Journal of Progressive Research in Mathematics(JPRM)
ISSN: 2395-0218

SCITECH
RESEARCH ORGANISATION|

Volume 6, Issue 1
Published online: December 04, 2015|

Journal of Progressive Research in Mathematics
www.scitecresearch.com/journals

Fuzzy Compact Fuzzy Distance Space

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Abstract.

In this paper we recall the definition of fuzzy distance space on a fuzzy set then we define a compact fuzzy distance space and fuzzy totally bounded after that we prove that fuzzy totally bounded fuzzy complete fuzzy distance space is fuzzy compact. Moreover we recall the definition of fuzzy continuous and uniform fuzzy continuous function to prove that fuzzy continuous function and uniform fuzzy continuous functions are equivalent on a fuzzy compact fuzzy distance spaces .

Keywords: Fuzzy distance space; fuzzy compact; fuzzy distance space; fuzzy bounded; fuzzy totally bounded; fuzzy continuous function.

1. Introduction

Theory of fuzzy sets was introduced by Zadeh in 1965 [21]. Many authors have introduced the concept of fuzzy metric in different ways [1,2,3,7,8,9,10,13,16,17]. Kramosil and Michalek in 1975 [6] introduced the definition of fuzzy metric space which is called later KM-fuzzy metric space .George and Veeramani in 1994[3] introduced the definition of continuous $*$ t-norm to modify the concept of KM-fuzzy metric space which was introduced by Kramosil and Michalek which is called later GV-fuzzy metric space. In section two of this paper we recall the definition of fuzzy distance space on fuzzy set [9] which is a modification of the definition GV-fuzzy metric space after that we introduce basic definitions ,basic concepts and properties of fuzzy distance space .

In section three the notion of fuzzy compact fuzzy distance space is introduced, we try to prove results similar to that in the ordinary case. The aim of studying a fuzzy continuous function on fuzzy compact spaces in section four is to prove that fuzzy continuous function and uniform fuzzy continuous functions are equivalent on fuzzy compact fuzzy distance space.

2. FUZZY DISTANCE SPACE ON FUZZY SET

Definition 2.1:[21]

Let X be a nonempty set of elements, a fuzzy set $\tilde{A}$ in X is characterized by a member ship function, $\mu_{\tilde{A}}(x)$: X$\to$ [0,1]. Then we can write $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)): x\in X, 0 \leq \mu_{\tilde{A}}(x) \leq 1\}$.

We now recall an example of a continuous fuzzy set.

Example 2.2:[18]

Let X = $\mathbb{R}$ and let $\tilde{A}$ be a fuzzy set in $\mathbb{R}$ with membership function by: $\mu_{\tilde{A}}(x) = \frac{1}{1+10x^2}$.

Definition 2.3:[4]

Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy sets in X. then

1- $\tilde{A} \subseteq \tilde{B}$ if and only if $\mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x)$ for all $x \in X$

2- $\tilde{A} = \tilde{B}$ if and only if $\mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x)$ for all $x \in X$

3- $\tilde{C} = \tilde{A} \cup \tilde{B}$ if and only if $\mu_{\tilde{C}}(x) = \mu_{\tilde{A}}(x) \vee \mu_{\tilde{B}}(x)$ for all $x \in X$

4- $\tilde{D} = \tilde{A} \cap \tilde{B}$ if and only if $\mu_{\tilde{D}}(x) = \mu_{\tilde{A}}(x) \wedge \mu_{\tilde{B}}(x)$ for all $x \in X$

5- $\mu_{\tilde{A}^c}(x) = 1 - \mu_{\tilde{A}}(x)$ for all $x \in X$

Definition 2.4:[18]

If $\tilde{A}$ and $\tilde{B}$ are fuzzy sets in a nonempty sets X and Y respectively then the Cartesian product

$\tilde{A} \times \tilde{B}$ of $\tilde{A}$ and $\tilde{B}$ is defined by: $\mu_{\tilde{A} \times \tilde{B}}(x, y) = \mu_{\tilde{A}}(x) \wedge \mu_{\tilde{B}}(y)$ for all $(x, y) \in X \times Y$

Definition 2.5:[20]

A fuzzy point p in X is a fuzzy set with member $p(x) = \alpha$ if $x = y$ and $p(x) = 0$ otherwise.

For all y in X where $0 < \alpha < 1$. We denote this fuzzy point by x_α. Two fuzzy points x_α and

y_β are said to be distinct if and only if $x \neq y$.

Definition 2.6:[21]

Let x_α be a fuzzy point and $\tilde{A}$ be a fuzzy set in X. then x_α is said to be in $\tilde{A}$ or belongs to

$\tilde{A}$ which is denoted by $x_\alpha \in \tilde{A}$ if and only if $\mu_{\tilde{A}}(x) > \alpha$.

Definition 2.7:[11]

Let f be a function from a nonempty set X into a nonempty set Y. If $\tilde{B}$ is a fuzzy set in Y

then $f^{-1}(\tilde{B})$ is a fuzzy set in X defined by:

$\mu_{f^{-1}(\tilde{B})}(x) = (\mu_{\tilde{B}} \circ f)(x)$ for all x in X. Also if $\tilde{A}$ is a fuzzy set in X then $f(\tilde{A})$ is a fuzzy set in Y defined by:

$\mu_{f(\tilde{A})}(y) = \vee \{ \mu_{\tilde{A}}(x) : x \in f^{-1}(y) \}$, if $f^{-1}(y) \neq \emptyset$ and $\mu_{f(\tilde{A})}(y) = 0$, otherwise.

Proposition 2.8:[12]

Let $f: X \rightarrow Y$ be a function. Then for a fuzzy point x_α in X, $f(x_\alpha)$ is a fuzzy point

in Y and $f(x_\alpha) = (f(x))_\alpha$.

Definition 2.9:[3]

A binary operation $*: [0,1] \times [0,1] \rightarrow [0,1]$ is a continuous t-norm if $*$ satisfies the following conditions:

1- $*$ is associative and commutative.

2- $*$ is continuous.

3- $a*1 = a$ for all $a \in [0,1]$.

4- $a*b \leq c*d$ whenever $a \leq c$ and $b \leq d$ where $a, b, c, d \in [0,1]$.

Remark 2.10:[3]

For any $a > b$ we can find c such that $a*c \geq b$ and for any d we can find an e such that $e*e \geq d$ where a, b, c ,d ,e $\in (0,1)$.

We introduce the following definition.

Definition 2.11:[14]

A triple $(\widetilde{A},\widetilde{D},*)$ is said to be fuzzy distance space if $\widetilde{A}$ is a fuzzy set of the nonempty set X, $*$ is a continuous t- norm and $\widetilde{D}$ is a fuzzy set on $\widetilde{A}^2$ satisfying the following conditions:

(FD_1) $\widetilde{D}(x_\alpha,y_\beta) > 0$ for all $x_\alpha, y_\beta \in \widetilde{A}$

(FD_2) $\widetilde{D}(x_\alpha,y_\beta) = 1$ if and only if $x_\alpha = y_\beta$

(FD_3) $\widetilde{D}(x_\alpha,y_\beta) = \widetilde{D}(y_\beta, x_\alpha)$ for all $x_\alpha, y_\beta \in \widetilde{A}$

(FD_4) $\widetilde{D}(x_\alpha,z_\sigma) \geq \widetilde{D}(x_\alpha,y_\beta) * \widetilde{D}(y_\beta, z_\sigma)$ for all x_α, y_β and $z_\sigma \in \widetilde{A}$

(FD_5) $\widetilde{D}(x_\alpha,y_\beta)$ is a continuous fuzzy set

Example 2.12:[14]

Let X= $\mathbb{N}$, and let $\widetilde{A}$ be a fuzzy set in X. Suppose that $a*b = a.b$ for all a, b $\in[0,1]$.

