normal operator.

2) Let $S = UTU^*$, where U is a unitaty operator. Thus,

$$S^n(S^{*m}S) = (U^*TU)^n\left((U^*TU)^{*m}(U^*TU)\right) = U^*T^nU(U^*T^{*m}UU^*TU) = U^*T^n(T^{*m}T)U = U^*(T^{*m}T)T^nU$$

$$= (U^*T^{*m}UU^*TU)U^*T^nU = \left((U^*TU)^{*m}(U^*TU)\right)(U^*TU)^n = (S^{*m}S)S^n.\blacksquare$$

3) By **[3:p159]** we have,$(T|M)^n\big((T|M)^{m*}(T|M)\big) = (T^n|M)\big((T^{m*}|M)(T|M)\big) = (T^n(T^{m*}T)|M) = \big((T^{m*}T)T^n\big)|M$

$= \big((T^{m*}|M)(T|M)\big)(T^n|M) = \big((T|M)^{m*}(T|M)\big)(T|M)^n.$

The following theorem proves that the class of (n,m)-normal powers is subclass of (n,m)-power quasi-normal.

Theorem 2.3: If $T \in B(H)$ is a (n,m)-normal powers operator, then T is (n,m)-power quasi-normal.

Proof: Since T is (n,m)-normal powers operator, then $T^n(T^m)^* = (T^m)^*T^n$.Note that,

$T^nT^{*m}T = T^{*m}T^nT = T^{*m}T^{n+1} = T^{*m}TT^n$.■

Theorem 2.4: Let T and S are (n,m)-power quasi-normal operators, such that T commutes with S and S^*. Then ST is a (n,m)-power quasi-normal operator.

Proof:$(ST)^n\big((ST)^{*m}(ST)\big) = S^nT^n\big((S^*T^*)^m(ST)\big) = (S^nT^n)\big((S^{*m}T^{*m})(ST)\big) = S^nT^nS^{*m}ST^{*m}T$

$$= S^n(S^{*m}S)T^n(T^{*m}T) = (S^{*m}S)S^n(T^{*m}T)T^n = S^{*m}S(T^{*m}T)S^nT^n$$

$= (S^{*m}T^{*m}ST)S^nT^n = \big((S^*T^*)^m(ST)\big)S^nT^n = \big((ST)^{*m}(ST)\big)(ST)^n$.■

Theorem 2.5: Let T and S are (n,m)-power quasi-normal operators, such that $ST = TS = T^*S = ST^* = 0$, then $S+T$ is (n,m)-power quasi-normal operator.

Proof:

$$(S+T)^n\big((S+T)^{*m}(S+T)\big) = (S^n+T^n)\big((S^*+T^*)^m(S+T)\big) = (S^n+T^n)(S^{*m}S+T^{*m}T)$$
$$= S^nS^{*m}S + T^nT^{*m}T$$

$$= S^{*m}SS^n + T^{*m}TT^n = (S^{*m}S+T^{*m}T)(S^n+T^n) = \big((S^*+T^*)^m(S+T)\big)(S^n+T^n)$$

$= \big((S+T)^{*m}(S+T)\big)(S+T)^n$.■

Proposition 2.6: Let $T_1, \cdots, T_k$ are (n,m)-power quasi-normal operators. Then $(T_1 \oplus \cdots \oplus T_k)$ and $(T_1 \otimes \cdots \otimes T_k)$ are(n,m)-power quasi-normal operators.

Proof:

$(T_1 \oplus \cdots \oplus T_k)^n\big((T_1 \oplus \cdots \oplus T_k)^{*m}(T_1 \oplus \cdots \oplus T_k)\big)$

$$= ({T_1}^n \oplus \cdots \oplus {T_k}^n)\big(({T_1}^{*m} \oplus \cdots \oplus {T_k}^{*m})(T_1 \oplus \cdots \oplus T_k)\big)$$

$$= T_1{}^n (T_1{}^{*m} T_1) \oplus \cdots \oplus T_k{}^n (T_k{}^{*m} T_k)$$

$$= (T_1{}^{*m} T_1) T_1{}^n \oplus \cdots \oplus (T_k{}^{*m} T_k) T_k{}^n$$

$$= \big((T_1{}^{*m} \oplus \cdots \oplus T_k{}^{*m})(T_1 \oplus \cdots \oplus T_k)\big)(T_1{}^n \oplus \cdots \oplus T_k{}^n)$$

$$= \big((T_1 \oplus \cdots \oplus T_k)^{*m} (T_1 \oplus \cdots \oplus T_k)\big)(T_1 \oplus \cdots \oplus T_k)^n.$$

