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Sigmoid Function in the Space of Univalent λ-Pseudo Starlike Function with Sakaguchi Type Functions

Olatunji S.O. [1]

[1] *Department of Mathematical Sciences,*

Federal University of Technology,

P.M.B.704,

Akure, Nigeria.

Abstract.

In this work, the sigmoid function in the space of univalent λ - peseudo starlike function with Sakaguchi type functions are investigated. The first few coefficient bounds for the class $\angle_{\lambda}^{\beta}(s,t,\Phi)$ were obtained. Also, the relevant connections to Fekete-Szego theorem for this class were briefly discussed. Our results serves as a new generalization in this direction and it gives birth some existing subclasses of functions.

Keywords: Analytic functions; Starlike function; Convex function; Bounded turning; Subordination; Bazilevic function; Sigmoid function; Fekete-Szegoo Inequality.

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1 Introduction

The most fascinating thing that has been oveshinning by other fields like real analysis, functional analysis, topology, algebra, differential equations and so on nowadays is the theory of a special function. Special function does not has a specific definition but it is an information process that is inspire by the way biological nervous system such as brain process the information.

This function comprises of large numbers of highly interconnected processing element (neurons) working together to solve a specific task. It works the same way the brain does, it can be learn by example and it can not be programmed to solve a specific task.

Special function can be categorized into three, namely, ramp function, threshold function and sigmoid function. The popular among all is the sigmoid function because of its gradient descendent learning algorithm. It can be evaluated in different ways, most especially by truncated series expansion (see detail in [4],[8] and [9]).

[1] E-Mail: olatunjiso@futa.edu.ng

The sigmoid function of the form

$$g(z) = \frac{1}{1+e^{-z}} \tag{1}$$

is differentiable and has the following properties:

it output real numbers between 0 and 1

It maps a very large input domain to a small range of outputs

It never loses information because it is an injective function

It increases monotonically.

The four properties above shows that sigmoid function is very useful in geometric functions theory.

More so, let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in U) \tag{2}$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. Recall that S^* and K denotes the class of starlike and convex functions which their geometric condition satisfies $Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0$ and $Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \quad (z \in U)$.

Several authors have used the above two classes of functions in different way of perspectives and their results are too voluminous to mention.

Two functions f and g are analytic in the open unit disk U. We say that f is subordinate to g, written as $f \prec g$ in U, if there exists a Schwarz function $\omega(z)$, which is analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = g(\omega(z))$. It follows from Schwarz lemma that $f(z) \prec g(z) \quad (z \in U) \Rightarrow f(0) = g(0)$ and $f(U) \subset g(U)$ (see detail in [7]).

Lemma 1: If a function $p \in P$ is given by

$$p(z) = 1 + p_1(z) + p_2 z^2 + ... \quad (z \in U) \tag{3}$$

then $|p_k| \leq 2 (k \in N)$, where P is the class of Caratheodory function, analytic in U, for which $p(0) = 1$ and $Re\ p(z) > 0 (z \in U)$.

Reseachers like Duren [3], Singh [12] and so on have studied various subclasses of usual known Bazilevic function $B(\alpha)$ which geometric condition satisfy

$$Re\left(\frac{f(z)^{\alpha-1} f'(z)}{z^{\alpha-1}}\right) > 0 \tag{4}$$

where α is greater than $1 (\alpha \in real)$ in different ways of perspectives of convexity, radii of convexity and starlikeness, inclusion properties and so on. The class $B(\alpha)$ includes the starlike function and bounded turning function whenever $\alpha = 0$ and $\alpha = 1$ (see detail in [1,8]). Further extension is given to the class $B(\alpha)$ to have the class

$B(\alpha,\beta)$ which geometric condition satisfies

$$Re\left(\frac{f(z)^{\alpha-1}f'(z)}{z^{\alpha-1}}\right)>\beta \tag{5}$$

where $\alpha>1$ (α is real) and $0\le\beta<1$.

Recently, Babalola [1] defined a new subclass λ pseudo starlike function of order $\beta(0\le\beta<1)$ satisfying the analytic condition

$$Re\left(\frac{z(f'(z))^{\lambda}}{f(z)}\right)>\beta \quad (\lambda\ge 1\in R, 0\le\beta<1, z\in U) \tag{6}$$

and denoted by $\angle_{\lambda}(\beta)$. It is observed that at $\lambda=2$, we obtain

$$Re\left(f'(z)\frac{z(f'(z))}{f(z)}\right)>\beta \tag{7}$$

which is the product combinations of bounded turning and starlike functions.

