Journal of Progressive Research in Mathematics(JPRM)
ISSN: 2395-0218

SCITECH
RESEARCH ORGANISATION|

Volume 7, Issue 1
Published online: March 17, 2016|

Journal of Progressive Research in Mathematics
www.scitecresearch.com/journals

Some theorem on common fixed points and points of coincidence for mappings in metric space

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Abstract.

The paper includes theorem giving the sufficient condition for existence of common point of coincidence and common fixed point for 2n + 1 mappings in metric space.

2000 Mathematics Subject Classification: Primary 47H10; Secondary 54H25, 55M20.

Key words and phrases: Fixed point; point of coincidence; weakly compatible mappings; contraction; Banach fixed point theorem; Kannan type condition.

Introduction

Let (X,d) be a metric space. By contraction ([3], [4]) we understand a mapping $F: X \to X$ for which there exists $L \in [0,1[$ such that $d(Fx,Fy) \leq L \cdot d(x,y)$, for all $x,y \in X$. The very good known Banach Fixed Point Theorem reads as follows ([3], [4]). Let X be a complete metric space with metric d. Let $F: X \to X$ be a contraction. The above suppositions imply the existing of a fixed point, the solution of $Fx = x$. The Banach Fixed Point Theorem is an important tool in mathematical analysis and has been investigated under various conditions and developed in different directions. Among others, many authors consider a variety of contractive conditions (see for example [5], [6], [7], [8]). In the presented paper we obtain a sufficient condition for existence of common fixed points and points of coincidence for $2n+1$ $(n \in \mathbb{N})$ mappings in metric space (X,d). The condition which is given in the paper is analogous to condition included in [2] and named by authors - Kannan type condition. Summarizing, we will prove theorem giving sufficient condition (Kannan type condition) for existence of common fixed point for $2n+1, n \in \mathbb{N}$ mappings $X \to X$ where (X,d) is a metric space.

1 Notations, definitions, lemma

Definition 1.1. ([7], [1]) A mapping $T: X \to X$ - for a metric space (X,d) - is called Kannan if there exists $\alpha \in [0, \frac{1}{2})$ such that for all $x,y \in X$

$$d(Tx,Ty) \leq \alpha[d(x,Tx) + d(y,Ty)]$$

Definition 1.2. A point $y \in X$ is called point of coincidence of a family $\{T_j\}_{j\in J}$ of self-mappings on X if there exists a point $x \in X$ such that $y = T_j x$ for all $j \in J$.

Definition 1.3. A pair (F, T) of self-mappings on X is said to be weakly compatible if $FTx = TFx$ whenever $Fx = Tx$.

The lemma below is a generalization of the result from the paper [1].

Lemma 1.4. *Let $n \in \mathbb{N}$. Let X be a nonempty set and the mappings*

$$F, S_1, ..., S_n, T_1, ..., T_n$$

have a unique point of coincidence v in X. If all pairs (F, T_i), (F, S_i), for $i \in \{1, 2, ..., n\}$ are weakly compatibles, then v is the unique common fixed point of mappings $F, S_1, ..., S_n$, $T_1, ..., T_n$.

Proof. Take $u \in X$ such that $Fu = S_1u = ... = S_nu = T_1u = ... = T_nu = v$. By weakly compatibility of all pairs (F, T_i), (F, S_i), for $i \in \{1, 2, ..., n\}$ we have

$$\Big(S_iv = S_iFu = FS_iu = Fv, \text{ and } T_iv = T_iFu = FT_iu = Fv\Big), \text{ for } i \in 1, ..., n.$$

Therefore the point w such that $w = Fv = S_1v = ... = S_nv = T_1v = ... = T_nv$ is a point of coincidence for mappings $F, S_1, ..., S_n, T_1, ..., T_n$, so $w = v$ by uniqueness. From the above v is a unique common fixed point of mappings $F, S_1, ..., S_n$, $T_1, ..., T_n$. □

