Journal of Progressive Research in Mathematics(JPRM)
ISSN: 2395-0218

SCITECH
RESEARCHORGANISATION

Volume 7, Issue1

Published online: March 12,2016

Journal of Progressive Research in Mathematics
www.scitecresearch.com/journals

QUASI-IQC-INJECTIVITY

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ABSTRACT. In this work, the notion of injectivity relative to a class of IQC submodules (namely, IQC-injectivity) has been introduced and studied, which is a generalization quasi-injective module. This notion is closed under direct summands. Several properties and characterizations have been given. We provide a characterization of semi simple Artinian ring, SI-ring and Dedekind domain in terms of IQC-injective $\mathcal{R}$-module.

Indexing terms/Keywords: Quasi -injective modules; IQC -injective R-module; Quasi-closed submodules; fully continuous modules; divisible modules.

Academic Discipline And Sub-Disciplines:

Mathematic: Algebra.

MATHEMATICSSUBJECT CLASSIFICATION:

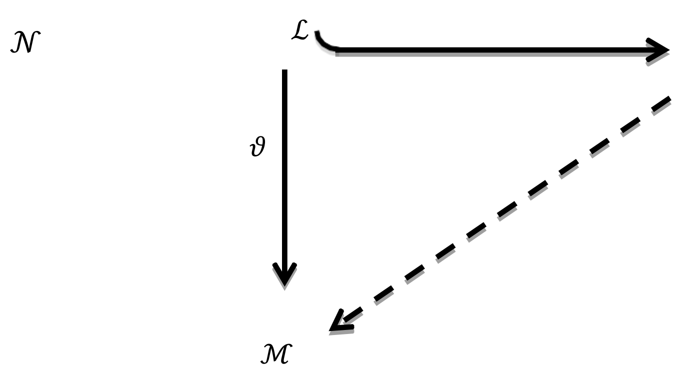
16D50, 16D70.

INTRODUCTION

Throughout, Rrepresents an associative ring with identity and $\mathcal{R}$ -modules are unitary left $\mathcal{R}$ -modules. For an $\mathcal{R}$-modules $\mathcal{M}$ and $\mathcal{N}$, $Hom_{\mathcal{R}}(\mathcal{M},\mathcal{N})$will denote the set of $\mathcal{R}$ -module homomorphisms from $\mathcal{M}$ to $\mathcal{N}$. The kernel of any $\beta \in Hom_R(\mathcal{M},\mathcal{N})$is denoted by $ker(\beta)$ and its image by$\beta(\mathcal{M})$. $S = End_R(\mathcal{M})$will denote the ring of $\mathcal{R}$-endomorphisms of $\mathcal{M}$ [1]. A submodule $\mathcal{N}$ of $\mathcal{R}$ -module $\mathcal{M}$ is said to be an essential submodule of an $\mathcal{R}$ -module $\mathcal{M}$, if $\mathcal{N}$ has nonzero intersection with every nonzero sub module of $\mathcal{M}$ [2]. A sub module $\mathcal{N}$ of $\mathcal{R}$ -module $\mathcal{M}$ is said to be a closed in $\mathcal{M}$, if $\mathcal{N}$ has no proper essential extensions in $\mathcal{M}$ ([3], P.5). We shall use $\vartheta(\mathcal{R})$ to stand for the set of all essential right ideals of the ring $\mathcal{R}$. Given any $\mathcal{R}$ -module $\mathcal{M}$, we set $Z(\mathcal{M}) = \{ x \in \mathcal{M} | \, xI = 0, for\ some\ I \in \vartheta(R)\}$ ([2], P. 30). An $\mathcal{R}$-module $\mathcal{M}$, is singular provided $Z(\mathcal{M}) = \mathcal{M}$. At the other extreme, we say $\mathcal{M}$ is a nonsingular provided Z($\mathcal{M}$)=0 ([2], P.31). A sub module $\mathcal{N}$ of $\mathcal{R}$ -module $\mathcal{M}$ is said to be a direct summand of $\mathcal{R}$-module $\mathcal{M}$, if $\mathcal{M} = \mathcal{N} \oplus \mathcal{L}$, for some submodule $\mathcal{L}$ of $\mathcal{M}$[2]. An R-module $\mathcal{M}$ is said to be semi simple, if every sub module of $\mathcal{M}$ is direct summand ([2], P.27). An $\mathcal{R}$ -module $\mathcal{M}$ is called CS-module (or extending ((C_1)-condition)), if $\mathcal{M}$ satisfies any one of the following equivalent conditions (1) for every submodule $\mathcal{N}$ of $\mathcal{M}$, there is a decomposition $\mathcal{M} = \mathcal{L} \oplus \mathcal{B}$ such that $\mathcal{N}$ is essential in $\mathcal{L}$, (2) every closed submodule of $\mathcal{M}$ is a direct summand [4]. A CS-module $\mathcal{M}$ which satisfies (C_2)-condition: every sub module of $\mathcal{M}$ which is isomorphic to a direct summand of $\mathcal{M}$ is itself direct summand, is called continuous[4]. Let $\mathcal{M}$ and $\mathcal{N}$ be two $\mathcal{R}$ -modules, $\mathcal{N}$ is called $\mathcal{M}$-injective, if for every submodule $\mathcal{L}$ of $\mathcal{M}$, any $\mathcal{R}$ -homomorphism from $\mathcal{L}$ to $\mathcal{N}$ can be extended to an $\mathcal{R}$-homomorphism from $\mathcal{M}$ to $\mathcal{N}$ ([5], P.28). An $\mathcal{R}$ -module $\mathcal{N}$ is called injective, if it is $\mathcal{M}$ -injective for all $\mathcal{R}$ -module $\mathcal{M}$. A right $\mathcal{R}$ -module $\mathcal{M}$ is (minimal) quasi-injective, if every homomorphism from a (simple) submodule of $\mathcal{M}$ to $\mathcal{M}$ can be extended to an endomorphism of $\mathcal{M}$ [6]([7]). A submodule $\mathcal{N}$ of $\mathcal{M}$ is called Quasi-closed submodule, if $\forall$ x $\in \mathcal{M}$ with x $\notin \mathcal{N}$, there exists a closed submodule $\mathcal{L}$ of $\mathcal{M}$ containing $\mathcal{N}$ and x $\notin \mathcal{L}$. it is clear that every closed submodule is a Quasi-closed –submodule[8]. Let $\mathcal{M}$ be an $\mathcal{R}$-module. A submodule $\mathcal{N}$ of $\mathcal{M}$ is called IQC-submodule (simply $\mathcal{N} \preccurlyeq^{\text{IQC}} \mathcal{M}$) , if $\mathcal{N}$ is $\mathcal{R}$ -isomorphic to a Quasi-closed submodule of $\mathcal{M}$. It is clear that, every Quasi-closed submodule (and hence direct summand) is IQC-submodule, but the converse generally is not true, $n\mathbb{Z}$ is IQC-submodule of the $\mathbb{Z}$-module $\mathbb{Z}$ which is not Quasi-closed for each positive integer $n > 2$. It is easy to prove that every submodule which is $\mathcal{R}$ -isomorphic to IQC-submodule in $\mathcal{M}$ is itself IQC-submodule in $\mathcal{M}$. Every IQC-submodule in a Quasi-closed submodule (direct summand) of $\mathcal{M}$ is IQC-submodule in $\mathcal{M}$. Let $\mathcal{M}$ and $\mathcal{N}$ be two R-modules. If $\mathcal{L} \preccurlyeq^{\text{IQC}} \mathcal{M}$, then f($\mathcal{L}$) $\preccurlyeq^{\text{IQC}}$ N where f : $\mathcal{M} \to \mathcal{N}$ is an R-isomorphism [9]. An $\mathcal{R}$ -module $\mathcal{M}$ is fully (extending) continuous , if every I(QC)-submodule of $\mathcal{M}$ is a direct summand [9] , ([8]).