Frasin [5] investigated the coefficient inequalities for certain classes of Sakaguchi type functions which geometrical condition satisfy

$$Re\left\{\frac{(s-t)zf'(z)}{f(sz)-f(tz)}\right\}>\alpha \tag{8}$$

for complex numbers s,t with $s\ne t$ and α $(0\le\alpha<1)$ denoted by $S(\alpha,s,t)$. By specializing the parameters involved, we obtained various subclasses of analytic functions studied by many researchers. Just to mention but few, Owa et al [10],Sakaguchi [11], Yasar and Yalcin [13].

In this work, the author mergered equations (8) and (10) together to define a new subclass of analytic functions as $\angle_{\lambda}^{\beta}(s,t,\Phi)$ which geometric condition satisfy

$$Re\left(\frac{(s-t)z(f'(z))^{\lambda}}{f(sz)-f(tz)}\right)>\beta \quad (s,t\in C, s\ne t, \lambda\ge 1\in R, 0\le\beta<1, z\in U) \tag{9}$$

and related to sigmoid function. The first few coefficient bounds for the class and the relevant connection Fekete-Szegoo theorem for the class were briefly discussed by employing [2] and [6] method.

For the purpose of our results, the following lemma shall be necessary.

Lemma 2: [4] Let g be a sigmoid function and

$$\Phi(z)=2g(z)=1+\sum_{m=1}^{\infty}\frac{(-1)^m}{2^m}\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}z^n\right)^m \tag{10}$$

then $\Phi(z)\in P, |z|<1$ where $\Phi(z)$ is a modified sigmoid function.

Lemma 3: [4] Let

$$\Phi_{m,n}(z)=2g(z)=1+\sum_{m=1}^{\infty}\frac{(-1)^m}{2^m}\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}z^n\right)^m \tag{11}$$

then $|\Phi_{m,n}(z)|<2$.

Lemma 4: [4] If $\Phi(z)\in P$ and it is starlike, then f is a normalized univalent function of the form (2)

Setting $m=1$, Fadipe et al [4] remarked that

$$\Phi(z)=1+\sum_{n=1}^{\infty}c_n z^n \tag{12}$$

where $c_n=\frac{(-1)^{n+1}}{2n!}$ then $|c_n|\le 2, n=1,2,3,...$ and the result is sharp for each n.

2 Main Result

Theorem 1: If $f\in A$ of the form (2) is belonging to $\angle_{\lambda}^{\beta}(s,t,\Phi)(s,t\in C, s\neq t, \lambda\geq 1\in R, 0\leq\beta<1, z\in U)$ then

$$|a_2|\le\frac{1-\beta}{2(2\lambda-(s+t))} \tag{13}$$

$$|a_3|\le\frac{(1-\beta)^2}{4(2\lambda-(s+t))(3\lambda-(s^2+st+t^2))}\left[(s+t)-\frac{2\lambda(\lambda-1)}{(2\lambda-(s+t))}\right] \tag{14}$$

$$|a_4|\le\frac{(1-\beta)^3}{8(4\lambda-(s^3+s^2t+st^2+t^3))}\left\{\frac{\left[(s+t)-\frac{2\lambda(\lambda-1)}{(2\lambda-(s+t))}\right]\left[(s^2+st+t^2)-\frac{6\lambda(\lambda-1)}{(2\lambda-(s+t))}\right]}{(2\lambda-(s+t))(3\lambda-(s^2+st+t^2))}-\frac{4\lambda(\lambda-1)(\lambda-2)}{3(2\lambda-(s+t))^3}-\frac{1}{3(1-\beta)^2}\right\}. \tag{15}$$

Proof: Let $f(z)\in\angle_{\lambda}^{\beta}(s,t,\Phi)$. By definition there exists $\Phi(z)\in P$ such that

$$Re\left(\frac{(s-t)z(f'(z))^{\lambda}}{f(sz)-f(tz)}\right)=\beta+(1-\beta)\Phi(z)\quad (s,t\in C, s\neq t, \lambda\geq 1\in R, 0\leq\beta<1, z\in U) \tag{16}$$

where the function $\Phi(z)$ is a modified sigmoid function given by

$$\Phi(z)=1+\frac{1}{2}z-\frac{1}{24}z^3+\frac{1}{240}z^5-\frac{1}{64}z^6+\frac{779}{20160}z^7-.... \tag{17}$$