Definition 1.5. Let (X, d) be a metric space. Let $F, S_1, ..., S_n, T_1, ..., T_n : X \to X$ be the mappings such that $S_i(X) \cup T_i(X) \subset F(X)$ for $i = 1, ...n$. We define the sequence $\{x_m\}_{m\geq 0}$ of elements of X as follows. Choose an arbitrary point x_0 in X. Let x_1, x_2 be the point of X such that $Fx_1 = T_1x_0$ and $Fx_2 = S_1x_1$. Continuing, $Fx_3 = T_2x_2$ and $Fx_4 = S_2x_3$,...,$Fx_{2n-1} = T_nx_{2n-2}$, $Fx_{2n} = S_nx_{2n-1}$. Generally, if we have defined x_{kn} for a $k \in \{0, 2, 4, ...\}$, we put

$$Fx_{kn+i} = \begin{cases} T_{\frac{i+1}{2}}x_{kn+i-1}, & \text{for } i \in \{1, 3, ..., 2n-1\}, \\ S_{\frac{i}{2}}x_{kn+i-1}, & \text{for } i \in \{2, 4, ..., 2n\}. \end{cases}$$

2 Main theorem

Theorem 2.1. *Let $n \in \mathbb{N}$. Let (X, d) be a complete metric space and*

$$F, S_1, ..., S_n, T_1, ..., T_n$$

be self mappings of the space (X, d) such that $F(X)$ is the closed subset of X and $\bigcup_{i=1}^n S_i(X) \subset F(X)$, $\bigcup_{i=1}^n T_i(X) \subset F(X)$. Let us suppose that the following condition is satisfied:

$$d(S_ix, T_jy) \leq A_id(Fx, S_ix) + B_jd(Fy, T_jy), \quad \textit{for } i, j \in \{1, 2, ..., n\} \tag{2.1}$$

for all $x, y \in X$ where A_i, B_j are non-negative real numbers with $A_i + B_j < 1$, for $i, j \in \{1, 2, ..., n\}$. Then $F, S_1, ..., S_n, T_1, ..., T_n$ have a unique point of coincidence. If additionally all pairs (F, T_i), (F, S_i), for $i \in \{1, 2, ..., n\}$ are weakly compatibles, then $F, S_1, ..., S_n, T_1, ..., T_n$ have the unique common fixed point.

Proof. Firstly, we will prove that, if $F, S_1, ..., S_n, T_1, ..., T_n$ have a point of coincidence, then it is unique. Assume that v, v^* are two distinct point of coincidence for mappings $F, S_1, ..., S_n, T_1, ..., T_n$. Then, there exists u, u^* such that

$$Fu = S_1 u = ... = S_n u = T_1 u = ... = T_n u = v$$

and

$$Fu^* = S_1 u^* = ... = S_n u^* = T_1 u^* = ... = T_n u^* = v^*.$$

By (2.1) we have

$$d(v, v^*) = d(S_1 u, T_1 u^*) \leq A_1 d(Fu, S_1 u) + B_1 d(Fu^*, T_1 u^*) = 0,$$

so $v = v^*$. We will prove the existence of a point of coincidence of mappings $F, S_1, ..., S_n, T_1, ..., T_n$. Choose an arbitrary point x_0 in X. The sequence $\{x_m\}_{m \geq 0}$ of elements of X is defined by Definition 1.5. Let us consider two cases:

a) there exists $k \in \{0, 2, 4, ...\}$ for which $Fx_{kn} = Fx_{kn+1}$.

b) for every $k \in \{0, 2, 4, ...\}$ $Fx_{kn} \neq Fx_{kn+1}$.

In the case a) - by (2.1) - for every $i \in \{1, 2, ..., n\}$ - we have

$$\begin{aligned} d(S_i x_{kn}, Fx_{kn}) = d(S_i x_{kn}, Fx_{kn+1}) &= d(S_i x_{kn}, T_1 x_{kn}) \\ &\leq A_i d(Fx_{kn}, S_i x_{kn}) + B_1 d(Fx_{kn}, T_1 x_{kn}) \\ &= A_i d(Fx_{kn}, S_i x_{kn}) + B_1 d(Fx_{kn}, Fx_{kn+1}) \\ &= A_i d(Fx_{kn}, S_i x_{kn}) \end{aligned}$$

and

$$\begin{aligned} d(Fx_{kn}, T_i x_{kn}) &= d(S_n x_{kn}, T_i x_{kn}) = \\ &\leq A_n d(Fx_{kn}, S_n x_{kn}) + B_i d(Fx_{kn}, T_i x_{kn}) \\ &= B_i d(Fx_{kn}, T_i x_{kn}). \end{aligned}$$