Quasi - IQC-injective module

Definition(2.1): Let $\mathcal{M}$ and $\mathcal{N}$ be two $\mathcal{R}$ -modules. $\mathcal{M}$is said to be anIQC-$\mathcal{N}$-injective, if for each IQC-submodule $\mathcal{L}$ of $\mathcal{N}$, every $\mathcal{R}$ -homomorphism ϑ from $\mathcal{L}$ to $\mathcal{M}$ can be extended to an $\mathcal{R}$ -homomorphism from $\mathcal{N}$ into $\mathcal{M}$, see (1). The $\mathcal{R}$-module $\mathcal{M}$ is called Quasi- IQC -injective, if it is IQC - $\mathcal{M}$-injective.



-(1)-

Examples and remarks (2.2):

(1) Every fully continuous $\mathcal{R}$ -module is Quasi- IQC -injective. But the converse may not be true, in general.

(2) Every quasi-injective $\mathcal{R}$ -module is Quasi- IQC -injective. But the converse may not be true, in general. For example see [10, Remark (2.9)]. It may be fully continuous (Quasi- IQC - injective) , but not quasi-injective.

(3) Let $\mathcal{M}$ and $\mathcal{N}$ be two $\mathcal{R}$ -modules. If $\mathcal{M}$ is an IQC -$\mathcal{N}$-injective, then $\mathcal{M}$ is an IQC - $\mathcal{L}$ -injective for each Quasi-closed $\mathcal{R}$-submodule $\mathcal{L}$ of $\mathcal{N}$.

Proof: Let $\mathcal{L}$ be any Quasi-closed $\mathcal{R}$ -submodule of $\mathcal{N}$, $\mathcal{B}$ be any IQC -submodule of $\mathcal{L}$ and $\vartheta: \mathcal{B} \to \mathcal{M}$ be any $\mathcal{R}$ -homomorphism. Let $\iota_{\mathcal{B}}$ be the inclusion $\mathcal{R}$ -homomorphism from $\mathcal{B}$ into $\mathcal{L}$ and $\iota_{\mathcal{L}}$ be the inclusion $\mathcal{R}$ -homomorphism from Quasi-closed $\mathcal{R}$ -submodule $\mathcal{L}$ into $\mathcal{N}$. $\mathcal{M}$ is an IQC - $\mathcal{N}$-injective, thus there exists an R-homomorphism $\zeta: \mathcal{N} \to \mathcal{M}$ such that $(\zeta\iota_{\mathcal{L}}\iota_{\mathcal{B}})(b) = \vartheta(b)$, for all b $\in\mathcal{B}$. Put $\psi = \zeta\iota_{\mathcal{L}}: \mathcal{L} \to \mathcal{M}$. For each $b\in\mathcal{B}$, then $\psi(b) = (\zeta\iota_{\mathcal{L}})(b) = (\zeta\iota_{\mathcal{L}})(\iota_{\mathcal{B}}(b)) = (\zeta\iota_{\mathcal{L}}\iota_{\mathcal{B}})(b) = \vartheta(b)$. Therefore $\mathcal{M}$ is an IQC - $\mathcal{L}$ -injective $\mathcal{R}$ -module.

(4) Let $\mathcal{M}$ be an $\mathcal{R}$ -module and $\{\mathcal{N}_i\}_{i\in I}$ a family of $\mathcal{R}$ -modules. If $\prod_{i\in I}\mathcal{N}_i$ is an IQC - $\mathcal{M}$ - injective, then for each i $\in$ I, $\mathcal{N}_i$ is an IQC - $\mathcal{M}$ -injective.