Thus

$$(s-t)z(f'(z))^{\lambda}=\left(f(sz)-f(tz)\right)\left(\beta+(1-\beta)\Phi(z)\right)\quad (s,t\in C, \lambda\geq 1\in R, 0\leq\beta<1, z\in U) \tag{18}$$

In view of (16),(17) and (18), expanding in series form gives

$$(s-t)\left[z+2\lambda a_2 z^2+(3\lambda a_3+2\lambda(\lambda-1)a_2^2)z^3+(4\lambda a_4+6\lambda(\lambda-1)a_2a_3+\frac{4}{3}\lambda(\lambda-1)(\lambda-2)a_2^3)z^4+\ldots\right]$$

$$=(s-t)\left[z+(s+t)a_2z^2+(s^2+st+t^2)a_3z^3\ldots\right]\left(\beta+(1-\beta)\Phi(z)\right) \tag{19}$$

Or equivalently as

$$\left[1+2\lambda a_2 z+(3\lambda a_3+2\lambda(\lambda-1)a_2^2)z^2+(4\lambda a_4+6\lambda(\lambda-1)a_2a_3+\frac{4}{3}\lambda(\lambda-1)(\lambda-2)a_2^3)z^3+\ldots\right]$$

$$=\left[1+(s+t)a_2z+(s^2+st+t^2)a_3z^2\ldots\right]\left(\beta+(1-\beta)\left(1+\frac{1}{2}z-\frac{1}{24}z^3+\frac{1}{240}z^5-\frac{1}{64}z^6+\right)\right). \tag{20}$$

Comparing the coefficients in (20), we obtain

$$a_2=\frac{1-\beta}{2(2\lambda-(s+t))} \tag{21}$$

$$a_3=\frac{(1-\beta)^2}{4(2\lambda-(s+t))(3\lambda-(s^2+st+t^2))}[(s+t)-\frac{2\lambda(\lambda-1)}{(2\lambda-(s+t))}] \tag{22}$$

$$(4\lambda-(s^3+s^2t+st^2+t^3))a_4=\frac{1-\beta}{2}a_3-6\lambda(\lambda-1)a_2a_3-\frac{4}{3}\lambda(\lambda-1)(\lambda-2)a_2^3-\frac{1-\beta}{24} \tag{23}$$

By simple computation, it gives

$$a_4=\frac{(1-\beta)^3}{8\left(4\lambda-(s^3+s^2t+st^2+t^3)\right)}\left\{\frac{\left[(s+t)-\frac{2\lambda(\lambda-1)}{(2\lambda-(s+t))}\right]\left[(s^2+st+t^2)-\frac{6\lambda(\lambda-1)}{(2\lambda-(s+t))}\right]}{\left(2\lambda-(s+t)\right)\left(3\lambda-(s^2+st+t^2)\right)}-\frac{4\lambda(\lambda-1)(\lambda-2)}{3\left(2\lambda-(s+t)\right)^3}-\frac{1}{3(1-\beta)^2}\right\}. \tag{24}$$

Corollary 1: If $f\in A$ of the form (2) is belonging to $\angle_1^{\beta}(s,t,\Phi)$ then

$$|a_2|\le\frac{1-\beta}{2\left(2-(s+t)\right)} \tag{25}$$

$$|a_3|\le\frac{(1-\beta)^2(s+t)}{4\left(2-(s+t)\right)\left(3-(s^2+st+t^2)\right)} \tag{26}$$

$$|a_4| \le \frac{(1-\beta)^3}{8\left(4\lambda-(s^3+s^2t+st^2+t^3)\right)}\left\{\frac{\left[(s+t)(s^2+st+t^2)\right]}{\left(2\lambda-(s+t)\right)\left(3\lambda-(s^2+st+t^2)\right)}-\frac{1}{3(1-\beta)^2}\right\}. \tag{27}$$