This yields that the point y defined as

$$y := Fx_{kn} = S_1 x_{kn} = S_2 x_{kn} = ... = S_n x_{kn} = T_1 x_{kn} = T_2 x_{kn} = ... = T_n x_{kn}$$

is the required unique point of coincidence for mappings $F, S_1, ..., S_n, T_1, ..., T_n$.

In the case b) the reasoning is as follows. Let $k \in \{0, 2, 4, ...\}$. We have $Fx_{kn} \neq Fx_{kn+1}$. By (2.1) we get

$$d(Fx_{kn}, Fx_{kn+1}) = d(S_n x_{kn-1}, T_1 x_{kn})$$

$$\leq A_n d(Fx_{kn-1}, S_n x_{kn-1}) + B_1 d(Fx_{kn}, T_1 x_{kn})$$
$$= A_n d(Fx_{kn-1}, Fx_{kn}) + B_1 d(Fx_{kn}, Fx_{kn+1}).$$

Hence

$$d(Fx_{kn}, Fx_{kn+1}) \leq \frac{A_n}{1 - B_1} d(Fx_{kn-1}, Fx_{kn})$$
$$\leq \lambda d(Fx_{kn-1}, Fx_{kn}),$$

where

$$0 < \lambda := max\left\{ \frac{A_i}{1 - B_j}, \frac{B_j}{1 - A_i} : \ i, j = 1, 2, ...n \right\} < 1.$$

Moreover

$$d(Fx_{kn}, Fx_{kn-1}) = d(S_n x_{kn-1}, T_n x_{kn-2})$$
$$\leq A_n d(Fx_{kn-1}, S_n x_{kn-1}) + B_n d(Fx_{kn-2}, T_n x_{kn-2})$$
$$= A_n d(Fx_{kn-1}, Fx_{kn}) + B_n d(Fx_{kn-2}, Fx_{kn-1}).$$

Therefore

$$d(Fx_{kn}, Fx_{kn-1}) \leq \frac{B_n}{1 - A_n} d(Fx_{kn-1}, Fx_{kn-2})$$
$$\leq \lambda d(Fx_{kn-1}, Fx_{kn-2}).$$

From the above we get easily for any $m \in \mathbb{N}$

$$d(Fx_m, Fx_{m+1}) \leq \lambda^m d(Fx_0, Fx_1)$$

and for any $m_2 > m_1$

$$d(Fx_{m_1}, Fx_{m_2}) \leq d(Fx_{m_1}, Fx_{m_1+1}) + d(Fx_{m_1+1}, Fx_{m_1+2}) + ... + d(Fx_{m_2-1}, Fx_{m_2})$$
$$\leq [\lambda^{m_1} + \lambda^{m_1+1} + ... + \lambda^{m_2}] d(Fx_0, Fx_1)$$
$$\leq \left[\frac{\lambda^{m_1}}{1 - \lambda} \right] d(Fx_0, Fx_1),$$

so $(F_{x_m})_{m \in \mathbb{N}}$ is a Cauchy sequence. Let us define $v := \lim_{m \to \infty} F_{x_m}$. Since $F(X)$ is a closed subset of X then there exists $u \in X$ such that $F(u) = v$. Let $j \in \{1, 2, ..., n\}$. We have for a $k \in \{0, 2, 4, ...\}$ and $i_0 \in \{2, 4, ..., 2n\}$

$$d(Fu, T_j u) \leq d(Fu, Fx_{kn+i_0}) + d(Fx_{kn+i_0}, T_j u)$$
$$= d(Fu, Fx_{kn+i_0}) + d(S_{\frac{i_0}{2}} x_{kn+i_0-1}, T_j u)$$
$$\leq d(Fu, Fx_{kn+i_0}) + A_{\frac{i_0}{2}} d(Fx_{kn+i_0-1}, S_{\frac{i_0}{2}} x_{kn+i_0-1})$$
$$+ B_j d(Fu, T_j u).$$