Define $\widetilde{D}(x_\alpha,y_\beta) = \frac{x}{y}$ If $x \leq y$ and $\widetilde{D}(x_\alpha,y_\beta) = \frac{y}{x}$ If $y \leq x$, for all x, y $\in \mathbb{N}$.

Then $(\widetilde{A},\widetilde{D},*)$ is a fuzzy distance space.

Example 2.13:[14]

Let X=$\mathbb{R}$ and let $\widetilde{A}$ be a fuzzy set in X. Suppose that $a*b = a.b$ for all a, b $\in [0,1]$.

Define $\widetilde{D}(x_\alpha,y_\beta) = \frac{1}{e^{|x_\alpha - y_\beta|}}$ for all $x_\alpha, y_\beta \in \widetilde{A}$.

Then $(\widetilde{A},\widetilde{D},*)$ is a fuzzy distance space.

Definition 2.14:[14]

Let $(\tilde{A},\widetilde{D},*)$ be a fuzzy distance space then $\widetilde{D}$ is continuous fuzzy set if whenever $(x_n,\alpha_n) \to x_\alpha$ and $(y_n,\beta_n) \to y_\beta$ in $\tilde{A}$ then $\widetilde{D}((x_n, \alpha_n),(y_n, \beta_n)) \to \widetilde{D}(x_\alpha, y_\beta)$ that is $\lim_{n\to\infty}\widetilde{D}((x_n, \alpha_n),(y_n, \beta_n))$ $= \widetilde{D}(x_\alpha, y_\beta)$.

Lemma 2.15:[14]

Suppose that (X,d) is an ordinary metric space and assume that $\tilde{A}$ is a fuzzy set in X. Define $d(x_\alpha, y_\beta) = d(x, y)$ for all $x_\alpha, y_\beta \in \tilde{A}$. Then $(\tilde{A}, d)$ is a metric space.

Example 2.16:[14]

Let X= $\mathbb{R}$ and let $\tilde{A}=[2,\infty]$ be a fuzzy set in X. consider the mapping $\widetilde{D} : \tilde{A} \times \tilde{A} \to [0, 1]$ defined by :

$\widetilde{D}(a_\alpha,b_\beta) = 1$ if a = b and $\widetilde{D}(a_\alpha,b_\beta) = \left(\frac{1}{a}\right).\alpha + \left(\frac{1}{b}\right).\beta$ if $a \neq b$, where

$\alpha * \beta = \alpha.\beta$ for all $\alpha, \beta \in[0,1]$

(FM_4) We show that $\widetilde{D}(a_\alpha,c_\sigma) \geq \widetilde{D}(a_\alpha,b_\beta) * \widetilde{D}(b_\beta, c_\sigma)$ is not satisfied for all $a_\alpha, b_\beta, c_\sigma \in \widetilde{A}$.

Let a=10, b = 3 and c= 100 where $\alpha=\frac{1}{a}$, $\beta = \frac{1}{b}$,

$\sigma = \frac{1}{c}$ Since $a \neq b \neq c$

Then $\widetilde{D}(a_\alpha, b_\beta) = (\frac{1}{a}).\alpha + \left(\frac{1}{b}\right).\beta = \frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{100} + \frac{1}{9}$ =0.01+0.111= 0.121

And $\widetilde{D}(b_\beta, c_\sigma) = \left(\frac{1}{b}\right).\beta + \left(\frac{1}{c}\right).\sigma = \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{9} + \frac{1}{10000}$ =0.111+0.0001= 0.1112

$\widetilde{D}(a_\alpha, c_\sigma) = (\frac{1}{a}).\alpha + \left(\frac{1}{c}\right).\sigma = \frac{1}{a^2} + \frac{1}{c^2} = \frac{1}{100} + \frac{1}{10000}$ =0.01+0.0001= 0.0101

Therefore $\widetilde{D}(a_\alpha, b_\beta) * \widetilde{D}(b_\beta, c_\sigma) > \widetilde{D}(a_\alpha, c_\sigma)$=(0.121)+(0.1112) =0.0134552 >0.0101

Thus $(\tilde{A}, \widetilde{D}, *)$ is not a fuzzy distance space ■

Proposition 2.17:[14]

Suppose that (X,d) is an ordinary metric space and assume that $a*b = a.b$ for all $a, b \in [0,1]$.

Then by lemma 2.15, $(\tilde{A}, d)$ is a metric space. Define $\widetilde{D}_d(x_\alpha, y_\beta) = \frac{t}{t + d(x_\alpha, y_\beta)}$, then $(\tilde{A}, \widetilde{D}_d, *)$ is a fuzzy distance space and it is called the fuzzy distance on the fuzzyset $\tilde{A}$ induced by d.

Definition 2.18:[14]

Let $(\tilde{A}, \widetilde{D}, *)$ be a fuzzy distance space on the fuzzy set $\tilde{A}$, we define $\widetilde{B}(x_\alpha, r) = \{y_\beta \in \tilde{A}: \widetilde{D}(x_\alpha, y_\beta) > (1-r)\}$ then $\widetilde{B}(x_\alpha, r)$ is called an fuzzy open fuzzy ball with center the fuzzy point $x_\alpha \in \tilde{A}$ and radius $0 < r < 1$.

Proposition 2.19:[14]

Suppose that $\widetilde{B}(x_\alpha, r_1)$ and $\widetilde{B}(x_\alpha, r_2)$ be two fuzzy open fuzzy balls with the same center $x_\alpha \in \tilde{A}$ and with radiuses $r_1, r_2 \in (0,1)$. Then we either have $\widetilde{B}(x_\alpha, r_1) \subseteq \widetilde{B}(x_\alpha, r_2)$ or $\widetilde{B}(x_\alpha, r_2) \subseteq \widetilde{B}(x_\alpha, r_1)$.

Definition 2.20:[14]

A sequence $\{(x_m, \alpha_m)\}$ of fuzzy points in a fuzzy distance space $(\tilde{A}, \widetilde{D}, *)$ is called fuzzy converges to a fuzzy point $x_\alpha \in \tilde{A}$ if whenever $0 < \varepsilon < 1$, we can find a positive integer K with, $\widetilde{D}((x_m, \alpha_m), x_\alpha) > (1-\varepsilon)$ whenever $m \geq K$.

Definition 2.21:[14]

A sequence $\{(x_n, \alpha_n)\}$ of fuzzy points in a fuzzy distance space $(\tilde{A}, \widetilde{D}, *)$ is called fuzzy converges to a fuzzy point $x_\alpha \in \tilde{A}$ if $\lim_{n \to \infty} \widetilde{D}((x_n, \alpha_n), x_\alpha) = 1$.

Theorem 2.22:[14]

Definition 2.21 and definition 2.20 are equivalent.

Proposition 2.23:[14]

Suppose that (X,d) is a metric space and assume that $(\tilde{A}, \widetilde{D}_d, *)$ is the fuzzy distance space induced by d. Let $\{(x_n, \alpha_n)\}$ be a sequence of fuzzy points in $\tilde{A}$. Then $\{(x_n, \alpha_n)\}$ converges to $x_\alpha \in \tilde{A}$ in $(\tilde{A}, d)$ if and only if $\{(x_n, \alpha_n)\}$fuzzy converges to x_α in $(\tilde{A}, \widetilde{D}_d, *)$.

