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Fixed Point Theorems in Ordered Generalized Metric Spaces with Integral Type

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Abstract.

In this paper, we prove fixed point theorems for weakly compatible self mappings satisfying certain contractive conditions of integral type in G - metric spaces.

Keywords: Weakly compatible mapping; G −metric space; common fixed point.

1. Introduction and Preliminaries

Fixed points of mappings in ordered metric space are of great use in many mathematical problems in applied and pure mathematics. The first result in this direction was obtained by Ran and Reurings [9], in this study the authors presented some applications of their obtained results to matrix equations. In [6, 7], Nieto and Lopez extended the result of Ran and Reurings [9] for non-decreasing mappings and applied their result to get a unique solution for a first order differential equation. While Agarwal et al. [1] and O'Regan and Petrutel [8] studied some results for generalized contractions in ordered metric spaces.

In 2002, **Branciari** [2] established a fixed point theorem for a single-valued mapping satisfying a contractive inequality of integral type. Recently, Liu et al. [3] (see also [4,5] obtained fixed point theorems for general classes of contractive mappings of integral type in complete metric spaces. In this paper, using auxiliary functions, we establish some fixed point theorems for self-mappings satisfying a certain contractive inequality of integral type.

In 2013, Liu et al. [3] introduced the following three contractive mappings of integral type:

$$\psi\left(\int_0^{d(fx,fy)} \varphi(t)\, dt\right) \leq \psi\left(\int_0^{d(x,y)} \varphi(t)\, dt\right) - \phi\left(\int_0^{d(x,y)} \varphi(t)\, dt\right) \quad (1)$$

Where $(\varphi, \phi, \psi) \in \phi_1 \times \phi_2 \times \phi_3$

$$\psi\left(\int_0^{d(fx,fy)}\varphi(t)\,dt\right)\le \alpha(d(x,y))\psi\left(\int_0^{d(x,y)}\varphi(t)dt\right)\ \forall x,y\in X \tag{2}$$

Where $(\varphi,\psi,\alpha)\in\phi_1\times\phi_3\times\phi_5$ and

$$\psi\left(\int_0^{d(fx,fy)}\varphi(t)\,dt\right)\le \alpha(d(x,y))\phi\left(\int_0^{d(x,y)}\varphi(t)dt\right)+\beta(d(x,y)\psi\left(\int_0^{d(x,y)}\varphi(t)dt\right)\ \forall x,y\in X \tag{3}$$

Definition 1.1: Let f and g on a G −metric space be two self mappings(X,G). The mappings $f\ and\ g$ are said to be compatible if

$$\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n)=0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z \text{ for some } z\in X.$$

Definition 1.2 Let A and S be mappings from a G-metric space (X,G) into itself. Then the mappings are said to be weakly compatible if they commute at their coincidence point, that is,

$Ax=Sx$ implies than $ASx=SAx$.

Definition 1.3 Let (X,G) be a G −metric space, for $A,B,C\subset X$ define

$$\delta_G(A,B,C)=\sup\{G(a,b,c); a\in A, b\in B, c\in C\}$$

If A consists of a single point $a, we\ write$

$$\delta_G(A,B,C)=\delta_G(a,B,C)$$

If B and C also consist of a single point $b\ and\ c, we\ write$

$$\delta_G(A,B,C)=G(a,b,c)$$

In particular for $A=B=C\in X$, $a_n=\delta_G(A)=\sup\{G(a,b,c): a,b,c\in A\}$

It follows from the definition that

(i) If $A\subseteq B$ then $\delta_G(A)\le\delta_G(B)$

For a sequence $A_n=\{x_n, x_{n+1}, x_{n+2} \dots\dots\dots\}$ in G-metric space(X,G), let

$a_n=\delta_G(A_n)$, for all $n\in N$, then

(a) Since $A_n\supseteq A_{n+1}$,then $a_{n+1}\le a_n$

(b) $G(x_l,x_m,x_k)\le\delta_G(A_n)=a_n$ for every $l,m,k\ge n$

(c) $0\le\delta_G(A_n)=a_n$ and $a_{n+1}\le a_n$ for every $n\ge l$

Therefore$\{a_n\}$ is decreasing and bounded for all $n\in N$ and so there exist an $0\le a$ such that $\lim_{n\to\infty} a_n=a$.