Proof: Put $\mathcal{N} = \prod_{i\in I}\mathcal{N}_i$, suppose that $\mathcal{N}$ is an IQC - $\mathcal{M}$ -injective and $\mathcal{A}$ is an IQC-submodule of $\mathcal{M}$, and $f: \mathcal{A} \to \mathcal{N}_i, \forall i \in I$. There exists h: $\mathcal{M} \to \mathcal{N}$ such that $hi_{\mathcal{A}} = \varphi_i f$ where $i_{\mathcal{A}}: \mathcal{A} \to \mathcal{M}$ is inclusion mapping and $\varphi_i: \mathcal{N}_i \to \mathcal{N}$ is injection mapping. We now define h': $\mathcal{M} \to \mathcal{N}_i$, by $h'(m) = \pi_i h(m), \forall m \in \mathcal{M}$ where $\pi_i: \mathcal{N} \to \mathcal{N}_i$ is projection mapping, $\forall$ i=1,2. Then h' is an $\mathcal{R}$ -homomorphism and if $\forall a \in \mathcal{A}$, $h'i_{\mathcal{A}}(a) = \pi_i hi_{\mathcal{A}}(a) = \pi_i\varphi_i f(a) = f(a)$, this shows that $\mathcal{N}_i$ is an IQC - $\mathcal{M}$ -injective.

(5) Let $\mathcal{M}$ and $\mathcal{N}_i$ be $\mathcal{R}$ -modules where i $\in$ I and I is finite index set, if $\bigoplus_{i\in I}\mathcal{N}_i$ is an IQC - $\mathcal{M}$ -injective $\forall$ i $\in$ I, then $\mathcal{N}_i$ is an IQC - $\mathcal{M}$ -injective. In particular every direct summand of IQC- $\mathcal{N}$-injective $\mathcal{R}$ -module is IQC- $\mathcal{N}$ -injective.

Proof: Let $\mathcal{M}$ be any IQC- $\mathcal{N}$-injective $\mathcal{R}$ -module and $\mathcal{L}$ be any direct summand $\mathcal{R}$ -submodule of $\mathcal{M}$. Thus there exists an $\mathcal{R}$ -submodule $\mathcal{A}$ of $\mathcal{M}$ such that $\mathcal{M} = \mathcal{L}\oplus\mathcal{A}$. Let $\mathcal{B}$ be any IQC-submodule of $\mathcal{N}$ and $f: \mathcal{B} \to \mathcal{L}$ be any $\mathcal{R}$-homomorphism. Define g: $\mathcal{B} \to \mathcal{M} = \mathcal{L}\oplus\mathcal{A}$ by g(b)=(f(b),0), for all b$\in\mathcal{B}$. It is clear that g is an $\mathcal{R}$ -homomorphism, since $\mathcal{M}$ is an IQC- $\mathcal{N}$-injective $\mathcal{R}$ -module, thus there exists an $\mathcal{R}$ -homomorphism $h: \mathcal{N} \to \mathcal{M}$ such that $h(b) = g(b)$ for all b $\in\mathcal{B}$. Let $\pi_{\mathcal{L}}$ be the natural projection $\mathcal{R}$-homomorphism of $\mathcal{M} = \mathcal{L}\oplus\mathcal{A}$ into $\mathcal{L}$. Put $h_1 = \pi_{\mathcal{L}}h: \mathcal{N} \to \mathcal{L}$. Thus h_1 is an $\mathcal{R}$ -homomorphism and for each b$\in\mathcal{B}$, then $h_1(b) = (\pi_{\mathcal{L}}h)(b) = \pi_{\mathcal{L}}(g(b)) = \pi_{\mathcal{L}}((f(b),0)) = f(b)$. Therefore $\mathcal{L}$ is an IQC-$\mathcal{N}$ -injective $\mathcal{R}$ -module.

(6) Let $\mathcal{M}$ be an $\mathcal{R}$ -module and $\{N_i\}_{i\in I}$ a family of $\mathcal{R}$ -modules. if $\mathcal{M}$ is IQC -$\oplus_{i\in I}\mathcal{N}_i$ -injective $\forall$ i $\in$ I, then $\mathcal{M}$ is IQC -$\mathcal{N}_i$ -injective.

Proof: Suppose that $\mathcal{M}$ is an IQC $-\oplus_{i=1}^{n}\mathcal{N}_i$-injective $\mathcal{R}$ -module. Let $\mathcal{A}$ is an IQC-submodule of $\mathcal{N}_i$ (inclusion homomorphism $\iota_{\mathcal{A}}: \mathcal{A}\rightarrow\mathcal{N}_i$)and μ: $\mathcal{A}\rightarrow\mathcal{M}$ be an $\mathcal{R}$-homomorphism. By $\iota_{\mathcal{N}_i}:\mathcal{N}_i \rightarrow \oplus_{i=1}^{n}\mathcal{N}_i$ is inclusion homomorphism and hypothesis , there exists $\mathcal{R}$-homomorphism $\gamma:\oplus_{i=1}^{n}\mathcal{N}_i \mapsto \mathcal{M}$ such that $\gamma\iota_{\mathcal{N}_i}\iota_{\mathcal{A}} = \mu$. Put $g = \gamma\iota_{\mathcal{N}_i}: \mathcal{N}_i \rightarrow \mathcal{M}$ such that $g\iota_{\mathcal{A}} = \mu$.

(7) Isomorphic to Quasi- IQC -injectivity is Quasi- IQC -injectivity.

(8) Let $\mathcal{N}$ be any IQC -submodule of $\mathcal{L}$ such that $\mathcal{N}$ is IQC -$\mathcal{M}$ -injective. Then every $\mathcal{R}$ -monomorphism from $\mathcal{N}$ into $\mathcal{M}$ splits. In particular, if $\mathcal{M}$ is an $\mathcal{R}$ -module whose Quasi-closed submodules are IQC -$\mathcal{M}$-injective, then $\mathcal{M}$ is fully extending module.