Definition 2.24:[14]

A fuzzy subset $\tilde{C}$ of a fuzzy distance space $(\tilde{A}, \tilde{D},*)$ is called fuzzy open if for each $x_\alpha \in \tilde{C}$ there is $\tilde{B}(x_\alpha,q) \subset \tilde{C}$ with $0 < q < 1$. A fuzzy set $\tilde{E} \subseteq \tilde{A}$ is said to be fuzzy closed if its complement is fuzzy open that is $\tilde{E}^c = \tilde{A} \setminus \tilde{E}$ is fuzzy open.

Theorem 2.25:[14]

If $\tilde{B}(x_\alpha,q)$ is fuzzy open fuzzy ball in a fuzzy distance space $(\tilde{A}, \tilde{D},*)$ on a fuzzy set $\tilde{A}$ then $\tilde{B}(x_\alpha,q)$ is a fuzzy open fuzzy set with $0 < q < 1$.

Definition 2.26:[14]

Suppose that $(\tilde{A}, \tilde{D},*)$ is a fuzzy distance space on a fuzzy set $\tilde{A}$ and let $\tilde{C}\subset\tilde{A}$ then the fuzzy closure of $\tilde{C}$ is denoted by $\bar{\tilde{C}}$ or FCL($\tilde{C}$) and is defined to be the smallest fuzzy closed fuzzy set contains $\tilde{C}$.

Definition 2.27:[14]

A fuzzy subset $\tilde{C}$ of a fuzzy distance space $(\tilde{A}, \tilde{D},*)$ on a fuzzy set $\tilde{A}$ is said to be fuzzy dense in $\tilde{A}$ if $\bar{\tilde{C}} = \tilde{A}$.

Lemma 2.28:[14]

Let $\tilde{C}$ be a fuzzy subset of $\tilde{A}$ and let $(\tilde{A}, \tilde{D},*)$ be a fuzzy distance space on the fuzzy set $\tilde{A}$ then $a_\alpha \in \bar{\tilde{C}}$ if and only if there is a sequence $\{(a_n,\alpha_n)\}$ in $\tilde{C}$ such that $(a_n,\alpha_n) \rightarrow a_\alpha$, where $\alpha,\alpha_n \in [0,1]$.

Theorem 2.29:[14]

Suppose that $\tilde{C}$ is a fuzzy subset of a fuzzy distance space $(\tilde{A}, \tilde{D},*)$ then $\tilde{C}$ is fuzzy dense in $\tilde{A}$ if and only if for every $x_\alpha \in \tilde{A}$ there is $a_\beta \in \tilde{C}$ such that $\tilde{D}(x_\alpha, a_\beta) > (1-\varepsilon)$ for some $0 < \varepsilon < 1$.

Definition 2.30:[14]

A sequence $\{(x_n, \alpha_n)\}$ of fuzzy points in a fuzzy distance space $(\tilde{A}, \tilde{D},*)$ is said to be fuzzy Cauchy if whenever $0 < \varepsilon < 1$ we can find K with $\tilde{D}((x_n,\alpha_n), (x_m, \alpha_m)) > (1-\varepsilon)$ for all n, m $\geq$K.

Theorem 2.31:[14]

Let $(\tilde{A}, \tilde{D},*)$ be a fuzzy distance space on the fuzzy set $\tilde{A}$ if $\{(x_n, \alpha_n)\}$ is a sequence of fuzzy points in $\tilde{A}$ that is fuzzy converges to $x_\alpha \in \tilde{A}$ then $\{(x_n, \alpha_n)\}$ is fuzzy Cauchy.

Proposition 2.32:[14]

Suppose that (X,d) is a metric space and let $\tilde{D}_d(x_\alpha,y_\beta) = \frac{t}{t+d(x_\alpha,y_\beta)}$ where t= min{α,β}.

Then $\{(x_n, \alpha_n)\}$ is a Cauchy sequence in $(\tilde{A},d)$ if and only if $\{(x_n, \alpha_n)\}$ is a fuzzy

Cauchy sequence in $(\tilde{A}, \tilde{D}_d, *)$.

Definition 2.33:[14]

Suppose that $(\tilde{A},\tilde{D},*)$ be a fuzzy distance space. A fuzzy subset $\tilde{C}$ of $\tilde{A}$ is called fuzzy bounded if we can find $0 < q < 1$ with, $\tilde{D}(x_\alpha,y_\beta) > (1-q)$, whenever $x_\alpha, y_\beta \in \tilde{C}$.

Proposition 2.34:[14]

Let (X,d) be a metric space and let $\tilde{D}_d(x_\alpha, y_\beta) = \frac{t}{t+d(x_\alpha,y_\beta)}$ where $t = \alpha \wedge \beta$ then a fuzzy subset $\tilde{C}$ of $\tilde{A}$ is fuzzy bounded if and only if it is bounded.

Definition 2.35:[14]

Let $(\tilde{A},\tilde{D},*)$ be a fuzzy distance space, then we define a fuzzy closed fuzzy ball with center $x_\alpha \in \tilde{A}$ and radius r, $0 < r < 1$ by $\tilde{B}[x_\alpha,r] = \{y_\beta \in X: \tilde{D}(x_\alpha,y_\beta) \geq (1- r)\}$.

Lemma 2.36:[14]

If $\tilde{B}[x_\alpha,q]$ is fuzzy closed fuzzy ball in a fuzzy distance space $(\tilde{A}, \tilde{D},*)$ on a fuzzy set $\tilde{A}$ then $\tilde{B}[x_\alpha,q]$ is a fuzzy closed fuzzy set with $0 < q < 1$.

Theorem 2.37:[14]

Suppose that $(\tilde{A},\tilde{D},*)$ is a fuzzy distance space . Put $\tau_{\tilde{D}} = \{\tilde{C}\subset\tilde{A} : x_\alpha \in \tilde{C}$ if and only if there is $0 < q < 1$ with $\tilde{B}(x_\alpha, q)\subset\tilde{C}\}$.Then $\tau_{\tilde{M}}$ is a fuzzy topology on $\tilde{A}$.

Proposition 2.38:[14]

Suppose that (X,d) is an ordinary metric space. Let $\tilde{D}_d(x_\alpha,y_\beta)= \frac{t}{t+d(x_\alpha,y_\beta)}$ be the fuzzy distance induced by d. Then the topology τ_d induced by d and the fuzzy topology $\tau_{\tilde{D}_d}$ induced by $\tilde{D}_d$ are the same. That is $\tau_d = \tau_{\tilde{D}_d}$.

Theorem 2.39:[14]

Every fuzzy distance space on a fuzzy set is a fuzzy Hausdorff space.

Definition 2.40:[14]

Suppose that $(\tilde{A}, \tilde{D}_{\tilde{A}},*)$ and $(\tilde{E}, \tilde{D}_{\tilde{E}},*)$ are fuzzy distance spaces and $\tilde{C} \subseteq \tilde{A}$.The mapping h: $\tilde{C} \to \tilde{E}$ is said to be fuzzy continuous at $a_\beta \in \tilde{C}$, if whenever $0 < \varepsilon < 1$, we can find $0 < \delta < 1$, with $\tilde{D}_{\tilde{E}}(h(x_\alpha),h(a_\beta)) > (1- \varepsilon)$ whenever $x_\alpha \in \tilde{C}$ and $\tilde{D}_{\tilde{A}}(x_\alpha, a_\beta) > (1- \delta)$. When f is fuzzy continuous at every fuzzy point of $\tilde{C}$, then it is called to be fuzzy continuous on $\tilde{C}$.