Lemma 1.1 Let (X, G) be a G −metric space. If $\text{Lim}_{n\to\infty} a_n = 0$ then sequence $\{x_n\}$ is a Cauchy sequence.

Theorem 1.1: Let S, R, T, U be self mapping of a complete $G -$ metric space (X, G) satisfying

(i) $SR \subseteq TU, TU$ is closed subset of X.

(ii) The pair (SR, TU) is weakly compatible,

(iii) $G(SRx, SRy, SRz) \leq \phi\big(G(TUx, TUy, TUz)\big)$ for every $x, y, z \in X$, where $\emptyset: [0, \infty) \to [0, \infty)$ is non-decreasing continuous function with $\Phi(t) < t$ for every $t > 0$

(iv) $(S, R), (T, U)$ are commutative, then S,R,T,U have a common fixed point in X.

2. Main Results

Theorem 2.1: Let (X, G) be a complete G-metric space and S, R, T,U be self mappings on (X, G) satisfying

(i) $SR \subseteq TU, TU$ is closed subset of X.

(ii) The pair (SR, TU) is weakly compatible,

(iii) $\int_0^{G(SRx,SRy,SRz)} \varphi(t)dt \leq \Phi\left(\int_0^{G(TUx,TUy,TUz)} \varphi(t)dt\right)$ for every $x, y, z \in X$, where $\emptyset: [0, \infty) \to [0, \infty)$ is non-decreasing continuous function with $\Phi(t) < t$ for every $t > 0$

(iv) $(S, R), (T, U)$ are commutative, then S,R,T,U have a common fixed point in X.

Proof: Let x_0 be an arbitrary point in X, by (i) we can choose a point x_1 in X such that

$y_0 = SRx_0 = TUx_1$ and $y_1 = SRx_1$=TUx_2 . In general, there exist a sequence $\{y_n\}$ such that $y_n = SRx_n = TUx_{n+1},$ for $n = 0,1,2,3$ ....

We prove that the sequence $\{y_n\}$ is a Cauchy sequence.

Let $A_n = \{y_n, y_{n+1}, y_{n+2}, \ldots\ldots\}$ and $a_n = \delta_G(A_n), n \in N$. Then we know that $\lim_{n\to\infty} a_n = a$ for some $a \geq 0$.

Taking $x = x_{n+k}, y = y_{m+k}$ *and* $z = z_{l+k}$ in (iii) for $k \geq 1$ *and* $m, n, l \geq 0$, we have

$$\int_0^{G(y_{n+k},y_{m+k},y_{l+k})} \varphi(t)dt = \int_0^{G(SRx_{n+k},SRx_{m+k},SRx_{l+k})} \varphi(t)dt \leq \Phi\left(\int_0^{G(TUx_{n+k},TUx_{m+k},TUx_{l+k})} \varphi(t)dt\right)$$

$$= \Phi \int_0^{G(y_{n+k-1},y_{m+k-1},y_{l+k-1})} \varphi(t)dt \qquad (1.1)$$

Now we claim that $\int_0^{G(y_{n+k-1},y_{m+k-1},y_{l+k-1})} \varphi(t)dt \leq \int_0^{a_{k-1}} \varphi(t)dt$ for every $n, m, l \geq 0$.

Since $A_{k-1} = \{y_{k-1}, y_k, y_{k+1} \ldots\ldots..\}$, $a_k = \sup\{G(a,b,c), a, b.c \in A_{k-1}\}$

Also $y_{k+n-1}, y_{k+m-1}, y_{k+l-1} \in A_{k-1}, implies\ \int_0^{G(y_{n+k-1}, y_{m+k-1}, y_{l+k-1})} \varphi(t)dt \quad \leq \int_0^{a_{k-1}} \varphi(t)dt$

Also Φ is increasing in t,

From (1.1) we get $\int_0^{\sup\ G(y_{k+n-1}, y_{k+m-1}, y_{k+l-1})} \varphi(t)dt \ \leq \Phi(\int_0^{a_{k-1}} \varphi(t)dt)$
Therefore we have $\int_0^{a_k} \varphi(t)dt \ \leq \Phi(\int_0^{a_{k-1}} \varphi(t)dt)$.
Letting $k \to \infty$, we get $\int_0^{a} \varphi(t)dt \ \leq \Phi(\int_0^{a} \varphi(t)dt)$. If $a \neq 0$, then

A$\int_0^{a} \varphi(t)dt \ \leq \Phi\left(\int_0^{a} \varphi(t)dt\right) < \int_0^{a} \varphi(t)dt$

which is a contradiction. Thus $a = 0$. Hence $\lim_{n\to\infty} a_n = 0$.