Proof: Let $\gamma: \mathcal{N} \rightarrow \mathcal{M}$ be an $\mathcal{R}$-monomorphism, and $\gamma^{-1}: \gamma(\mathcal{N})\rightarrow \mathcal{N}$. As $\mathcal{N}$ is an IQC -$\mathcal{M}$ -injective module, there exists an $\mathcal{R}$-homomorphism $\beta: \mathcal{M}\rightarrow \mathcal{N}$, such that $\beta\gamma = I_{\mathcal{N}}$. For m$\in \mathcal{M}$ then $\beta(m) \in \mathcal{N}$, there exists $\gamma(n)\in \gamma(\mathcal{N})$ such that $\gamma^{-1}(\gamma(n)) = \beta(m) = \beta(\gamma(n))$ and hence $m-\gamma(n) \in \ker(\beta)$. It follows that $m = \gamma(n) + (m - \gamma(n)) \in \gamma(\mathcal{N}) + \ker(\beta)$. Moreover, $\gamma(N) \cap \ker(\beta) = \ker(\gamma^{-1}) = 0$. Thus $\mathcal{M} = \gamma(\mathcal{N})\oplus\ker(\beta)$.

(9) If $\mathcal{M}$ is Quasi- IQC -injective $\mathcal{R}$-module then any $\mathcal{R}$-monomorphism $\gamma: \mathcal{M} \rightarrow \mathcal{M}$ splits.

Proposition(2.3): Every Quasi-IQC-injective $\mathcal{R}$ -module $\mathcal{M}$ has C_2-condition.

Proof: Let $\mathcal{M}$ be a Quasi- IQC -injective $\mathcal{R}$ -module, $\mathcal{A}$ and $\mathcal{B}$ two sub modules of $\mathcal{M}$ with $\mathcal{A}$ is a direct summand in $\mathcal{M}$ and $\mathcal{B}$ is $\mathcal{R}$ -isomorphic to $\mathcal{A}$. Let $f:\mathcal{B} \rightarrow \mathcal{A}$ be an $\mathcal{R}$ -isomorphism. Then $\mathcal{A}$ is an IQC -$\mathcal{M}$ -injective, Examples and remarks (2.2), $\mathcal{B}$is an IQC -$\mathcal{M}$ -injective. The inclusion mapping $\iota_{\mathcal{B}}: \mathcal{B} \rightarrow \mathcal{M}$, there exists an $\mathcal{R}$ -homomorphism g: $\mathcal{M} \rightarrow \mathcal{B}$ such that $g\iota_{\mathcal{B}} = I_{\mathcal{B}}$. Then $\mathcal{M} = \mathcal{B}\oplus\ker(g)$. That is; $\mathcal{B}$ is a direct summand in $\mathcal{M}$, then $\mathcal{M}$ has C_2-condition.

The submodule $n\mathbb{Z}$ (where $n \geq 2$) of $\mathbb{Z}$ as $\mathbb{Z}$ -module which is isomorphic to $\mathbb{Z}$ is not a directsummand in $\mathbb{Z}$ as $\mathbb{Z}$ -module.

Corollary(2.4): Let $\mathcal{M}$ be a Quasi-IQC-injective $\mathcal{R}$-module. Then every submodule of $\mathcal{M}$ whichis $\mathcal{R}$ -isomorphic to $\mathcal{M}$ is a direct summand in $\mathcal{M}$.

Proposition(2.5): Let $\mathcal{M}$ bea Quasi- IQC -injective $\mathcal{R}$- module. Then every submodule of $\mathcal{M}$ which is isomorphic to closed submodule in $\mathcal{M}$ is closed in $\mathcal{M}$.

Proof: Let $\mathcal{M}$ be a Quasi- IQC – injective $\mathcal{R}$ - module, $\mathcal{K}$ a closed in $\mathcal{M}$ and $\mathcal{A}$ a submodule of $\mathcal{M}$ with An $\mathcal{R}$ - isomorphism $f : \mathcal{A} \rightarrow \mathcal{K}$. Consider the following diagram where $\iota_{\mathcal{A}}: \mathcal{A} \rightarrow \mathcal{M}$, $\iota_{\mathcal{K}}:\mathcal{K} \rightarrow \mathcal{M}$ are two inclusion homomorphism. Then f extends to some g in End($\mathcal{M}$) such that $\iota_{\mathcal{K}} f = g\iota_{\mathcal{A}}$, by a Quasi -IQC -$\mathcal{M}$ - injectivity of $\mathcal{M}$. Now let Ω be collection of the set of all essential extension of $\mathcal{A}$ in

$\mathcal{M}$. $\Omega \neq \phi$, since $\mathcal{A} \in \Omega$. By Zorn's lemma , there exists maximal essential member $\mathcal{A}'$. That is; $\mathcal{A}'$ ismaximal essential extension sub module in $\mathcal{M}$, which is evidently, it is closed submodule of $\mathcal{M}$. Thus $g_{|_{\mathcal{A}'}}$ is an $\mathcal{R}$ -homomorphism. Since $g(\mathcal{A}) = f(\mathcal{A})$, hence $\mathcal{K} = g(\mathcal{A})$ is essential in $g(\mathcal{A}')$, by $\mathcal{A}$ is essential sub module in $\mathcal{A}'$. Since $\mathcal{K}$ is a closed in $\mathcal{M}$. This implies $\mathcal{K} = g(\mathcal{A})$, whence $\mathcal{A} = \mathcal{A}'$. The conclusion follows.

An $\mathcal{R}$- module $\mathcal{M}$ is multiplication, if each submodule is of the form $\mathcal{M}\mathcal{A}$ for some rightideal $\mathcal{A}$ of $\mathcal{R}$ [13].