Theorem 2.41:[14]

Let $(\tilde{A}, \tilde{D}_{\tilde{A}},*)$ and $(\tilde{E}, \tilde{D}_{\tilde{E}},*)$ be fuzzy distance spaces and $\tilde{C} \subseteq \tilde{A}$. The mapping h: $\tilde{C} \to \tilde{E}$ is fuzzy continuous at $a_\beta \in \tilde{C}$ if and only if whenever a sequence of fuzzy points $\{(x_n, \alpha_n)\}$ in $\tilde{C}$ fuzzy converge to a_β, then sequence of fuzzy points $\{(h(x_n, \alpha_n))\}$ fuzzy converges to

$h(a_\beta)$.

Proposition 2.42:[11]

Let $\tilde{A}$ be a fuzzy set in X and let $\tilde{B}$ be a fuzzy set in Y. let f: $\tilde{A} \to \tilde{B}$ be a function and let $\tilde{C} \subseteq \tilde{A}$ and $\tilde{E} \subseteq \tilde{B}$ .Then $f(\tilde{C}) \subseteq \tilde{E}$ if and only if $\tilde{C} \subseteq f^{-1}(\tilde{E})$.

Theorem 2.43:[14]

A mapping f: $\tilde{A} \to \tilde{E}$ is fuzzy continuous on $\tilde{A}$ if and only if the inverse image of $\tilde{C}$ is fuzzy open in $\tilde{A}$ for all fuzzy open fuzzy subset $\tilde{C}$ of $\tilde{E}$. Where $\tilde{A}$ and $\tilde{E}$ are fuzzy distance spaces.

Theorem 2.44:[14]

A mapping f: $\tilde{A} \to \tilde{E}$ is fuzzy continuous on $\tilde{A}$ if and only if the inverse image of $\tilde{C}$ is fuzzy closed in $\tilde{A}$ for all fuzzy closed fuzzy subset $\tilde{C}$ of $\tilde{E}$.

3. FUZZY COMPACT FUZZY DISTANCE SPACE

Definition 3.1:

Suppose that $(\tilde{A},\tilde{D},*)$ is a fuzzy distance space and $\tilde{F} \subseteq \tilde{A}$. Let $\tilde{\hat{C}} = \{ \tilde{E} : \tilde{E}$ is a fuzzy open fuzzy sets in $\tilde{A}$ $\}$ such that $\tilde{F} \subseteq \bigcup_{\tilde{E} \in \hat{\tilde{C}}} \tilde{E}$. That is for each $t_\alpha \in \tilde{F}$ there is $\tilde{E} \in \hat{\tilde{C}}$ such that $t_\alpha \in \tilde{E}$. Then $\hat{\tilde{C}}$ is said to be a fuzzy open fuzzy cover of $\tilde{F}$.

Definition 3.2:

A fuzzy distance space $(\tilde{A}, \tilde{D},*)$ is called fuzzy compact if for all fuzzy open fuzzy covering $\hat{\tilde{C}}$ of $\tilde{A}$ we can find $\{\tilde{E}_1,\tilde{E}_2,\tilde{E}_3,\ldots,\tilde{E}_n\} \subseteq \tilde{\hat{C}}$ such that $\tilde{A} = \bigcup_{i=1}^{n}\tilde{E}_i$.

Definition 3.3:

Suppose that $(\tilde{A},\tilde{D},*)$ is a fuzzy distance space and $\tilde{F} \subseteq \tilde{A}$ with $\tilde{F} \neq \emptyset$ is called a fuzzy compact if $\tilde{F}$ is fuzzy compact with the fuzzy distance induced on it by $\tilde{D}$.

Example 3.4:

Let X= $\mathbb{R}$ and let $\tilde{A}$ be a fuzzy set in X. let $\tilde{B} =\{((0,1), \mu_{\tilde{B}})\}$ be a fuzzy subset of $\tilde{A}$. then the fuzzy set $\tilde{B}$ in the fuzzy distance space $(\tilde{A},\tilde{D}_d,*)$ where $\tilde{D}_d(x_\alpha,y_\beta) = \frac{t}{t+|x_\alpha - y_\beta|}$, a$*$b = a.b for all a, b$\in$[0,1], is not fuzzy compact. The fuzzy open fuzzy covering $\{(\frac{1}{n},\frac{1}{n})$: n = 2, 3...$\}$ of $\tilde{B}$ does not contains a finite set that we can cover $\tilde{B}$.

Remark 3.5:

Suppose that $(\tilde{A}, \tilde{D},*)$ is fuzzy distance space and assume that $\tilde{E} \subseteq \tilde{A}$. If $\tilde{E}$ is finite then $\tilde{E}$ is fuzzy compact.

Definition 3.6:

Suppose that $(\tilde{A}, \tilde{D},*)$ is fuzzy distance space and $\tilde{F} = \{ \tilde{E}_j : j \in I \}$ where $\tilde{E}_j$ are fuzzy sets in

$\widetilde{A}$. Then $\widetilde{F}$ is said to has the finite intersection property if for every $\widetilde{E}_1, \widetilde{E}_2, \ldots \widetilde{E}_m$ [finite], we have $\cap_{j=1}^{m} \widetilde{E}_j \neq \emptyset$.

Definition 3.7:

Suppose that $(\widetilde{A}, \widetilde{D}, *)$ is a fuzzy distance space and $\tilde{S} \subseteq \widetilde{A}$. Then $\tilde{S}$ is said to be fuzzy totally bounded if for each $0< r <1$, we can find a finite fuzzy set of fuzzy points $\{(t_1,\beta_1), (t_2,\beta_2), (t_3,\beta_3),\ldots, (t_n ,\beta_n)\} \subset \tilde{S}$ with the property for any z_α in $\tilde{S}$, $\widetilde{D}(z_\alpha,(t_i,\beta_i))>(1- r)$ for some $(t_i,\beta_i) \in \{(t_1,\beta_1), (t_2,\beta_2), (t_3,\beta_3),\ldots, (t_n ,\beta_n)\}$.This fuzzy set of fuzzy points $\{(t_1 ,\beta_1), (t_2 ,\beta_2), (t_3 ,\beta_3)\ldots(t_n ,\beta_n)\}$ is called fuzzy r-net.

Proposition 3.8:

Suppose that $(\widetilde{A}, \widetilde{D}, *)$ is a fuzzy totally bounded fuzzy distance space. Then $\widetilde{A}$ is fuzzy bounded.