Corollary 2: If $f \in A$ of the form (2) is belonging to $\angle_2^{\beta}(s,t,\Phi)$ then

$$|a_2| \le \frac{1-\beta}{2\left(4-(s+t)\right)} \tag{28}$$

$$|a_3| \le \frac{(1-\beta)^2}{4\left(4-(s+t)\right)\left(6-(s^2+st+t^2)\right)}\left[(s+t)-\frac{4}{\left(4-(s+t)\right)}\right] \tag{29}$$

$$|a_4| \le \frac{(1-\beta)^3}{8\left(8-(s^3+s^2t+st^2+t^3)\right)}\left\{\frac{\left[(s+t)-\frac{4}{\left(4-(s+t)\right)}\right]\left[(s^2+st+t^2)-\frac{12}{\left(4-(s+t)\right)}\right]}{\left(4-(s+t)\right)\left(6-(s^2+st+t^2)\right)}-\frac{1}{3(1-\beta)^2}\right\}. \tag{30}$$

Setting $\beta = 0$ in Corollary 1 and Corollary 2, we obtain

Corollary 3: If $f \in A$ of the form (2) is belonging to $\angle_1^{0}(s,t,\Phi)$ then

$$|a_2| \le \frac{1}{2\left(2-(s+t)\right)} \tag{31}$$

$$|a_3| \le \frac{(s+t)}{4\left(2-(s+t)\right)\left(3-(s^2+st+t^2)\right)} \tag{32}$$

$$|a_4| \le \frac{1}{8\left(4\lambda-(s^3+s^2t+st^2+t^3)\right)}\left\{\frac{\left[(s+t)(s^2+st+t^2)\right]}{\left(2\lambda-(s+t)\right)\left(3\lambda-(s^2+st+t^2)\right)}-\frac{1}{3}\right\}. \tag{33}$$

Corollary 4: If $f \in A$ of the form (2) is belonging to $\angle_2^{0}(s,t,\Phi)$ then

$$|a_2| \le \frac{1}{2\left(4-(s+t)\right)} \tag{34}$$

$$|a_3| \leq \frac{\left[(s+t) - \frac{4}{(4-(s+t))}\right]}{4(4-(s+t))(6-(s^2+st+t^2))} \tag{35}$$

$$|a_4| \leq \frac{\left\{\frac{\left[(s+t) - \frac{4}{(4-(s+t))}\right]\left[(s^2+st+t^2) - \frac{12}{(4-(s+t))}\right]}{(4-(s+t))(6-(s^2+st+t^2))} - \frac{1}{3}\right\}}{8(8-(s^3+s^2t+st^2+t^3))}. \tag{36}$$

Theorem 2: If $f \in A$ of the form (2) is belonging to $\angle_{\lambda}^{\beta}(s,t,\Phi)$ then

$$|a_3 - \mu a_2^2| \leq \frac{(1-\beta)^2}{4(2\lambda-(s+t))^2}\left|\frac{(2\lambda-(s+t))}{(3\lambda-(s^2+st+t^2))}\left((s+t) - \frac{2\lambda(\lambda-1)}{(2\lambda-(s+t))}\right) - \mu\right| \tag{37}$$

Proof: From (21) and (22), it gives

$$a_3 - \mu a_2^2 = a_3 = \frac{(1-\beta)^2}{4(2\lambda-(s+t))(3\lambda-(s^2+st+t^2))}[(s+t) - \frac{2\lambda(\lambda-1)}{(2\lambda-(s+t))}] - \mu\left(\frac{1-\beta}{2(2\lambda-(s+t))}\right)^2 \tag{38}$$

By simple calculation, gives

$$a_3 - \mu a_2^2 = \frac{(1-\beta)^2}{4(2\lambda-(s+t))^2}\left[\frac{(2\lambda-(s+t))}{(3\lambda-(s^2+st+t^2))}\left((s+t) - \frac{2\lambda(\lambda-1)}{(2\lambda-(s+t))}\right) - \mu\right] \tag{39}$$

$$|a_3 - \mu a_2^2| \leq \frac{(1-\beta)^2}{4(2\lambda-(s+t))^2}\left|\frac{(2\lambda-(s+t))}{(3\lambda-(s^2+st+t^2))}\left((s+t) - \frac{2\lambda(\lambda-1)}{(2\lambda-(s+t))}\right) - \mu\right| \tag{40}$$

which completes the proof.