Hence, $(T_1 \oplus \cdots \oplus T_k)$ is a (n, m)-power quasi-normal operator.Now, let $x_1, \cdots, x_k \in H$, then

$$(T_1 \otimes \cdots \otimes T_k)^n \big((T_1 \otimes \cdots \otimes T_k)^{*m} (T_1 \otimes \cdots \otimes T_k)\big)(x_1 \otimes \cdots \otimes x_k)$$

$$= (T_1{}^n \otimes \cdots \otimes T_k{}^n)\big((T_1{}^{*m} \otimes \cdots \otimes T_k{}^{*m})(T_1 \otimes \cdots \otimes T_k)\big)(x_1 \otimes \cdots \otimes x_k)$$

$$= \big(T_1{}^n (T_1{}^{*m} T_1) \otimes \cdots \otimes T_k{}^n (T_k{}^{*m} T_k)\big)(x_1 \otimes \cdots \otimes x_k)$$

$$= \big((T_1{}^{*m} T_1) T_1{}^n \otimes \cdots \otimes (T_k{}^{*m} T_k) T_k{}^n\big)(x_1 \otimes \cdots \otimes x_k)$$

$$= \big((T_1{}^{*m} \otimes \cdots \otimes T_k{}^{*m})(T_1 \otimes \cdots \otimes T_k)\big)(T_1{}^n \otimes \cdots \otimes T_k{}^n)(x_1 \otimes \cdots \otimes x_k)$$

$$= \big((T_1 \otimes \cdots \otimes T_k)^{*m} (T_1 \otimes \cdots \otimes T_k)\big)(T_1 \otimes \cdots \otimes T_k)^n (x_1 \otimes \cdots \otimes x_k).$$

Hence, $(T_1 \otimes \cdots \otimes T_k)$ is a (n, m)-power quasi-normal.

3. On (n, m)-Power Class (Q) Operators.

Recall that **[5]**, an operator $T \in B(H)$ is called *class*(Q)operator if $T^{*2}T^2 = (T^*T)^2$. In **[7]** the authors introduced the n-*power class*(Q) operator as follows: $T \in B(H)$ is called n-power class (Q) operator if and only if $T^{*2}T^{2n} = (T^*T^n)^2$ for some nonnegative integer n. It is clear that every class (Q) operator is 1-power class. In this section we introduce a new class of operator, which is called (n, m)-power class (Q) is introduced as follows:

Definition 3.1: Let $T \in B(H)$, T is called (n, m)-*power class*(Q) operator if and only if $T^{*2m}T^{2n} = (T^{m*}T^n)^2$ for some nonnegative integers n, m.

It is clear that, every class (Q) operator is $(1,1)$-power class (Q) operator. Moreover, one can see that every n-power class (Q) operator is $(n, 1)$-power class (Q) operator. But the converse is not true in general. It is easily seen that,$T = \begin{bmatrix} 2 & 1 \\ 0 & -2 \end{bmatrix}$ is (n, m)-powers class (Q) operator but it is not class (Q) and not 3-power class (Q).

Theorem 3.2: If T is (n, m)-power class (Q) operator, then so are:

1) kT for every scalar k.

2) Any $S \in B(H)$ that is unitary equivalent to T.

3) The restriction $T|M$ of T to any closed subspace M of H that reduces T.

Proof : Since T is (n,m)-power class (Q) operator, then $T^{*2m}T^{2n} = (T^{m*}T^n)^2$

1) $$(kT)^{*2m}(kT)^{2n} = \bar{k}^{2m}T^{*2m}k^{2n}T^{2n} = \bar{k}^{2m}k^{2n}T^{*2m}T^{2n} = \left(\bar{k}^m k^n\right)^2(T^{*m}T^n)^2$$

$$= \left(\bar{k}^m k^n T^{*m}T^n\right)^2 = \left(\bar{k}^m T^{*m} k^n T^n\right)^2 = ((kT)^{*m}(kT)^n)^2.$$

Therefore, kT is (n,m)-power class (Q) operator.

2) Let $S = UTU^*$, where U is a unitaty operator. Thus,

$$S^{*2m}S^{2n} = (U^*TU)^{*2m}(U^*TU)^n = U^*T^{*2m}UU^*T^{2n}U = U^*T^{*2m}T^{2n}U = U^*(T^{*m}T^n)^2U$$

$$= U^*(T^{*m}T^n)UU^*(T^{*m}T^n)U = (U^*T^{*m}T^nU)^2 = (U^*T^{*m}UU^*T^nU)^2 = (S^{*m}S^n)^2.$$

Hence, S is (n,m)-powers class (Q) operator.