Proof:

For $0< q <1$ we can find a finite fuzzy q- net for $\widetilde{A}$, say $\tilde{C}$. Since $\tilde{C}$ is a finite fuzzy set of fuzzy points $0 < \widetilde{D}(\tilde{C}) < 1$, where $\widetilde{D}(\tilde{C}) = \sup\{\widetilde{D}(y_\beta,z_\sigma): y_\beta, z_\sigma \in \tilde{C}\}$. Now let (x_1, α_1) and (x_2, α_2) be any two fuzzy points of $\widetilde{A}$. There exists fuzzy points y_β and z_σ in $\tilde{C}$ such that $\widetilde{D}((x_1,\alpha_1), y_\beta) > (1-q)$ and $\widetilde{D}((x_2, \alpha_2),z_\sigma) > (1-q)$. Now for $\widetilde{D}(\tilde{C})$ and q there is $(1- r)$, where $0 < r <1$ such that $\widetilde{D}(\tilde{C}) * (1-q) * (1-q) \geq (1-r)$. It follows that $\widetilde{D}((x_1, \alpha_1), (x_2, \alpha_2)) \geq \widetilde{D}((x_1, \alpha_1),y_\beta)* \widetilde{D}(y_\beta,z_\sigma)* \widetilde{D}(z_\sigma,(x_2, \alpha_2)) \geq (1-q) * \widetilde{D}(\tilde{C})*(1-q) \geq (1-r)$ So, $\widetilde{D}(\widetilde{A})= \sup\{\widetilde{D}[(x_1, \alpha_1),(x_2, \alpha_2)]:(x_1, \alpha_1),(x_2, \alpha_2) \in \widetilde{A}\} \geq (1-r)$. Hence, $\widetilde{A}$ is fuzzy bounded ■

Theorem 3.9:

Suppose that $(\widetilde{A}, \widetilde{D}, *)$ is fuzzy metric space and assume that $\widetilde{E} \subseteq \widetilde{A}$. Then $\widetilde{E}$ is fuzzy totally bounded if and only if for all $\{(x_n, \beta_n)\}$ in $\widetilde{E}$ has $\{(x_{n_k}, \beta_{n_k})\}$ which is a fuzzy Cauchy.

Proof:

Let $\widetilde{E}$ be a fuzzy totally bounded. Suppose that $\{(x_n, \beta_n)\}$ be a sequence of fuzzy points in $\widetilde{E}$. Take fuzzy $\frac{1}{2}$–net $\{(t_1, \alpha_1), (t_2, \alpha_2),\ldots(t_n, \alpha_n)\}$ in $\widetilde{E}$. Choose $\widetilde{B}(t_i,\frac{1}{2})$ then $\widetilde{B}(t_i,\frac{1}{2})$ contains $\{(x_n^{(1)}, \beta_n^{(1)})\}$ of $\{(x_n, \beta_n)\}$. Nowtake fuzzy $\frac{1}{4}$–net $\{(t_1, \alpha_1), (t_2, \alpha_2),\ldots(t_n, \alpha_n)\}$ in $\widetilde{E}$. Choose $\widetilde{B}(t_i,\frac{1}{4})$ then $\widetilde{B}(t_i,\frac{1}{4})$ contains $\{(x_n^{(2)}, \beta_n^{(2)})\}$ of $\{(x_n^{(1)}, \beta_n^{(1)})\}$. After kth steps we have $\{(x_n^{(k)}, \beta_n^{(k)})\}$ is in the fuzzy ball $\widetilde{B}(t_i,\frac{1}{2^k})$. Now $\{(x_n^{(n)}, \beta_n^{(n)})\}$ is a subsequence of $\{(x_n, \beta_n)\}$. Let $0 < \varepsilon < 1$ be given. Choose K so large that $(1-\frac{1}{2^n}) * (1-\frac{1}{2^{n+1}}) * \ldots *(1-\frac{1}{2^{m-1}}) > (1- \varepsilon)$. Whenever $m > n > K$, we have

$\widetilde{D}[(x_n^{(n)},\beta_n^{(n)}),\ (x_m^{(m)},\beta_m^{(m)})] \geq \widetilde{D}[(x_n^{(n)},\beta_n^{(n)}),\ (x_{n+1}^{(n+1)},\beta_{n+1}^{(n+1)})] * \widetilde{D}[(x_{n+1}^{(n+1)},\beta_{n+1}^{(n+1)}),\ (x_{n+2}^{(n+2)},\beta_{n+2}^{(n+2)})] * \ldots * \widetilde{D}[(x_{m-1}^{(m-1)},\beta_{m-1}^{(m-1)}),(x_m^{(m)},\beta_m^{(m)})]$

$\geq (1-\frac{1}{2^n}) * (1-\frac{1}{2^{n+1}}) * \ldots (1-\frac{1}{2^{m-1}}) > (1-\varepsilon)$

Hence $\{(x_n^{(n)},\beta_n^{(n)})\}$ is a fuzzy Cauchy sequence.

For the converse, suppose that every $\{(x_n,\beta_n)\}$ in $\widetilde{E}$ contains $\{(x_{n_k},\beta_{n_k})\}$ which is a fuzzy Cauchy.We will prove that $\widetilde{E}$ is fuzzy totally bounded. Assume that $0 < \varepsilon < 1$ and let (x_1,β_1) $\in\widetilde{E}$. If $\widetilde{E}\backslash\widetilde{B}((x_1,\beta_1),\varepsilon) = \emptyset$, we get a fuzzy ε-net, which is, the fuzzy set $\{(x_1,\beta_1)\}$. If it is not take $(x_2,\beta_2)\in\widetilde{E}\backslash\widetilde{B}((y_1,\beta_1),\varepsilon)$. If $\widetilde{E}\backslash[\widetilde{B}((x_1,\beta_1),\varepsilon) \cup \widetilde{B}((x_2,\beta_2),\varepsilon)]= \emptyset$. We get a fuzzy ε-net, which is $\{(x_1,\beta_1),(x_2,\beta_2)\}$. After finite steps this process stop. If it dose not stop, we will get an infinite sequence $\{(x_n,\beta_n)\}$ with property that $\widetilde{D}[(x_n,\beta_n),(x_m,\beta_m)] \leq (1-\varepsilon)$, $n \neq m$. Consequently, the sequence $\{(x_n,\beta_n)\}$ does not contains a fuzzy Cauchy subsequence of fuzzy points, this is a contradiction.■

Proposition 3.10:

Suppose that $(\widetilde{A},\widetilde{D},*)$ is a fuzzy compact fuzzy distance space. Then $\widetilde{A}$ is fuzzy totally bounded.

Proof:

Whenever $0 < r < 1$, the set $\{\widetilde{B}(x_\alpha,r) : x_\alpha\in\widetilde{A}\}$ is a fuzzy open fuzzy cover of $\widetilde{A}$. But $\widetilde{A}$ is fuzzy compact we have $\{\widetilde{B}(x_j,r): j=1, 2,\ldots n\}$ with $\widetilde{A} = \cup_{j=1}^{n}\widetilde{B}(x_j,r)$. Hence the centers $\{(x_1,\alpha_1),(x_2,\alpha_2),\ldots(x_n,\alpha_n)\}$ form a finite fuzzy r-net for $\widetilde{A}$. Therefore, $\widetilde{A}$ is fuzzy totally bounded. ■

Proposition 3.11:

If $(\widetilde{A},\widetilde{D},*)$ be a fuzzy compact fuzzy distance space. Then $(\widetilde{A},\widetilde{D},*)$ is fuzzy complete.

