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Families of Multivalent Analytic Functions Associated with the Convolution Structure

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Abstract.

The main aim of the present paper is to introduce a new class of multivalent analytic functions by using the familiar concept's of convolution structure. The results investigated in the present paper include the characterization properties for this class of analytic functions. Some new and interesting consequences of our results are also pointed out.

Keywords: Analytic functions; Convolution; Characterization properties; linear operators.

1 Introduction

Let A_p denote the class of functions that are analytic in the unit disk $U = \{z : z \in C, |z| < 1\}$ and consisting of the functions f of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, (p, \in N = \{1,2,3...\}), \tag{1.1}$$

where f is analytic and p-valent in U. If $f \in A_p$ is given by (1.1) and $g \in A_p$ is given by

$$g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n, (p \in N = \{1,2,3...\}) \tag{1.2}$$

then the Hadamard product (or convolution) $f * g$ of f and g, defined by

$$(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n = (g * f)(z). \tag{1.3}$$

In this article we study the class $S_{\gamma}^{p}(g;\alpha)$ introduced in the following:

Definition 1.1 *For a given function* $g(z) \in A_p$ *defined by (1.2), where* $b_n \geq 0$, $(n \geq p+1)$, $p = 1,2,\ldots$. *We say that* $f(z) \in A_p$ *is in* $S_\gamma^p(g;\alpha)$, *provided that* $(f*g)(z) \neq 0$, *and*

$$Re\left\{p + \frac{1}{\gamma}\left(\frac{z(f*g)'(z)}{(f*g)(z)} - p\right)\right\} > \alpha \; (z \in U; \gamma \in C/\{0\}; 0 \leq \alpha < p) \tag{1.4}$$

Note that $S_1^1\left(\frac{z}{1-z};\alpha\right) = S^*(\alpha)$ and $S_1^1\left(\frac{z}{(1-z)^2};\alpha\right) = K(\alpha)$, are respectively, the familiar classes of starlike and convex functions of order α in U (see for example, [16]).

Also, $S_\gamma^p\left(\frac{z^p}{1-z};0\right) = S_\gamma^{p*}$ and $S_\gamma^p\left(\frac{z^p}{(1-z)^2};0\right) = K_\gamma^p$.

For $p = 1$, the classes $S_\gamma^{1*} = S_\gamma^*$ and $K_\gamma^1 = K_\gamma$, where the classes S_γ^* and K_γ stand essentially for the classes of starlike and convex functions of complex order, which were considered earlier by Nasr and Aouf [11] and Wiatrowski [17], respectively (see also [9] and [10]).

Remark: When

$$g(z) = z^p + \sum_{n=p+1}^{\infty} \frac{\Pi_{i=1}^s \Gamma(\beta_i)\Pi_{i=1}^q \Gamma(\alpha_i + A_i(n-p))}{\Pi_{i=1}^q \Gamma(\alpha_i)\Pi_{i=1}^s \Gamma(\beta_i + B_i(n-p))} \frac{z^n}{(n-p)!},$$

where $\alpha_i \in C (i = 1,\ldots,q), \beta_i \in C (i = 1,\ldots,s)$ and the coefficients $A_i \in R_+ (i = 1,\ldots,q)$ and $B_i \in R_+ (i = 1,\ldots,s)$ being so chosen that the coefficients b_n in (1.2) satisfying the following condition:

$$b_n = \frac{\Pi_{i=1}^s \Gamma(\beta_i)\Pi_{i=1}^q \Gamma(\alpha_i + A_i(n-p))}{\Pi_{i=1}^q \Gamma(\alpha_i)\Pi_{i=1}^s \Gamma(\beta_i + B_i(n-p))} \frac{1}{(n-p)!} \geq 0, \tag{1.5}$$

then the class $S_\gamma^p(g;\alpha)$ is transformed into a (presumably) new class $S_\gamma^p(q,s,\alpha)$ defined by

$$S_\gamma^p(q,s,\alpha) = \left\{ f : f \in A_p \; and \; Re\left\{p + \frac{1}{\gamma}\left(\frac{z(L_{q,s}^p[\alpha_i]f)'(z)}{(L_{q,s}^p[\alpha_i]f)(z)} - p\right)\right\} > \alpha \right\} \tag{1.6}$$

$z \in U; \gamma \in C\{0\}$ and $1 + \sum_{i=1}^s B_i - \sum_{i=1}^q A_i \geq 0, (q,s \in N_0 = N \cup \{0\})$.

The operator

$$L_{q,s}^p(\alpha_i)f(z) = L_{q,s}^p(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;B_1,\ldots,B_s)f(z), \quad (i = 1,\ldots,q)$$

involved in (1.6) is defined by the Chaurasia and Parihar (see for details [3]).

Special cases of the operator $L_{q,s}^p(\alpha_i)f(z)$ includes Dziok-Srivastava linear operator (cf. [4, 5, 15]), Hohlov linear operator [15], the Carlson-Shaffer linear operator [2], the Ruscheweyh derivative operator [14], the Barnardi-Libra -Livingston linear integral operator (cf. [8, 7, 1]), and the Srivastava - Owa fractional derivative operators(cf. [12, 13])

2 Characterization Properties

In this section, we establish two results, Theorem 2.1 and Theorem 2.3, which gives the sufficient conditions for a function $f(z)$ defined by (1.1) and belongs to the class $f(z)\in S_{\gamma}^{p}(g;\alpha)$.