3) By [3:p159] we have,

$$(T|M)^{*2m}(T|M)^{2n} = (T^{*2m}|M)(T^{2n}|M) = (T^{*2m}T^{2n}|M) = (T^{*m}T^n)^2|M = ((T|M)^{*m}(T|M)^n)^2.\blacksquare$$

Theorem 3.3: If $T \in B(H)$ is (n,m)-normal powers operator, then T is (n,m)-power class (Q).

Proof : Since T is (n,m)-normal powers operator, then $T^n(T^m)^* = (T^m)^*T^n$.Note that,

$$T^{*2m}T^{2n} = T^{*m}T^{*m}T^nT^n = T^{*m}T^nT^{*m}T^n = (T^{*m}T^n)^2.\blacksquare$$

The above theorem proves that the class of all (n,m)-normal powers operator is subclass of the class of (n,m)-power class (Q).In addition that, the next theorem proves that the class of all (n,m)-powers quasi-normal operator is subclass of the class of (n,m)-power class (Q).

Theorem 3.4: If $T \in B(H)$ is (n,m)-powers quasi-normal operator, then T is (n,m)-power class (Q).

Proof:

Since T is (n,m)-powers quasi-normal operator, then $T^n(T^{m*}T) = (T^{m*}T)T^n$.Note that,

$$T^{*2m}T^{2n} = T^{*m}T^{*m}T^nT^n = T^{*m}T^{*m}TT^nT^{n-1} = T^{*m}T^nT^{*m}TT^{n-1} = T^{*m}T^nT^{*m}T^n = (T^{*m}T^n)^2.\blacksquare$$

Proposition 3.5: Let $T_1, \cdots, T_k$ are (n,m)-power class (Q) operators. Then $(T_1 \oplus \cdots \oplus T_k)$ and $(T_1 \otimes \cdots \otimes T_k)$ are(n,m)-power class (Q) operators.

Proof :

$$(T_1 \oplus \cdots \oplus T_k)^{*2m}(T_1 \oplus \cdots \oplus T_k)^{2n} = \left({T_1}^{*2m} \oplus \cdots \oplus {T_k}^{*2m}\right)\left({T_1}^{2n} \oplus \cdots \oplus {T_k}^{2n}\right)$$
$$= {T_1}^{*2m}{T_1}^{2n} \oplus \cdots \oplus {T_k}^{*2m}{T_k}^{2n}$$

$$= ({T_1}^{*m}{T_1}^{n})^2 \oplus \cdots \oplus ({T_k}^{*m}{T_k}^{n})^2 = ({T_1}^{*m}{T_1}^{n})({T_1}^{*m}{T_1}^{n}) \oplus \cdots \oplus ({T_k}^{*m}{T_k}^{n})({T_k}^{*m}{T_k}^{n})$$

$$= ({T_1}^{*m}{T_1}^{n} \oplus \cdots \oplus {T_k}^{*m}{T_k}^{n})^2 = \left(({T_1}^{*m} \oplus \cdots \oplus {T_k}^{*m})({T_1}^{n} \oplus \cdots \oplus {T_k}^{n})\right)^2$$

$$= ((T_1 \oplus \cdots \oplus T_k)^{*m}(T_1 \oplus \cdots \oplus T_k)^{n})^2.$$

Hence, $(T_1 \oplus \cdots \oplus T_k)$ is a (n, m)-power class (Q) operator.Now, let $x_1, \cdots, x_k \in H$, then

$$(T_1 \otimes \cdots \otimes T_k)^{*2m}(T_1 \otimes \cdots \otimes T_k)^{2n}(x_1 \otimes \cdots \otimes x_k)$$
$$= \left({T_1}^{*2m} \otimes \cdots \otimes {T_k}^{*2m}\right)\left({T_1}^{2n} \otimes \cdots \otimes {T_k}^{2n}\right)(x_1 \otimes \cdots \otimes x_k)$$

$$= \left({T_1}^{*2m}{T_1}^{2n} \otimes \cdots \otimes {T_k}^{*2m}{T_k}^{2n}\right)(x_1 \otimes \cdots \otimes x_k) = {T_1}^{*2m}{T_1}^{2n}x_1 \otimes \cdots \otimes {T_k}^{*2m}{T_k}^{2n}x_k$$

$$= ({T_1}^{*m}{T_1}^{n}x_1)^2 \otimes \cdots \otimes ({T_k}^{*m}{T_k}^{n}x_k)^2 = ({T_1}^{*m}{T_1}^{n}x_1 \otimes \cdots \otimes {T_k}^{*m}{T_k}^{n}x_k)^2$$

$$= \left(({T_1}^{*m} \otimes \cdots \otimes {T_k}^{*m})({T_1}^{n} \otimes \cdots \otimes {T_k}^{n})(x_1 \otimes \cdots \otimes x_k)\right)^2$$
$$= \left((T_1 \otimes \cdots \otimes T_k)^{*m}(T_1 \otimes \cdots \otimes T_k)^{n}(x_1 \otimes \cdots \otimes x_k)\right)^2$$

$$= ((T_1 \otimes \cdots \otimes T_k)^{*m}(T_1 \otimes \cdots \otimes T_k)^{n})^2(x_1 \otimes \cdots \otimes x_k).$$