Proof:

Let $(\widetilde{A},\widetilde{D},*)$ be a fuzzy compact fuzzy metric space and $\widetilde{A}$ is not fuzzy complete. So we can find a fuzzy Cauchy sequence $\{(x_n,\alpha_n)\}$ of fuzzy points in $(\widetilde{A},\widetilde{D},*)$ not having a fuzzy limit in $\widetilde{A}$. Let $y_\beta\in\widetilde{A}$, since $\{(x_n,\alpha_n)\}$ dose not fuzzy converge to y_β there exists $0 < r <1$ such that $\widetilde{D}((x_n,\alpha_n),y_\beta) \leq (1-r)$ for infinitely many values of n, since $\{(x_n,\alpha_n)\}$ is fuzzy Cauchy, we can find K with $k, j \geq K$. With $\widetilde{D}((x_k,\alpha_k),(x_j,\alpha_j)) > (1-r)$. Choose $k \geq K$ for which $\widetilde{D}((x_k,\alpha_k),y_\beta) > (1-\varepsilon)$. So, the fuzzy open fuzzy ball $\widetilde{B}(y_\beta,\varepsilon)$ contains a finite number of (x_n,α_n). In this step, we can put for each $x_\alpha\in\widetilde{A}$ a fuzzy ball $\widetilde{B}(x_\alpha,r(x_\alpha))$,where $0 < r(x_\alpha)$ < 1 depends on x_α, and the fuzzy ball $\widetilde{B}(x_\alpha,r(x_\alpha))$ contains a finite number of (x_n,α_n). Notice that $\widetilde{A} = \bigcup_{x_\alpha \in \widetilde{A}} \widetilde{B}(x_\alpha, r(x_\alpha))$ which means that $\{\widetilde{B}(x_\alpha, r(x_\alpha)) : x_\alpha \in \widetilde{A}\}$ is a fuzzy covering of $\widetilde{A}$. But $\widetilde{A}$ is fuzzy compact so we can find $\widetilde{B}((y_i, \beta_i), \varepsilon(y_i, \beta_i))$, i = 1, 2, …, n. But each fuzzy ball has finite number of (x_n, α_n), hence, $\widetilde{A}$, will contains (x_n, α_n) for only a finite number of values of n. This is impossible. Therefore $\widetilde{A}$ is a fuzzy complete ■

Theorem 3.12:

If $(\widetilde{A}, \widetilde{D}, *)$ be fuzzy totally bounded and fuzzy complete fuzzy metric space then $(\widetilde{A}, \widetilde{D}, *)$ is fuzzy compact.

Proof:

Let $(\widetilde{A}, \widetilde{D}, *)$ be fuzzy totally bounded and fuzzy complete but not fuzzy compact. So we can find a fuzzy open fuzzy cover $\{\widetilde{G}_\lambda : \lambda \in \Lambda\}$ of $\widetilde{A}$ dose not have a finite fuzzy subcovering.

But $(\widetilde{A}, \widetilde{D}, *)$ is fuzzy totally bounded, hence it is fuzzy bounded, so for some $0 < r < 1$ and some $x_\alpha \in \widetilde{A}$, we have $\widetilde{A} \subseteq \widetilde{B}(x_\alpha, r)$. Observe that $\widetilde{A} \subseteq \widetilde{B}(x_\alpha, r)$ implies $\widetilde{A} = \widetilde{B}(x_\alpha, r)$.

Let $\varepsilon_n = \frac{r}{2^n}$. Now $\widetilde{B}((x_1, \alpha_1), \varepsilon_1)$ is fuzzy totally bounded ["any nonempty fuzzy subset of fuzzy totally bounded fuzzy set is fuzzy totally bounded"], then there exists $(x_2, \alpha_2) \in \widetilde{B}[(x_1, \alpha_1), \varepsilon_1]$ such that $\widetilde{B}[(x_2, \alpha_2), \varepsilon_2]$ cannot be fuzzy covered by a finite number of fuzzy sets $\widetilde{G}_\lambda$. By this process, a sequence of fuzzy points $\{(x_n, \alpha_n)\}$ can be found with the conduction that for all n, $\widetilde{B}((x_n, \alpha_n), \varepsilon_n)$ will not be fuzzy covered by a finite number of fuzzy sets $\widetilde{G}_\lambda$ and $(x_{n+1}, \alpha_{n+1}) \in \widetilde{B}((x_n, \alpha_n), \varepsilon_n)$.We next show that the sequence of fuzzy points $\{(x_n, \alpha_n)\}$ is fuzzy convergent.

Since$(x_{n+1}, \alpha_{n+1}) \in \widetilde{B}((x_n, \alpha_n), \varepsilon_n)$ it follows that $\widetilde{D}((x_n, \alpha_n), (x_{n+1}, \alpha_{n+1})) > (1 - \varepsilon_n)$.

Let $0 < \varepsilon < 1$ such that $(1 - \varepsilon_n) * (1 - \varepsilon_{n+1}) * \ldots * (1 - \varepsilon_m) > (1 - \varepsilon)$

Hence $\widetilde{D}((x_n, \alpha_n), (x_m, \alpha_m)) \geq \widetilde{D}((x_n, \alpha_n), (x_{n+1}, \alpha_{n+1})) * \ldots * \widetilde{D}((x_{m-1}, \alpha_{m-1}), (x_m, \alpha_m)) \geq$ $(1 - \varepsilon_n) * (1 - \varepsilon_{n+1}) * \ldots * (1 - \varepsilon_m) > (1 - \varepsilon)$

So $\{(x_n, \alpha_n)\}$ is a fuzzy Cauchy sequence in $\widetilde{A}$ but $\widetilde{A}$ is fuzzy complete, so $\widetilde{A}$ fuzzy converges to $y_\beta \in \widetilde{A}$. Since $y_\beta \in \widetilde{A}$ we can find $\lambda_0 \in \Lambda$ such that $y_\beta \in \widetilde{G}_{\lambda_0}$. Because $\widetilde{G}_{\lambda_0}$ is fuzzy open it contains $\widetilde{B}(y_\beta, \delta)$ for some $0 < \delta < 1$. Choose N so large that, $\widetilde{D}((x_n, \alpha_n), y_\beta) > (1 - \delta)$ and $(1 - \varepsilon_n) > (1 - \delta)$. Then, for any $x_\alpha \in \widetilde{A}$ such that $\widetilde{D}(x_\alpha, (x_n, \alpha_n)) > (1 - \varepsilon_n)$.It follows that $D(x_\alpha, y_\beta) \geq \widetilde{D}(x_\alpha, (x_n, \alpha_n)) * \widetilde{D}((x_n, \alpha_n), y_\beta) \geq (1 - \delta) * (1 - \delta) > (1 - r)$, for some $0 < r < 1$.

So that $\widetilde{B}((x_n, \alpha_n), \varepsilon_n) \subseteq \widetilde{B}(y_\beta, r)$. Therefore $\widetilde{B}((x_n, \alpha_n), \varepsilon_n)$ have a finite fuzzy subcovering, defined by the fuzzy set $\widetilde{G}_{\lambda_0}$. But this contradicts $\widetilde{B}((x_n,\alpha_n),\varepsilon_n)$ will not be fuzzy covered by a finite number of fuzzy sets $\widetilde{G}_\lambda$. ■

Theorem 3.13:

(X,d) is a compact metric space if and only if $(\widetilde{A},\widetilde{D}_d,*)$ is a fuzzy compact fuzzy distance space where $\widetilde{D}_d(x_\alpha,y_\beta)=\frac{t}{t+d(x_\alpha,y_\beta)}$.

Proof:

Suppose that (X,d) is compact. Let $\{(x_n,\alpha_n)\}$ be a sequence of fuzzy points in $(\widetilde{A},\widetilde{D}_d,*)$ then (x_n) is a sequence in (X,d). But (X,d) is compact hence (x_n) has a convergent subsequence. Then $\{(x_n,\alpha_n)\}$ has a convergent sequence in $(\widetilde{A},\widetilde{D}_d,*)$ by Proposition 2.23. Hence $(\widetilde{A},\widetilde{D}_d,*)$ is fuzzy compact. In similar way we can prove that if $(\widetilde{A},\widetilde{D}_d,*)$ is fuzzy compact then (X,d) is compact by using Proposition 2.23■

Proposition 3.14:

Suppose that $(\widetilde{A},\widetilde{D},*)$ is a fuzzy distance space. Then

(i) For any $\{(x_n,\alpha_n)\}$ in $\widetilde{A}$ there is $x_\alpha\in\widetilde{A}$ such that $(x_n,\alpha_n)\to x_\alpha$.