Then

$$d(Fu, T_j u) \leq \frac{1}{1 - B_j} d(Fu, Fx_{kn+i_0})$$

$$+\frac{A_{\frac{i_0}{2}}}{1-B_j}d(Fx_{kn+i_0-1}, Fx_{kn+i_0}),$$

so for sufficently large k - the distance $d(Fu, T_ju)$ can be arbitrarily small, then $T_ju = Fu$, for any $j \in \{1, 2, ..., n\}$. Moreover

$$d(Fu, S_ju) = d(S_ju, T_ju) \leq A_jd(Fu, S_ju) + B_jd(Fu, T_ju) = A_jd(Fu, S_ju),$$

whence $S_ju = Fu$, for any $j \in \{1, 2, ..., n\}$. We have proved that v is a unique point of coincidence of mappings $F, S_1, ..., S_n, T_1, ..., T_n$. If additionally all pairs (F, T_i), (F, S_i), for $i \in \{1, 2, ..., n\}$ are weakly compatibles, then by Lemma 2.1, $F, S_1, ..., S_n, T_1, ..., T_n$ have the unique common fixed point. □

3 Examples

Example 3.1. Let $X = [0, \infty[$ and d be the Euclidean metric on X. Let $n = 2$. We define the mappings $F, S_1, S_2, T_1, T_2 : X \to X$ as follows:

$$F(x) = \begin{cases} -\frac{7}{2}x + 21, & \text{for } x \in [0, 4], \\ 7, & \text{for } x \in [4, 7], \\ -\frac{1}{2}x + \frac{21}{2}, & \text{for } x \in [7, 21], \\ 0, & \text{for } x \in [21, \infty[. \end{cases}$$

and for $i = 1, 2$

$$S_i(x) = \begin{cases} (1 + \frac{1}{2i})x + i, & \text{for } x \in [0, 4], \\ 7, & \text{for } x \in [4, \infty[, \end{cases}$$

$$T_i(x) = \begin{cases} \frac{1}{2i}x + i + 4, & \text{for } x \in [0, 4], \\ 7, & \text{for } x \in [4, \infty[. \end{cases}$$

We have $F(X) = [0, 21]$, $S_i(X) = [i, 7]$, $T_i(X) = [i + 4, 7]$, for $i = 1, 2$. We put $A_1 = A_2 = \frac{1}{2}$ and $B_1 = B_2 = \frac{1}{3}$. One can easily observe that the condition (2.1) is satisfied. For $x_0 = 2$ - by Definition 1.5 - we get the sequence $(Fx_m)_{m \in \mathbb{N}}$ as follows: $-14, 6, 7, 7, 7, ...$ So $v = \lim_{m\to\infty} Fx_m = 7$ is a unique point of coincidence of mappings F, S_1, S_2, T_1, T_2. Since all pairs (F, T_i), (F, S_i), for $i \in \{1, 2\}$ are weakly compatibles, then - by Lemma 2.1 - $v = 7$ is the unique common fixed point for mappings F, S_1, S_2, T_1, T_2. Moreover, let us remark that if we change definition of the function F as follows

$$F(x) = \begin{cases} -\frac{7}{2}x + 21, & \text{for } x \in [0, 4], \\ 7, & \text{for } x \in [4, 7[, \\ 6, & \text{for } x = 7, \\ -\frac{1}{2}x + \frac{21}{2}, & \text{for } x \in]7, 21], \\ 0, & \text{for } x \in [21, \infty[, \end{cases}$$

then there is no common fixed point for mappings F, S_1, S_2, T_1, T_2. Since $7 = F(4) = S_1(4) = S_2(4) = T_1(4) = T_2(4)$, the point 7 is only the point of coincidence of a family F, S_1, S_2, T_1, T_2. In this case the condition (2.1) is still satisfied but all pairs (F, S_i), (F, T_i) for $i \in \{1, 2\}$ are not weakly compatibles.