Proposition(2.6): Every Quasi - closed submodule of a multiplication a Quasi – IQC -injective is aQuasi - IQC - injective.

Proof: Let $\mathcal{L}$ be an IQC- submodule of a Quasi- closed submodule $\mathcal{N}$ of $\mathcal{M}$ and let $\theta : \mathcal{L} \rightarrow \mathcal{N}$ be an $\mathcal{R}$ -homomorphism. Since $\mathcal{N}$ is an Quasi - closed submodule of $\mathcal{R}$ -module $\mathcal{M}$. By hypothesis,there exists $\xi: \mathcal{M} \rightarrow \mathcal{M}$, by multiplication property of $\mathcal{M}$, then $\mathcal{N} = \mathcal{M}\mathcal{A}$ for some right ideal $\mathcal{A}$ of $\mathcal{R}$, $\xi|_{\mathcal{N}} = \xi(\mathcal{N}) = \xi(\mathcal{M}\mathcal{A}) = \xi(\mathcal{M})\mathcal{A} \subseteq \mathcal{M}\mathcal{A} = \mathcal{N}$.

In the following, we characterize fully continuous modules in terms of IQC -$\mathcal{M}$-injectivity.

Proposition(2.7): The following statements are equivalent for an $\mathcal{R}$ -module $\mathcal{M}$:

(1) $\mathcal{M}$ is fully continuous.

(2) Every $\mathcal{R}$ -module is IQC - $\mathcal{M}$ - injective.

(3) Every IQC-submodule of $\mathcal{M}$ is IQC - $\mathcal{M}$ - injective.

(4) Every Quasi-closed submodule of $\mathcal{M}$ is IQC - $\mathcal{M}$ - injective.

Proof: (1) ⇒ (2) ⇒ (3) ⇒ (4) It is clear. (4) ⇒ (1). Let $\mathcal{K}$ be any submodule of $\mathcal{M}$ which is isomorphic to Quasi-closed submodule $\mathcal{L}$ of $\mathcal{M}$. By (4) $\mathcal{L}$ is IQC $- \mathcal{M} -$injective. Then $\mathcal{K}$ is IQC -$\mathcal{M}$-injective The identity mapping $i_{\mathcal{K}}: \mathcal{K} \rightarrow \mathcal{K}$, there exists an $\mathcal{R}$ -homomorphism $\mathfrak{G} : \mathcal{M} \rightarrow \mathcal{K}$ such that $\mathfrak{G} i_{\mathcal{K}} = I_{\mathcal{K}}$. Then $\mathcal{M} = \mathcal{K} \oplus \ker(\mathfrak{G})$. That is; $\mathcal{K} \leqslant^{\oplus} \mathcal{M}$.

An $\mathcal{R}$ -module $\mathcal{M}$ is said to be fully IQC- stable, if every IQC-submodule of $\mathcal{M}$ is stable [9].

Proposition(2.8): Every multiplication Quasi-IQC-injective is a fully IQC- stable.

Proof: Let $\mathcal{N}$ be an IQC-submodule of $\mathcal{M}$ and an $\mathcal{R}$-monomorphism g: $\mathcal{N} \rightarrow \mathcal{M}$. Since M is multiplication, then $\mathcal{N} = \mathcal{M}\mathcal{A}$ for some ideal $\mathcal{A}$ of $\mathcal{R}$. Then g can be extended to an $\mathcal{R}$- homomorphism h: $\mathcal{M} \rightarrow \mathcal{M}$, since $\mathcal{M}$ is Quasi-IQC -injective. Now $g(\mathcal{N}) = h(\mathcal{N}) = h(\mathcal{M}\mathcal{A}) = h(\mathcal{M})\mathcal{A} \subseteq \mathcal{M}\mathcal{A} = \mathcal{N}$.

Proposition(2.9): If $\mathcal{M}$ is a fully extending and fully IQC-stable, then $\mathcal{M}$ is Quasi- IQC – injective module.

Proof: It follows by [9 ,Proposition(2. 10)] and Proposition(2.3).

Theorem(2.10): The following statements are equivalent for an $\mathcal{R}$ -module $\mathcal{M}$:

(1) $\mathcal{M}$ is fully continuous.

(2)$\mathcal{M}$ is Quasi- IQC - injective module and fully extending.

Proof: (1) $\Longrightarrow$ (2). By Examples and remarks (2.2).(2) $\Longrightarrow$ (1). By Proposition(2.3).

According to the definition of anIQC-injectivity, every R-homomorphism of IQC-submodule of $\mathcal{M}$ to $\mathcal{M}$ is extendable to all $\mathcal{M}$. In the following, we consider a direct sum of IQC-submodules instead of individual IQC-submodule.

We consider the following condition for an $\mathcal{R}$ -module $\mathcal{M}$ and a positive integer n.

(ω_n) : For any submodule K of $\mathcal{M}$ such that K $=K_1 \oplus K_2 \oplus \cdots \oplus K_n$ where K_i is IQC-submodule of $\mathcal{M}$, $\forall$ i=1,2, ... ,n, every $\mathcal{R}$-homomorphism $\vartheta: K \to \mathcal{M}$ can be extended to an $\mathcal{R}$ -endomorphism of $\mathcal{M}$. It is clear that, if $\mathcal{M}$ satisfies (ω_n), then $\mathcal{M}$ satisfies (ω_{n-1}) ,$\forall\, n \geq 2$.

Theorem(2.11): The following statements are equivalent for a fully extending module $\mathcal{M}$:

(1) $\mathcal{M}$ is fully continuous.

(2) $\mathcal{M}$satisfies $(\omega_n)\ \forall\, n \in Z^+$.

(3) $\mathcal{M}$satisfies $(\omega_n)\ \forall\, (n \geq 2) \in Z^+$.

