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The properties of straight lines that issue from the point of intersection of the diagonals of a trapezoid, and are perpendicular to its legs

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Abstract.

We present four theorems with their proofs, which have to do with the dropping of perpendiculars to the sides of a trapezoid from the point of intersection of the diagonals. Two of the theorems are actually new indicators for the fact that the quadrilateral in which certain characteristics hold is a trapezoid. We also consider whether the converse theorem is also true, as well as different geometric properties that result from the theorems.

Keywords: Special properties in the trapezoid; Indicator for a trapezoid; Complete quadrangle; Harmonic quadruple.

1. Introduction

Different properties of segments, angles, triangles, etc., which arise in the trapezoid as a result of drawn lines or from certain constructions, have been known for many years. However, this shape continues to be of interest to those who study geometry, who occasionally discover in it unfamiliar properties or algebraic relations and formulas among the segments, diagonals and the sides of the trapezoid, in particular due to the fact that the trapezoid is a particular case of the quadrilateral.

The property of Steiner's theorem for the trapezoid (midpoints of the bases of the trapezoid, the point of intersection of the diagonals, the point of intersection of the continuations of the sides all lie on a single straight line [1]) allowed certain constructions to be carried out using a straightedge only [2]. The development of algebraic formulas for segments in the trapezoid using different methods [3-5] permitted the calculation of different lengths using faster method.

In the present paper we present theorems representing properties of straight lines that issue from the point of intersection of the diagonals of the trapezoid and are perpendicular to its sides.

In addition to these straight lines, we shall give indicators for identifying trapezoids from different data that are based on the proof of the converse theorem – which for certain data is not true and for other data is true, and whose proof is more involved.

2. Theorem 1

Given is some trapezoid ABCD, in which the point E is the point of intersection of the diagonals, and the points I and J are the midpoints of the bases AB and DC, respectively.

The straight segment EM is perpendicular to the leg BC

(EM ⊥ BC), as shown in Figure 1.

Prove that the segment EM bisects the angle ∡IMJ.

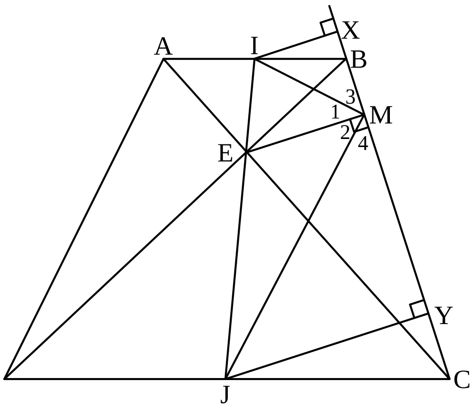

Figure 1

Proof

We denote AB = a, CD = b.
Auxiliary construction: we construct the perpendiculars IX and JY to the leg BC (as shown in Figure 1).
We make use of the known properties of the trapezoid:

(1) The point E lies on the straight line IJ which connects the midpoints of the bases (in accordance with Steiner's theorem for trapezoid).

(2) $\frac{IB}{JC} = \frac{IE}{EJ} = \frac{a}{b}$

The triangles $\triangle$ IBX and $\triangle$ JCY are similar (by two angles), and therefore there holds

$$(*)\ \frac{IX}{JY} = \frac{IB}{JC} = \frac{a}{b}$$

In the trapezoid IXYJ the segment EM is parallel to its bases IX and JY. Therefore there holds $(**)\ \frac{IE}{EJ} = \frac{XM}{MY}$.

From the relations $(*)$and $(**)$ it follows that $\frac{XM}{MY} = \frac{a}{b}$.

In the right-angled triangles $\triangle$ IXM and $\triangle$ JYM we obtain: $\frac{IX}{JY} = \frac{XM}{MY} = \frac{a}{b}$,
therefore these triangles are similar and in them we have: $\measuredangle M_3 = \measuredangle M_4$.

Hence it follows that $\measuredangle M_1 = \measuredangle M_2$ ($\measuredangle M_2 = 90° - \measuredangle M_4$, $\measuredangle M_1 = 90° - \measuredangle M_3$), which means that the segment ME bisects ∡IMJ.

A link to a GeoGebra applet which allows dynamic investigation of the theorem to be carried out.

Link: http://tube.geogebra.org/m/1464701

Demonstration applet for angle bisection by means of a perpendicular to the leg of a trapezoid.