Example 3.2. Let $X = [0, \infty[$ and d be the Euclidean metric on X. Let $n = 3$. We define the mappings $F, S_1, S_2, S_3, T_1, T_2, T_3 : X \to X$ as follows:

$$F(x) = x, \quad \text{for } x \in X.$$

and for $i = 1, 2, 3$

$$S_i(x) = \begin{cases} 3 - x^{i+1}, & \text{for } x \in [0,1], \\ 2, & \text{for } x > 1. \end{cases}$$

$$T_i(x) = \begin{cases} 3 - x^{i+4}, & \text{for } x \in [0,1], \\ 2, & \text{for } x > 1. \end{cases}$$

We have $F(X) = X$, $S_i(X) = T_i(X) = [2, 3]$, for $i = 1, 2, 3$. We put $A_1 = A_2 = A_3 = \frac{1}{2}$ and $B_1 = B_2 = B_3 = \frac{5}{12}$. One can easily observe that the condition (2.1) is satisfied. For $x_0 = \frac{1}{2}$ - by Definition 1.5 - we get the sequence $(Fx_m)_{m \in \mathbb{N}}$ as follows: $\frac{1}{2}, \frac{95}{32}, 2, 2, 2, \ldots$. So $v = \lim_{m \to \infty} Fx_m = 2$ is a unique point of coincidence of mappings $F, S_1, S_2, S_3, T_1, T_2, T_3$. Since all pairs (F, T_i), (F, S_i), for $i \in \{1, 2, 3\}$ are weakly compatibles, then - by Lemma 2.1 - $v = 2$ is the unique common fixed point for mappings $F, S_1, S_2, S_3, T_1, T_2, T_3$.

Example 3.3. Let $X = [\frac{1}{4}, 4]$ and d be the Euclidean metric on X. Let $n = 2$. We define the mappings $F, S_1, S_2, T_1, T_2 : X \to X$ as follows:

$$F(x) = \frac{1}{x}, \quad \text{for } x \in X$$

and for $i = 1, 2$

$$S_i(x) = x^{\frac{1}{i+3}}, \qquad T_i(x) = x^{\frac{1}{i+1}}.$$

All the pairs (F, T_i), (F, S_i), for $i \in \{1, 2\}$ are weakly compatibles. If we put $A_1 = A_2 = \frac{1}{4}$ and $B_1 = B_2 = \frac{5}{8}$ then the condition (2.1) is satisfied. By Definition 1.5 starting from $x_0 = 2$ we obtain the sequence $(Fx_m)_{m \in \mathbb{N}}$ as follows: $\frac{1}{2}, 2^{\frac{1}{2}}, 2^{-\frac{1}{8}}, 2^{\frac{1}{24}}, 2^{-\frac{1}{120}}, \ldots$, whence $v = \lim_{m \to \infty} Fx_m = 1$ is the unique common fixed point for mappings F, S_1, S_2, T_1, T_2.

References

[1] M. Arshad, A. Azam, P. Vetro, *Some common fixed point results in cone metric spaces*, Fixed Point Theory Appl., 2009 (2009), Article ID 493965,11 pages.

[2] A. Azam, I. Beg, *Kannan type mapping in its TVS-valued cone metric space and their applications to Uryshon integral equations*, Sarajevo Journal of Mathematics, Vol 9 (22) (2013), 243-255.

[3] S. Banach, *Sur les opération dans l'ensembles abstraits et leur application aux équations intégrales*, Fundam. Math., 3 (1922), 133-181.

[4] J. Dugundji and A. Granas, *Fixed Point Theory, Monografie Matematyczne*, Tom 61 vol. I, PWN- Polish Scientific Publishers (1982).

[5] J. Jachymski, *Common fixed point theorems for some families of mappings*, Indian J. Pure Appl. Math. 25 (1994), 925-937.

[6] K. Jha, R. P. Pant, S. L. Singh, *Common fixed points for comapatible mappings in metric spaces*, Radovi Matematički, vol. 12 (2003), 107-114.

[7] R. Kannan, *Some results on fixed points II*, Am. Math. Mon. 76(4) (1969), 405-408.

[8] R. P. Pant, P. C. Joshi, V. Gupta, *A Meir-Keeler type fixed point theorem*, Indian J. Pure Appl. Math. 32(6) (2001), 779-787.

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