Hence, $(T_1 \otimes \cdots \otimes T_k)$ is a (n, m)-power class (Q) operator. ■

Proposition 3.6: Let $T \in B(H)$. Then T is a (n, m)-power class (Q) operator if and only if T is a (m, n)-power class (Q)operator.

Proof : Let, T is (n, m)-power class (Q) operator, then $T^{*2m}T^{2n} = (T^{m*}T^{n})^2$.Note that,

$$T^{*2n}T^{2m} = (T^{*2m}T^{2n})^* = ((T^{*m}T^{n})^2)^* = ((T^{*m}T^{n})^*)^2 = (T^{*n}T^{m})^2.$$

Thus, T is a (m, n)-power class (Q).The converse of the proposition is similar.

■

Theorem 3.7: Let T and S are (n, m)-power class (Q) operators, such that T commutes with S and S^*. Then ST is a (n, m)-power class (Q) operator.

Proof :

$$(ST)^{*2m}(ST)^{2n} = T^{*2m}S^{*2m}S^{2n}T^{2n} = T^{*2m}T^{2n}S^{*2m}S^{2n} = T^{*2m}T^{2n}(S^{*m}S^{n})^2 = (T^{*m}T^{n})^2(S^{*m}S^{n})^2$$

$$= (T^{*m}T^{n}S^{*m}S^{n})^2 = (T^{*m}S^{*m}T^{n}S^{n})^2 = ((S^{m}T^{m})^*S^{n}T^{n})^2 = ((ST)^{*m}(ST)^{n})^2.$$ ■

Theorem 3.8: The class of all (n,m)-power class (Q) operators on H is closed subset of $B(H)$ under scalar multiplication.

Proof : Put, $Q(H) = \{T \in B(H): T$is a (n,m)-powers class (Q) operator on H for somenonnegative integer $n,m\}$.

One can show that from theorem (5.1), $\alpha T \in Q(H)$ for any scalar α, therefor the scalar multiplication is closed under $Q(H)$.Now let T_k be a sequance in $B(H)$ of (n,m)-power class (Q) converges to T, then after simple computation one can see that,$\|T^{2m*}T^{2n} - (T^{m*}T^n)^2\| = \|T^{2m*}T^{2n} - {T_k}^{2m*}{T_k}^{2n} + ({T_k}^{m*}{T_k}^n)^2 - (T^{m*}T^n)^2\|$

$\leq \|T^{2m*}T^{2n} - {T_k}^{2m*}{T_k}^{2n}\| + \|({T_k}^{m*}{T_k}^n)^2 - (T^{m*}T^n)^2\| \longrightarrow 0$ as $k \longrightarrow \infty$.

This implies that, $T^{2m*}T^{2n} = (T^{m*}T^n)^2$, therefore $T \in Q(H)$.Hence, $Q(H)$ is closed under scalar multiplication.

From the previous we get the following inclusions of classes:

Normal$\subsetneq(n,m)$-normal powers $\subsetneq(n,m)$-powers quasi-normal $\subsetneq(n,m)$-power class (Q).

References

[1] Abood E. H. and Al-loz M. A., On some generalization of normal operators on Hilbert space, Iraqi J. of Sci., 2C (56) (2015), 1786-1794.

[2]Al-loz M. A., On (n,m)-*normal powers* operators onHilbert spaces, Mc.S. thesis, Univ. of Baghdad, College of science, 2016.

[3] Alzuraiqi S. A., On n-normal operators, General Math. Notes, 1 (2) (2010), 61-73.

[4] Brown A., On a class of operators , Proc. Amer. Math. Soc., 4 (1953), 723-728.

[5] Jibril A. A., On operators for which$\mathrm{T}^{2*}\mathrm{T}^2 = (\mathrm{T}^*\mathrm{T})^2$, International Math. Forum, 5 (46) (2010), 2255 – 2262.

[6] Mecheri S., On n-Power quasi-normal on Hilbert space, Bull. Math. Ana. and App., 3 (2) (2011), 213-228.

[7] Panayappan S., On n-Power Class (Q) Operators, Int. J. Math. Ana., 6 (31) (2012), 1513 – 1518.