The applet illustrates the property by which the straight line issuing from E, the point of intersection of the diagonals of the trapezoid, and he is perpendicular to the leg BC, bisects the angle ∡IMJ (DJ=JC, AI=IB). When the vertices of the trapezoid A, B, C are dragged, the property is conserved. In any condition the points J, E, I, F remain on a straight line according to the Steiner's theorem.

Question

Is the converse statement also true?

In other words, let ABCD be some convex quadrilateral, E the point intersection of its diagonals, EM the perpendicular to the side BC (EM⊥BC), the segment ME bisects the angle ∡IMJ (I and J are the midpoints of the sides AB and CD, respectively). Is the claim that the quadrilateral ABCD is a trapezoid true or false?

Answer

The converse statement is not true, and the proof shall be given by presenting an example of the quadrilateral which is not a trapezoid, which meets the conditions of the specified claim.

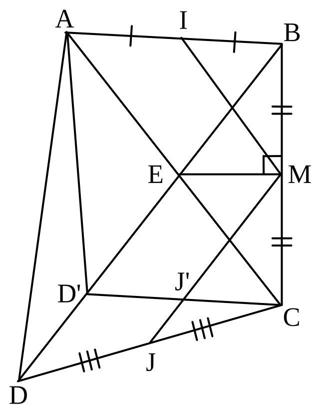

Figure 2

Disproving the claim (by counter example)

Let ABCD be a special convex quadrilateral in which AB ∦ CD, and the straight line that connects the point of intersection of the diagonals E with the point M, which is the midpoint of the side BC, is perpendicular to this side (EM⊥BC, BM = MC). Then the segment EM bisects the angle ∡IMJ, where the points I and J are the midpoints of the sides AB and DC, respectively (as shown in Figure 2).

In other words, we have to prove that the segment EM bisects the angle ∡IMJ.

Proof

We construct CD' parallel to AB (D' is a point on the diagonal BD).

The point E is also the point of intersection of the diagonals in the trapezoid ABCD'.

We denote by J' the point of intersection of the segments JM and CD'.

JM is the midline in the triangle $\triangle \mathrm{BCD}$, and therefore the point J' is the midpoint of the segment CD', as shown in Figure 2.

Therefore, in accordance with theorem 1, the perpendicular EM to the leg BC of the trapezoid ABCD' bisects the angle $\measuredangle \mathrm{IMJ}'$ (where I and J' are the midpoints of the sides AB and CD'), and therefore ME bisects the angle ∡IMJ in the quadrilateral ABCD, which is not a trapezoid.

Note

Theorem 1 can be proven using another method.

From Steiner's theorem for the trapezoid which states that the points F, I, E, J are located on one straight line (AI = IB, DJ = JC), and from the proportion ratio of parallel segments, we find that these 4 points, form a harmonic quadruplet, in other words: $\frac{\mathrm{FJ}}{\mathrm{FI}} = \frac{\mathrm{JE}}{\mathrm{EI}}$

The segment FE is the diameter of the Appolonius circle that intersects the legs of the trapezoid at the points M and N.

Therefore, and based on the converse theorem of the angle-bisector theorem, one obtains that ME and NE (Fig. 2a) bisect the angles ∡IMJ and ∡INJ respectively.

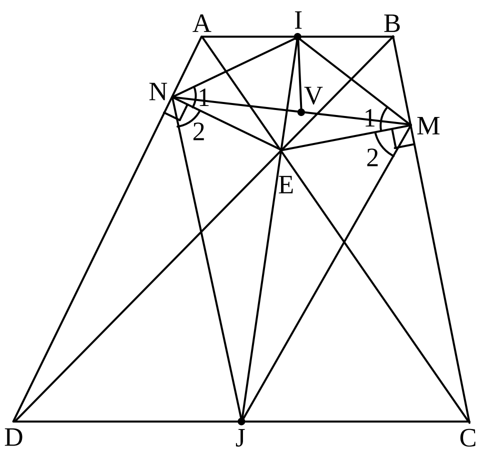

Figure 2a

3. Properties that result from theorem 1

When perpendiculars are dropped from the point of intersection of the diagonals to the sides of the trapezoid, an inscribed quadrilateral IMJN is formed (Fig. 2a). We give three properties of this quadrilateral.