(4) $\mathcal{M}$satisfies (ω_2).

(5) $\mathcal{M}$isQuasi- IQC-injective.

Proof: (1) $\Longrightarrow$ (2). [9 , Definition (2.2)] implies that $K_i$is direct summand of $\mathcal{M}$ for each i=1,2, ... ,n. So Kis direct summand of M, Theorem(2.10) and hence each $\mathcal{R}$ -homomorphism from K into $\mathcal{M}$ can be extended to an $\mathcal{R}$ -endomorphism.
(2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5). It is clear. (5) $\Longrightarrow$ (1): It follows from Proposition (2.3).

An $\mathcal{R}$-module $\mathcal{M}$ is said to be co-Hopfian if every injective endomorphism $f: \mathcal{M} \to \mathcal{M}$ is an automorphism [14]. An $\mathcal{R}$ -module $\mathcal{M}$ is directly finite, if $fg = I_{\mathcal{M}}$implies thatgf $= I_{\mathcal{M}}$ for all f ; g $\in$ End$(\mathcal{M})$ ([2], Lemma (6.9)). An $\mathcal{R}$ -module $\mathcal{M}$ is called weakly co-Hopfian, if any injective $\mathcal{R}$ -endomorphism $f: \mathcal{M} \to \mathcal{M}$ is essential, that is; $f(\mathcal{M})$ is an essential submodule of $\mathcal{M}$ [15]. In the following proposition, a sufficient condition for Quasi- IQC -injective modules to be co-Hopfian is given.

Proposition (2.12): A Quasi- IQC-injective $\mathcal{R}$ -module $\mathcal{M}$ is directly finite if and only if it is co-Hopfian.

Proof: Let f be injective $\mathcal{R}$-endomorphism of $\mathcal{M}$ and $I_{\mathcal{M}}: \mathcal{M} \rightarrow \mathcal{M}$ the identitymap. Since $\mathcal{M}$ is a Quasi-IQC -injective, there exists a map $g: \mathcal{M} \rightarrow \mathcal{M}$ such that, $gf = I_{\mathcal{M}}$. By directly finite of $\mathcal{M}$, we have $fg = I_{\mathcal{M}}$ which shows that f is an automorphism. Hence $\mathcal{M}$ is co-Hopfian. The converse is clear.

In the following proposition, we give a condition for weakly co-Hopfian modules to be co-Hopfian.

Proposition (2.13): The following conditions are equivalent for a Quasi-IQC-injective $\mathcal{R}$ -module $\mathcal{M}$:

(1) $\mathcal{M}$ is weakly co-Hopfian .

(2) $\mathcal{M}$ is co-Hopfian.

Proof: (1) $\Longrightarrow$(2) Let $f: \mathcal{M} \rightarrow \mathcal{M}$ be an $\mathcal{R}$ -monomorphism. By(1) we have $f(\mathcal{M})$ is essential in $\mathcal{M}$.f splits and hence $f(\mathcal{M})$ is a direct summand of $\mathcal{M}$, since $\mathcal{M}$ is a Quasi- IQC -injective. Therefore $f(\mathcal{M}) = \mathcal{M}$. This shows that $\mathcal{M}$ is co-Hopfian. (2) $\Longrightarrow$ (1) is obvious.

It is well-known that an $\mathcal{R}$ -module $\mathcal{M}$ is injective if and only if $\mathcal{M}$ is $\mathcal{N}$-injective for each $\mathcal{R}$-module $\mathcal{N}$.

Proposition(2.14): The following statements are equivalent for an $\mathcal{R}$ -module $\mathcal{M}$:

(1) $\mathcal{M}$ is injective.

(2) $\mathcal{M}$ is IQC - $\mathcal{N}$ -injective, for each $\mathcal{R}$ -module $\mathcal{N}$.

Proof: (1) $\Longrightarrow$ (2): Obvious, (2) $\Longrightarrow$ (1): Let $E = E(\mathcal{M})$ be the injective hull of $\mathcal{M}$. Let $\acute{\imath}: \mathcal{M} \rightarrow E$ be the inclusion mapping and $j: E \rightarrow \mathcal{M} \oplus E$ the natural injection. By IQC -$\mathcal{M} \oplus E$ – injectivityof $\mathcal{M}$, implies that the identity mapping $I_{\mathcal{M}}$ of $\mathcal{M}$, can be extended to an $\mathcal{R}$-homomorphism $f: \mathcal{M} \oplus E \rightarrow \mathcal{M}$ such that $g\acute{\imath} = I_{\mathcal{M}}$ where $g = fj$. Then $E = \mathcal{M} \oplus \ker(g)$, then $\mathcal{M}$ =E, hence $\mathcal{M}$ is injective.

It is well-known that if $\mathcal{R}$ is a semi simple Artinian ring , then every $\mathcal{R}$ -module is injective ([2], Theorem(1.18)). Also, Osofsky in [16] a proved that ring $\mathcal{R}$ is semi simple Artinian if and only if every cyclic $\mathcal{R}$ -module is injective. $\mathcal{R}$ecall that $\mathcal{R}$ is a right V-ring, if every simple $\mathcal{R}$ -module is injective [17].We now provide a characterization of semi simple Artinian rings in terms of Quasi- IQC -injective modules.

Theorem (2.15) : The following conditions are equivalent for a ring $\mathcal{R}$.