Thus by lemma $\{y_n\}$ is a Cauchy sequence in X. By completeness of X, there exist $y_1 \in X$ such that $\lim_{n\to\infty} y_n = \lim_{n\to\infty} SRx_n = \lim_{n\to\infty} TUx_{n+1} = y_1$. Also $TU(X)$ is closed, there exist $z \in X$ such that $TUz = y_{1.}$

Now we show that $SRz = y_1$. For this set x_n, x_n, z replacing x, y, z respectively in equation (iii), we get

$$\int_0^{G(SRx_n, SRx_n, SRz)} \varphi(t)dt \leq \Phi\left(\int_0^{G(TUx_n, TUx_{n,}, TUz)} \varphi(t)dt\right)$$

Taking $n \to \infty$, we get

$$\int_0^{G(y_{1.}, y_{1.}, SRz)} \varphi(t)dt \leq \Phi\left(\int_0^{G(y_{1.}, y_{1.}, y_{1.})} \varphi(t)dt\right) = 0$$

Implies $SRz = y_{1.}$. Since the pair (SR, TU) is weakly compatible $(SR)(TU) = (TU)(SR)$

$Thus\ SRy_{1.} = TUy_{1.}$.

Now we prove that $SRy_1 = y_1$. If we substitute x, y, z in (iii) by x_n, x_n, y_1 respectively

$$\int_0^{G(SRx_n, SRx_n, SRy_1)} \varphi(t)dt \leq \Phi\left(\int_0^{G(TUx_n, TUx_{n,}, TUy_1)} \varphi(t)dt\right),$$

Taking $n \to \infty$, we get

$$\int_0^{G(y_{1.}, y_{1.}, SRy_1)} \varphi(t)dt \leq \Phi\left(\int_0^{G(y_{1.}, y_{1.}, TUy_{1.})} \varphi(t)dt\right) = \Phi\left(\int_0^{G(y_{1.}, y_{1.}, SRy_1)} \varphi(t)dt\right.$$

If $SRy_1 \neq y_1$, then $G(y_{1.}, y_{1.}, SRy_1) < G(y_{1.}, y_{1.}, SRy_1)$ is a contradiction.

Therefore $SRy_1 = TUy_1 = y_{1.}$

For uniqueness let y_1 and y_2 b e two fixed points of SR, TU,

Taking $x = y = y_1$ and $z = y_2$ in (iii) we have

$$\int_0^{G(y_1,y_1,y_2)} \varphi(t)dt = \int_0^{G(SRy_1,SRy_1,SRy_2)} \varphi(t)dt \leq \Phi(\int_0^{G(TUy_1,TUy_1,TUy_2)} \varphi(t)dt)$$

$= \Phi(\int_0^{G(y_1,y_1,y_1)} \varphi(t)dt) < \int_0^{G(y_1,y_1,y_2)} \varphi(t)dt)$,a contradiction. Thus we have $y_1 = y_2$.

Now by (iv)$(S, R), (T, U)$ are mutually commutative pair of mapping.

Consider $Sy_1 = S(SRy_1) = S(RSy_1) = SR(Sy_1), implies$ Sy_1 is the unique fixed point of SR, but y_1 is the unique fixed point of SR hence $Sy_1 = y_1$.

Also $Ry_1 = R(SRy_1) = (RS)(Ry_1) = SR(Ry_1)$ implies Ry_1 is the fixed point of SR but y_1 is the unique fixed point of SR. Hence $Ry_1 = y_1$.

Thus $Sy_1 = Ry_1 = y_1$. In the same way we have $Ty_1 = Uy_1 = y_1$. Hence the result.

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