(ii) For any $\{(x_n,\alpha_n)\}$ in $\widetilde{A}$ there is $\{(x_{n_k},\alpha_{n_k})\}$ and $x_\alpha\in\widetilde{A}$ such that$(x_{n_k},\alpha_{n_k})\to x_\alpha$

are equivalent

Proof of (ii):

Suppose that $\{(x_n,\alpha_n)\}$ is a sequence of fuzzy points in $\widetilde{A}$. If the fuzzy set$\{(x_1,\alpha_1),(x_2,\alpha_2),\ldots,\}$ is finite, then the sequence of the fuzzy points, $(x_1,\alpha_1),(x_1,\alpha_1),\ldots$ or $(x_2,\alpha_2),(x_2,\alpha_2),\ldots$ or $(x_j,\alpha_j),(x_j,\alpha_j)$ are all constant sequences which is a subsequences of $\{(x_1,\alpha_1),(x_2,\alpha_2),\ldots,\}$fuzzy converges. Suppose that the fuzzy set $\{(x_1,\alpha_1),(x_2,\alpha_2),\ldots,\}$. By (i), the fuzzy set $\{(x_1,\alpha_1),(x_2,\alpha_2),\ldots,\}$ have a fuzzy limit fuzzy point $x_\alpha\in\widetilde{A}$. suppose $n_1\in\mathbb{N}$ with $\widetilde{D}((x_{n_1},\alpha_{n_1}),x_\alpha)>0$.Having defined n_k, let n_{k+1} be the smallest integer such that $n_{k+1}>n_k$And $\widetilde{D}((x_{n_{k+1}},\alpha_{n_{k+1}}),x_\alpha)>(1-\frac{1}{k+1})$. Then the sequence $\{(x_{n_k},\alpha_{n_k})\}$ of fuzzy points fuzzy converges to x_α ■

Proof of (i):

Suppose that $\widetilde{E}\subseteq\widetilde{A}$ and $\widetilde{E}$ is infinite fuzzy set. Now we can find a sequence $\{(y_n,\beta_n)\}$ of fuzzy points in $\widetilde{A}$ of distinct terms. By (ii) $\{(y_n,\beta_n)\}$ contains $\{(y_{n_i},\beta_{n_i})\}$ such that $\{(y_{n_i},\beta_{n_i})\}\to y_\beta$ and $y_\beta\in\widetilde{A}$. So every $\widetilde{B}(y_\beta,r)$ contains an infinite number of $\{(y_{n_i},\beta_{n_i})\}$. But the terms are distinct; hence every $\widetilde{B}(y_\beta,r)$ contains an infinite number of fuzzy points of $\widetilde{E}$. Hence $y_\beta \in \widetilde{A}$ is a fuzzy limit

point of $\widetilde{E}$ ■

Theorem 3.15:

A fuzzy distance space $(\widetilde{A}, \widetilde{D}, *)$ is fuzzy compact if and only if every sequence $\{(x_n, \alpha_n)\}$ in $\widetilde{A}$ has $\{(x_{n_i}, \alpha_{n_i})\}$ such that $(x_{n_i}, \alpha_{n_i}) \to x_\alpha \in \widetilde{A}$.

Proof:

Let $\widetilde{A}$ be a fuzzy compact and let $\{(x_n, \alpha_n)\}$ be any sequence of fuzzy points in $\widetilde{A}$. Since $\widetilde{A}$ is fuzzy totally bounded, it follows, using Theorem 3.9, that $\{(x_n, \alpha_n)\}$ contains a fuzzy Cauchy subsequence $\{(x_{n_i}, \alpha_{n_i})\}$. But $\{(x_{n_i}, \alpha_{n_i})\}$ fuzzy converges to a fuzzy point $x_\alpha \in \widetilde{A}$ because $\widetilde{A}$ is fuzzy complete.

Conversely, suppose that every sequence $\{(x_n, \alpha_n)\}$ in $\widetilde{A}$ has $\{(x_{n_i}, \alpha_{n_i})\}$ such that $(x_{n_i}, \alpha_{n_i}) \to x_\alpha \in \widetilde{A}$. Suppose that $\{(x_n, \alpha_n)\}$ is a fuzzy Cauchy sequence of fuzzy points in $\widetilde{A}$.By assumption $\{(x_n, \alpha_n)\}$ has a subsequence $\{(x_{n_i}, \alpha_{n_i})\}$ that fuzzy converges to a fuzzy point $x_\alpha \in \widetilde{A}$. We shall show that $(x_n, \alpha_n) \to x_\alpha$. Let $0 < q < 1$ be given by Remark 2.5, there is $0 < p < 1$ with $(1-p) * (1-p) > (1-q)$.

Now $(x_{n_i}, \alpha_{n_i}) \to x_\alpha$, there exist K_1 such that $\widetilde{D}((x_{n_i}, \alpha_{n_i}), x_\alpha) > (1-p)$ for all $n_i \geq K_1$.Since the sequence of fuzzy points $\{(x_n, \alpha_n)\}$ is fuzzy Cauchy, there exists K_2 with $\widetilde{D}((x_n, \alpha_n), (x_m, \alpha_m)) > (1-p)$ for all $m, n \geq K_2$. Let $K = \min\{K_1, K_2\}$ then

$\widetilde{D}((x_n, \alpha_n), x_\alpha) \geq \widetilde{D}((x_n, \alpha_n), (x_{n_i}, \alpha_{n_i})) * \widetilde{D}((x_{n_i}, \alpha_{n_i}), x_\alpha) > (1-p) * (1-p) > (1-q)$ with $n \geq K$. ■

The results of this section can be summed up as follows:

Corollary 3.16:

Suppose that $(\widetilde{A}, \widetilde{D}, *)$ is a fuzzy compact fuzzy distance space, and $\widetilde{E} \subseteq \widetilde{A}$. If $\widetilde{E}$ is a fuzzy closed fuzzy then $\widetilde{E}$ is fuzzy compact**.**

Proof:

Let $\{(y_n, \beta_n)\}$ be a sequence of fuzzy points in $\widetilde{E}$. Then $\{(y_n, \beta_n)\}$ is a sequence of fuzzy points in $\widetilde{A}$, contains $\{(x_{n_i}, \alpha_{n_i})\}$ such that $(x_{n_i}, \alpha_{n_i}) \to x_\alpha \in \widetilde{A}$. But then $x_\alpha \in \widetilde{E}$ since $\widetilde{E}$ is a fuzzy closed. But $\{(y_n, \beta_n)\}$ was arbitrary in $\widetilde{E}$. By Theorem 3.15 $\widetilde{E}$ is a fuzzy compact ■

Theorem 3.17:

Suppose that $(\widetilde{A}, \widetilde{D}, *)$ is a fuzzy distance space, and $\widetilde{E} \subseteq \widetilde{A}$ If $\widetilde{E}$ is fuzzy compact then $\widetilde{E}$ is a fuzzy closed.