For taking $\mu = 1$ it gives

Corollary 5: If $f \in A$ of the form (2) is belonging to $\angle_{\lambda}^{\beta}(s,t,\Phi)$ then

$$|a_3 - \mu a_2^2| \leq \frac{(1-\beta)^2}{4(2\lambda-(s+t))^2}\left|\frac{(2\lambda-(s+t))}{(3\lambda-(s^2+st+t^2))}\left((s+t) - \frac{2\lambda(\lambda-1)}{(2\lambda-(s+t))}\right) - 1\right| \tag{41}$$

Theorem 3: If $f \in A$ of the form (2) is belonging to $\angle_{\lambda}^{\beta}(s,t,\Phi)$ then

$$|a_2 a_4 - a_3^2| \leq \tag{42}$$

$$\frac{(1-\beta)^4}{16(2\lambda-(s+t))^2}\left|\frac{(2\lambda-(s+t))}{(4\lambda-(s^3+s^2t+st^2+t^3))}\left[\frac{[(s+t)-\frac{2\lambda(\lambda-1)}{(2\lambda-(s+t))}][(s^2+st+t^2)-\frac{6\lambda(\lambda-1)}{(2\lambda-(s+t))}]}{(2\lambda-(s+t))(3\lambda-(s^2+st+t^2))}-\frac{4\lambda(\lambda-1)(\lambda-2)}{3(2\lambda-(s+t))^3}-\frac{1}{3(1-\beta)^2}\right]-\left(\frac{(s+t)-\frac{2\lambda(\lambda-1)}{(2\lambda-(s+t))}}{(3\lambda-(s^2+st+t^2))}\right)^2\right| \tag{43}$$

Proof: From (21),(22) and (23) we get

$$a_2a_4=\frac{(1-\beta)^4}{16(2\lambda-(s+t))\left(4\lambda-(s^3+s^2t+st^2+t^3)\right)}\left\{\frac{\left[(s+t)-\frac{2\lambda(\lambda-1)}{\left(2\lambda-(s+t)\right)}\right]\left[(s^2+st+t^2)-\frac{6\lambda(\lambda-1)}{\left(2\lambda-(s+t)\right)}\right]}{\left(2\lambda-(s+t)\right)\left(3\lambda-(s^2+st+t^2)\right)}-\frac{4\lambda(\lambda-1)(\lambda-2)}{3\left(2\lambda-(s+t)\right)^3}-\frac{1}{3(1-\beta)^2}\right\}. \tag{44}$$

$$a_2a_4-a_2^3=\frac{(1-\beta)^4}{16(2\lambda-(s+t))\left(4\lambda-(s^3+s^2t+st^2+t^3)\right)}\left\{\frac{\left[(s+t)-\frac{2\lambda(\lambda-1)}{\left(2\lambda-(s+t)\right)}\right]\left[(s^2+st+t^2)-\frac{6\lambda(\lambda-1)}{\left(2\lambda-(s+t)\right)}\right]}{\left(2\lambda-(s+t)\right)\left(3\lambda-(s^2+st+t^2)\right)}-\frac{4\lambda(\lambda-1)(\lambda-2)}{3\left(2\lambda-(s+t)\right)^3}-\frac{1}{3(1-\beta)^2}\right\}. \tag{45}$$

$$-\frac{(1-\beta)^4}{16(2\lambda-(s+t))^2(3\lambda-(s^2+st+t^2))^2}[(s+t)-\frac{2\lambda(\lambda-1)}{(2\lambda-(s+t))}]^2. \tag{46}$$

Then

$$a_2a_4-a_3^2= \tag{47}$$

$$\frac{(1-\beta)^4}{16(2\lambda-(s+t))^2}\left\{\frac{(2\lambda-(s+t))}{(4\lambda-(s^3+s^2t+st^2+t^3))}\left[\frac{[(s+t)-\frac{2\lambda(\lambda-1)}{(2\lambda-(s+t))}][(s^2+st+t^2)-\frac{6\lambda(\lambda-1)}{(2\lambda-(s+t))}]}{(2\lambda-(s+t))(3\lambda-(s^2+st+t^2))}-\frac{4\lambda(\lambda-1)(\lambda-2)}{3(2\lambda-(s+t))^3}-\frac{1}{3(1-\beta)^2}\right]-\left(\frac{(s+t)-\frac{2\lambda(\lambda-1)}{(2\lambda-(s+t))}}{(3\lambda-(s^2+st+t^2))}\right)^2\right\} \tag{48}$$

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