Theorem 2.1 *Let* $f(z)\in A_p$ *such that*

$$\left|\frac{z(f*g)'(z)}{(f*g)(z)}-p\right|<p-\beta,\ (\beta<p;z\in U) \tag{2.1}$$

then $f(z)\in S_{\gamma}^{p}(g;\alpha)$ provided that

$$|\gamma|\geq\frac{p-\beta}{p-\alpha},\ (0\leq\alpha<p). \tag{2.2}$$

Proof: In view of (2.1), we write

$$\frac{z(f*g)'(z)}{(f*g)(z)}=p+(p-\beta)w(z)$$

where $|w(z)|<1$ for $z\in U$.

Now

$$\begin{aligned}Re\left\{p+\frac{1}{\gamma}\left(\frac{z(f*g)'(z)}{(f*g)(z)}-p\right)\right\} &= Re\left\{p+\frac{1}{\gamma}(p-\beta)w(z)\right\}\\ &= p+(p-\beta)Re\left\{\frac{w(z)}{\gamma}\right\}\\ &\geq p-(p-\beta)\left|\frac{w(z)}{\gamma}\right|>p-(p-\beta)\frac{1}{|\gamma|}\geq\alpha\end{aligned}$$

provided that $|\gamma|\geq\dfrac{p-\beta}{p-\alpha}$. This completes the proof.

If we set $\beta=p-(p-\alpha)|\gamma|(\gamma\in C\{0\};0\leq\alpha<p)$, in Theorem 2.1, we obtain

Corollary 2.2 *If* $f(z)\in A_p$ *such that*

$$\left|\frac{z(f*g)'(z)}{(f*g)(z)}-p\right|<(p-\alpha)|\gamma|,\quad (z\in U,\gamma\in C\{0\};0\leq\alpha<p) \tag{2.3}$$

then $f(z)\in S_{\gamma}^{p}(g;\alpha)$.

Theorem 2.3. *Let* $f(z)\in A_p$ *satisfying the following inequality*

$$\sum_{n=p+1}^{\infty}b_n[(n-p)+(p-\alpha)|\gamma|]|a_n|\leq(p-\alpha)|\gamma| \tag{2.4}$$

$$(z\in U,b_n\geq 0(n\geq p+1,p\in\{1,2,3,..\});\gamma\in C\{0\};0\leq\alpha<p)$$

then $f(z)\in S_{\gamma}^{p}(g;\alpha)$.

Proof: Suppose the inequality (2.4) holds true. Then in view of Corollary 2.2, we have

$$\begin{aligned}|z(f*g)'(z) &- p(f*g)(z)|-(p-\alpha)|\gamma||(f*g)(z)| \\ &= \left|\sum_{n=p+1}^{\infty} b_n(n-p)a_n z^n\right| - (p-\alpha)|\gamma|\left|z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n\right| \\ &\le \left\{\sum_{n=p+1}^{\infty} b_n(n-p)|a_n| - (p-\alpha)|\gamma| + (p-\alpha)|\gamma|\sum_{n=p+1}^{\infty} b_n|a_n|\right\}|z^p| \\ &\le \left\{\sum_{n=p+1}^{\infty} b_n[(n-p)+(p-\alpha)|\gamma|]|a_n| - (p-\alpha)|\gamma|\right\} \le 0\end{aligned}$$

This completes the proof.

Corollary 2.4 *If* $f(z)\in A_p$ *satisfying the following inequality*

$$\sum_{n=p+1}^{\infty}[(n-p)+|\gamma|p]|a_n| \le p|\gamma|, \quad z\in U; \gamma\in C/\{0\} \tag{2.5}$$

then $f(z)\in S_\gamma^p(g;\alpha)$.

Corollary 2.5 *If* $f(z)\in A_p$ *satisfying the following inequality*

$$\sum_{n=p+1}^{\infty} n[(n-p)+|\gamma|p]|a_n| \le p|\gamma|, \quad z\in U; \gamma\in C/\{0\} \tag{2.6}$$

Then $f(z)\in K^p$.

Corollary 2.6 *If* $f(z)\in A_p$ *satisfying the following inequality*

$$\sum_{n=p+1}^{\infty}[(n-p)+|\gamma|(p-\alpha)]\frac{\Pi_{i=1}^{s}\Gamma(\beta_i)\Pi_{i=1}^{q}\Gamma(\alpha_i+A_i(n-p))}{\Pi_{i=1}^{q}\Gamma(\alpha_i)\Pi_{i=1}^{s}\Gamma(\beta_i+B_i(n-p))}\frac{1}{(n-p)!}|a_n| \le (p-\alpha)|\gamma| \tag{2.7}$$

$(z\in U; \gamma\in C\{0\}; 0\le\alpha<p)$ and $1+\sum_{i=1}^{s}B_i - \sum_{i=1}^{q}A_i \ge 0, \quad (q,s\in N_0 = N\cup\{0\})$, then $f(z)\in S_\gamma^p(q,s,\alpha)$.

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