(1) $\mathcal{R}$ is semi simple Artinian,

(2) $\mathcal{R}$ is a right V-ring and every minimal quasi-injective right $\mathcal{R}$ -module is Quasi- IQC -injective,

(3) Every $\mathcal{R}$ -module is Quasi- IQC -injective,

(4) The direct sum of every two Quasi- IQC -injective modules is Quasi- IQC - injective. And every cyclic $\mathcal{R}$ -module is Quasi- IQC -injective,

Proof:(1)⟹(2).It follows from([2],Theorem(1.18)). (2) ⟹ (3). Since $\mathcal{R}$ is a right V-ring, every simple $\mathcal{R}$ -module is injective and hence every simple right $\mathcal{R}$ -module is a direct summand of each module containing it. So every $\mathcal{R}$ -module is minimal quasi-injective, hence is Quasi- IQC -injective $\mathcal{R}$ -module.(3) ⟹(4).It is clear. (4) ⟹ (1). Let $\mathcal{M}$ be Quasi- IQC -injective module and E the injective hull of $\mathcal{M}$. By(4) $\mathcal{M}\oplus$E is Quasi- IQC -injective. Then Examples and remarks (2.2), $\mathcal{M}$is IQC -$\mathcal{M}\oplus$E-injective and Proposition (2.14), hence $\mathcal{M}$ is injective. By every cyclic $\mathcal{R}$ -module is Quasi- IQC -injective, then every cyclic $\mathcal{R}$-module is injective , that is; $\mathcal{R}$ is semi simple Artinian , by Osofsky's theorem in [16].

Theorem (2.16): The following statements are equivalent for a ring :

(1) $\mathcal{R}$is a semi-simple Artinian ring .

(2) For each $\mathcal{R}$ -module , if $\mathcal{N}_1$ and $\mathcal{N}_2$ are Quasi- IQC -injective $\mathcal{R}$ -submodules of $\mathcal{M}$, then $\mathcal{N}_1 \cap \mathcal{N}_2$ is a Quasi- IQC -injective $\mathcal{R}$ -module .

(3) For each $\mathcal{R}$ -module , if $\mathcal{N}_1$ and $\mathcal{N}_2$ are quasi-injective $\mathcal{R}$ -submodules of $\mathcal{M}$, then $\mathcal{N}_1 \cap \mathcal{N}_2$ is a Quasi-IQC-injective $\mathcal{R}$ -module.

(4) For each R-module , if $\mathcal{N}_1$ and $\mathcal{N}_2$ are injective $\mathcal{R}$ -submodules of $\mathcal{M}$, then $\mathcal{N}_1 \cap \mathcal{N}_2$ is a Quasi- IQC -injective $\mathcal{R}$ -module.

Proof: (1)⟹(2).It follows from Theorem (2.15). (2) ⟹(3) and (3) ⟹ (4) are obvious. (4) ⟹(1)Let $\mathcal{M}$ be any $\mathcal{R}$-module and $\Xi = \Xi(\mathcal{M})$ is the injective envelope of $\mathcal{M}$,let $\mathcal{Q}$=Ξ⊕Ξ , $\mathcal{K}=\{(x, x) \in \mathcal{Q} \mid x \in \mathcal{M}\}$ and let $\mathcal{Q} = \mathcal{Q} / \mathcal{K}$ .Also, put $\mathcal{M}_1$={ $\mathcal{Y}$+$\mathcal{K}$ ∊ $\mathcal{Q}$ |y∊ Ξ⊕(0)} and $\mathcal{M}_2$={ $\mathcal{Y}$ +$\mathcal{K}$ ∊ $\mathcal{Q}$ | y∊(0) ⊕Ξ }. It is clear that $\mathcal{Q} = \mathcal{M}_1 + \mathcal{M}_2$Define τ_1: Ξ →$\mathcal{M}_1$by τ_1 (y) =(y,0) + $\mathcal{K}$, for all y ∊ Ξ and τ_2: Ξ →$\mathcal{M}_2$ by τ_2(y) = (0,y) + $\mathcal{K}$, for all y ∊ Ξ .Since (Ξ⊕(0))∩ $\mathcal{K}$=(0) and ((0) ⊕ Ξ)∩ $\mathcal{K}$ =(0) , thus we have $\tau_1$and $\tau_2$are $\mathcal{R}$-isomorphisms. Since Ξ is an injective $\mathcal{R}$ -module , therefore $\mathcal{M}_i$is injective $\mathcal{R}$ -submodule of $\mathcal{Q}$, for i=1,2. Thus by (4) , we have $\mathcal{M}_1 \cap \mathcal{M}_2$ is a Quasi- IQC -injective $\mathcal{R}$ -module. Define f: $\mathcal{M} \to \mathcal{M}_1 \cap \mathcal{M}_2$ by f(m)=(m,0)+ $\mathcal{K}$, for all m∊ $\mathcal{M}$. Since $\mathcal{M}_1 \cap \mathcal{M}_2$={$\mathcal{Y}$+$\mathcal{K}$ ∊ $\mathcal{Q}$ |y∊ $\mathcal{M}$⊕(0)}, thus it is easy to prove that f is an $\mathcal{R}$ -isomorphism. Thus $\mathcal{M}$ is a Quasi- IQC -injective $\mathcal{R}$ -module, by remark ((2.2),7) . Hence every $\mathcal{R}$-module is Quasi- IQC -injective and this implies that $\mathcal{R}$ is a semi-simple Artinian ring , by Theorem (2.15).

$\mathcal{R}$ecall that an $\mathcal{R}$ -module $\mathcal{M}$ is direct injective, if given any direct summand A of $\mathcal{M}$, an injection $i_A: \mathcal{A} \to \mathcal{M}$ and every $\mathcal{R}$ -monomorphism $f : \mathcal{A} \to \mathcal{M}$, there is an $\mathcal{R}$ -endomorphism g of $\mathcal{M}$ such that $gf = i_A$ [18].

Nicholson in([19], Theorem(7.13)) proved that direct injective$\mathcal{R}$ -module is equivalent to C_2-condition. Proposition(2.3) shows that every Quasi- IQC injective $\mathcal{R}$ -module is a direct injective and every direct injective $\mathcal{R}$ -module is divisible [18]. Then we have the following:

Proposition(2.17): Every Quasi- IQC -injective $\mathcal{R}$ -module is divisible.