Proof:

Assume that $x_\alpha \in \widetilde{A}$ is a fuzzy limit point of $\widetilde{E}$. So we can find a sequence $\{(y_n, \beta_n)\}$ of fuzzy points in $\widetilde{E}$ fuzzy converges to x_α. But then $\{(y_n, \beta_n)\}$ is a fuzzy Cauchy sequence in $\widetilde{E}$. Since $\widetilde{E}$ is a fuzzy complete, $(y_n, \beta_n) \to y_\beta$ in $\widetilde{E}$. Therefore $y_\beta = x_\alpha$ and so $x_\alpha \in \widetilde{E}$. Thus $\widetilde{E}$ contains all its fuzzy limit fuzzy points. Hence is fuzzy closed ■

4. FUZZY CONTINUOUS FUNCTION ON FUZZY COMPACT SPACE

Theorem 4.1:

Suppose that $(\widetilde{A},\widetilde{D}_{\widetilde{A}},*)$ is a fuzzy compact fuzzy distance space and assume that $(\widetilde{E},\widetilde{D}_{\widetilde{E}},*)$ is a fuzzy distance space. If $g:\widetilde{A} \to \widetilde{E}$ is a continuous mapping then $g(\widetilde{A})$ is fuzzy compact.

Proof:

Suppose that $\{\widetilde{G}_\lambda : \lambda \in \Lambda\}$ is a fuzzy open covering of $g(\widetilde{A})$. But g is continuous so the inverse image of $\widetilde{G}_\lambda$ is fuzzy open in $\widetilde{A}$. Moreover $\{g^{-1}(\widetilde{G}_\lambda):\lambda \in \Lambda\}$ is an fuzzy open covering of $\widetilde{A}$. Since $\widetilde{A}$ is fuzzy compact, there exists $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ in Λ such that $\widetilde{A} = \bigcup_{i=1}^{n} g^{-1}(\widetilde{G}_{\lambda_i})$.

Now $g(\widetilde{A}) = g(\bigcup_{i=1}^{n} g^{-1}(\widetilde{G}_{\lambda_i})) = \bigcup_{i=1}^{n} g(g^{-1}(\widetilde{G}_{\lambda_i})) \subseteq \bigcup_{i=1}^{n} \widetilde{G}_{\lambda_i}$ So $\{\widetilde{G}_{\lambda_i}$, i = 1, 2, …, n$\}$ is a finite fuzzy sub covering of $g(\widetilde{A})$. Hence, $g(\widetilde{A})$ is fuzzy compact ■

Corollary 4.2:

Suppose that $(\widetilde{A},\widetilde{D}_{\widetilde{A}},*)$ is a fuzzy distance space and assume that $(\widetilde{E},\widetilde{D}_{\widetilde{E}},*)$ is a fuzzy distance space. If $g:\widetilde{A} \to \widetilde{D}$ is a fuzzy homeomorphism. Then $\widetilde{A}$ is fuzzy compact if and only if $\widetilde{E}$ is fuzzy compact.

Corollary 4.3:

Suppose that $(\widetilde{A},\widetilde{D}_{\widetilde{A}},*)$ is a fuzzy compact fuzzy distance space and assume that $(\widetilde{E},\widetilde{D}_{\widetilde{E}},*)$ is a fuzzy distance space. If $g:\widetilde{A} \to \widetilde{E}$ is a fuzzy continuous function.Then $g(\widetilde{A})$ is fuzzy bounded and fuzzy closed fuzzy subset of $\widetilde{E}$.

Theorem 4.4:

Suppose that $(\widetilde{A},\widetilde{D}_{\widetilde{A}},*)$ is a fuzzy compact fuzzy distance space and assume that $(\widetilde{E},\widetilde{D}_{\widetilde{E}},*)$ is a fuzzy distance space. If $g:\widetilde{A} \to \widetilde{E}$ is a one-to-one, onto fuzzy continuous mapping then $g^{-1}: \widetilde{E} \to \widetilde{A}$ is fuzzy continuous and therefore, is fuzzy homeomorphism.

Proof:

Let $g: \widetilde{A} \to \widetilde{E}$ be one-to-one and onto. So its inverse exists. Assume that $\widetilde{F} \subseteq \widetilde{A}$ and $\widetilde{F}$ is a fuzzy closed. By Corollary 3.16, $\widetilde{F}$ is fuzzy compact. By Theorem 4.1 $g(\widetilde{F})$ is fuzzy compact and therefore, a fuzzy closed fuzzy subset of $\widetilde{E}$ by Theorem 3.17. Since $g(\widetilde{F}) = (g^{-1})^{-1}(\widetilde{F})$ and then $(g^{-1})^{-1}(\widetilde{F})$ is fuzzy closed in $\widetilde{E}$. Hence by Theorem 2.44 g^{-1} is fuzzy continuous ■

Definition 4.5:

Let $\{\widetilde{G}_\lambda : \lambda \in \Lambda\}$ be a fuzzy open covering of the fuzzy distance space $(\widetilde{A},\widetilde{D}_{\widetilde{A}},*)$. Any number $0 < \delta < 1$ such that for each $x_\alpha \in \widetilde{A}$ there exists $\lambda \in \Lambda$(dependent on x_α) for which $\widetilde{B}(x_\alpha,\delta) \subseteq \widetilde{G}_\lambda$ is said to be a fuzzy Lebesgue number of $\{\widetilde{G}_\lambda : \lambda \in \Lambda\}$.

Theorem 4.6:

Suppose that $(\widetilde{A},\widetilde{D}_{\widetilde{A}},*)$ is a fuzzy compact fuzzy distance space, and assume that $(\widetilde{E},\widetilde{D}_{\widetilde{E}},*)$ is another fuzzy distance space and g: $\widetilde{A} \to \widetilde{E}$ is fuzzy continuous. So for any $0 < r < 1$, there we can find a δ, $0 < \delta < 1$ (depending on r) with $g(\widetilde{B}(x_\alpha,\delta)) \subseteq \widetilde{B}(g(x_\alpha),r)$ for every $x_\alpha \in \widetilde{A}$. Hence g is uniformly fuzzy continuous.

Proof:

Let $0 < r < 1$ such that $(1- r) *(1- r) > (1- \varepsilon)$ .The collection of fuzzy ball $\{\widetilde{B}(y_\beta,r): y_\beta \in \widetilde{E}\}$ constitutes an fuzzy open fuzzy cover of $\widetilde{E}$. The fuzzy set $\{g^{-1}(\widetilde{B}(y_\beta,r)): y_\beta \in \widetilde{E}\}$ hence is a fuzzy open fuzzy cover of the fuzzy compact fuzzy distance space $\widetilde{A}$. Assume that δ is a Lebesgue number of $\{g^{-1}(\widetilde{B}(y_\beta,r)): y_\beta \in \widetilde{E}\}$. But each $\widetilde{B}(x_\alpha,\delta)$ is a subset of $g^{-1}(\widetilde{B}(y_\beta,r))$, so $g(\widetilde{B}(x_\alpha,\delta)) \subseteq \widetilde{B}(y_\beta,r)$ for some $y_\beta \in \widetilde{E}$. Because $g(x_\alpha) \in \widetilde{B}(y_\beta,r)$, we find for any $z_\sigma \in \widetilde{B}(x_\alpha,\delta)$ that $\widetilde{D}_{\widetilde{E}}(g(z_\sigma),g(x_\alpha)) \geq \widetilde{D}_{\widetilde{E}}(g(z_\sigma),y_\beta) * \widetilde{D}_{\widetilde{E}}(y_\beta,g(x_\alpha)) > (1- r) *(1- r) > (1- \varepsilon)$ i.e, $g(\widetilde{B}(x_\alpha,\delta)) \subseteq \widetilde{B}(g(x_\alpha),\varepsilon)$. ■

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