The converse of Proposition(2.17) may not be true.

Quasi- IQC - injectivity is not closed under direct sums in general, as we see in the following

$\mathcal{R}$=$\begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, $\mathcal{A}$= $\begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$, $\mathcal{B}$= $\begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$, C= $\begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$Where $F = \frac{Z}{2Z}$. It Is easy to see that the $\mathcal{R}$ -modules $\mathcal{A}$ and $\mathcal{B}$ are quasi-injective. And hence by Examples and remarks (2.2), they are Quasi- IQC -injective. However $\mathcal{R}$ = $\mathcal{A} \oplus \mathcal{B}$ is not Quasi- IQC -injective, since otherwise R satisfies (C2)-condition, by Proposition(2.3). But $\mathcal{A}$ is isomorphic to C and C is not a direct summand in $\mathcal{R}$, contradiction.

Since $\mathcal{A}$ and $\mathcal{B}$ are two divisible $\mathcal{R}$ -modules. And every direct sum of divisible $\mathcal{R}$ -modules is divisible. That is; $\mathcal{A} \oplus \mathcal{B}$ is divisible. But it is not Quasi- IQC -injective.

In the following, we show that the distinction between Quasi- IQC -injectivity and divisibility vanishes over Dedekind domain. A domain $\mathcal{R}$ is called Dedekind ring, if every divisible $\mathcal{R}$ -module is injective ([20], Theorem(4.24)). We now provide a characterizationof domain$\mathcal{R}$ is Dedekind rings in terms of Quasi- IQC -injective $\mathcal{R}$ -modules.

Theorem(2.18): The following conditions are equivalent for a ring $\mathcal{R}$.

(1) $\mathcal{R}$is Dedekind domain,

(2) Every divisible $\mathcal{R}$ -module is Quasi- IQC -injective.

Proof: (1) $\Longrightarrow$ (2). By ([20], Theorem(4.24)). (2) $\Longrightarrow$ (1). Let $\mathcal{M}$be a divisible $\mathcal{R}$-module and $\Xi(\mathcal{M})$ an injective hull of M. By ([5], proposition (2.6)), $\Xi(\mathcal{M})$ is divisible and by ([5], Lemma(2.5)), then $\mathcal{M} \oplus \Xi$ is divisible. By(2) $\mathcal{M} \oplus \Xi$ is Quasi- IQC -injective. Then Examples and remarks (2.2), $\mathcal{M}$is IQC-$\mathcal{M} \oplus \Xi$ -injective and Proposition(2.14). That is; $\mathcal{M}$ is injective, implies $\mathcal{R}$ is Dedekind domain [20].

Recall that a ring $\mathcal{R}$ is SI-ring, if every singular $\mathcal{R}$ -module is injective ([3], below Corollary (7.16)). Over non singular ring; we provide a characterization of SI-ring in terms of Quasi- IQC -injective $\mathcal{R} -$ modules.

Proposition(2.19): The following statements are equivalent for non singular ring $\mathcal{R}$:

(1) $\mathcal{R}$is SI-ring.

(2) Every singular $\mathcal{R}$ -module is Quasi- IQC -injective,

Proof: (1) $\Longrightarrow$ (2) is clear.(2) $\Longrightarrow$ (1). Let $\mathcal{M}$ be a singular $\mathcal{R}$ -module and $\Xi(\mathcal{M})$ the injective hull of $\mathcal{M}$. ([2], Proposition(1.23) and (1.22)), then $\mathcal{M}\oplus\Xi(\mathcal{M})$ is singular. By(2) $\mathcal{M}\oplus\Xi$ is Quasi- IQC -injective. Then Examples and remarks (2.2), $\mathcal{M}$is IQC -$\mathcal{M}\oplus\Xi$ -injective and Proposition(2.14), hence $\mathcal{M}$ is injective. That is; $\mathcal{R}$ is SI-ring .

In the next part we characterize some rings by Quasi- IQC -injectivity. In the following, Noetherian rings are characterize as in terms of Quasi- IQC -injective. $\mathcal{R}$ecall that a $\mathcal{R}$ -module $\mathcal{M}$ is F-injective, if for any finitely generated ideal $\mathcal{L}$ of $\mathcal{R}$, every $\mathcal{R}$ -homomorphism of $\mathcal{L}$ into $\mathcal{M}$, can be extended to an $\mathcal{R}$ -homomorphism $\mathcal{M}$ into $\mathcal{M}$ [21].

Proposition (2.20) : The following conditions are equivalent:

(1) $\mathcal{R}$is Noetherian ring ;

(2) Every F-injective $\mathcal{R}$ -modules are injective;

(3) Every F-injective $\mathcal{R}$ -module is Quasi- IQC -injective.

Proof: (1) implies (2) and (2) implies (3) are evidently.

Assume (3) . Let $\mathcal{M}$ be a F-injective $\mathcal{R}$ -module, E the injective hull of $\mathcal{M}$. Write Q=$\mathcal{M}\oplus\Xi$ is F-injective $\mathcal{R}$ -module. By(3) $\mathcal{M}\oplus\Xi$ is Quasi- IQC -injective. Then Examples and remarks (2.2) ,$\mathcal{M}$is IQC - $\mathcal{M}\oplus\Xi$ -injective and Proposition(2.14), hence $\mathcal{M}$ is injective. We have shown that every F-injective $\mathcal{R}$ -module is injective. Since any direct sum of F-injective $\mathcal{R}$ -modules is F-injective, then every direct sum of injective modules is injective which implies that $\mathcal{R}$ is Noetherian, by ([20], P.82). Thus (3) implies (2) and (2) implies (1).

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