3.1 Property 1

The quadrilateral IMJN is the quadrilateral with the smallest perimeter of all the quadrilaterals IXJY in which the points X and Y are located on the sides of the trapezoid, BC and AD, respectively. From theorem 1 we have: $\measuredangle \mathrm{N}_1 = \measuredangle \mathrm{N}_2 . \measuredangle \mathrm{M}_1 = \measuredangle \mathrm{M}_2$ It is known that when a ray issues from a single point, intersects a straight line and is reflected from it to another point, such that the angle of incidence (the angle between the ray and the perpendicular to the straight line at the point of incidence) is equal to the angle of reflection, the sum of the distances traveled by the ray is the smallest (this is also true for a ray of light in accordance with Fermat's principle). Therefore the path IMJ (IM + MJ) and the path INJ (IN + NJ) are the shortest, and therefore the perimeter of the quadrilateral IMJN is the smallest.

3.2 Property 2

The bisectors of the opposing angles ∡MIN and ∡MJN of the quadrilateral IMJN are concurrent on the diagonal of the quadrilateral. The bisectors of the angles ∡IMJ and ∡INJ are concurrent at the point E of the diagonal IJ, and therefore from the bisector theorem, we have: $\frac{\mathrm{IM}}{\mathrm{MJ}} = \frac{\mathrm{IN}}{\mathrm{NJ}} \left(= \frac{\mathrm{IE}}{\mathrm{EJ}} \right)$, or alternatively: $(*) \frac{\mathrm{IM}}{\mathrm{IN}} = \frac{\mathrm{JM}}{\mathrm{JN}}$.

We denote by V the point of intersection of the angle bisector ∡MIN with the diagonal MN (as shown in Figure 2a). Therefore from the bisector theorem, we have $(**) \frac{\mathrm{MI}}{\mathrm{IN}} = \frac{\mathrm{MV}}{\mathrm{VN}}$.

From the two proportions it follows that $\frac{MJ}{JN} = \frac{MV}{VN}$, and therefore it follows that JV is the bisector of the angle $\measuredangle MJN$.
In other words, the angle bisectors are concurrent on the diagonal MN of the quadrilateral.

3.3 Property 3

Another property of the quadrilateral IMJN arises from proportion$(*)$:

The product of the lengths of one pair of opposing sides equals the product of the length of the other pair of opposing sides.

4. Theorem 2 (Indicator for identifying a trapezoid)

Let ABCD be a convex quadrilateral in which $AD \nparallel BC$, E is the point of intersection of its diagonals, EM is a perpendicular to the side BC which is not a mid-perpendicular to this side ($BM \neq MC$). The points I and J are the middles of the sides AB and CD respectively, and ME is the bisector of the angle $\measuredangle IMJ$, as shown in Figure 3.
Then, $AB \parallel CD$, in other words – the quadrilateral ABCD is a trapezoid.

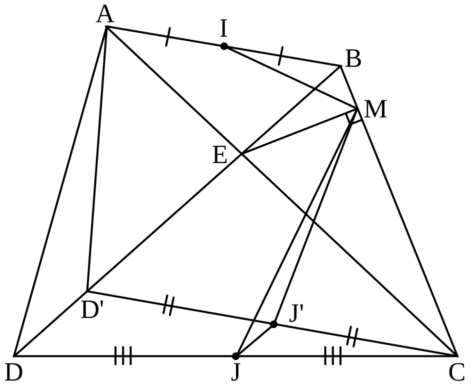

Figure 3

Proof (indirect)

Let us assume that $AB \nparallel CD$. We construct CD' parallel to AB (the point D' is on the diagonal BD).

We denote by J' the midpoint of the segment CD'.

All the conditions of theorem 1 hold in the trapezoid ABCD', therefore the segment EM is the bisector of the angle $\measuredangle IMJ'$. Hence it follows that $\measuredangle EMJ = \measuredangle EMJ'$, i.e., The rays JM and J'M coincide and the point J' belongs to the segments JM.

The segment JJ' is a midline in the triangle $\triangle CDD'$, therefore the segment JJ' is parallel to the straight line BD.

The continuation of the segment JJ' must intersect the side BC at its midpoint. This is the point M, since we have proven that J' belongs to the segments JM. Thus, we have obtained that the point M is the middle of the side BC, contradicting the datum that $BM \neq MC$. Therefore,$AB \parallel CD$ and the quadrilateral ABCD is a trapezoid.

5. Theorem 3

Let ABCD be a trapezoid ($AB \parallel CD$).E is the point of intersection of its diagonals. The segment MN passes through the point E and is perpendicular to the side BC at the point M on it (the point N is on the side AD).The segment KL also passes through the point E and is perpendicular to the side AD at the point K on it (the point L is on the side BC), as shown in Figure 4.

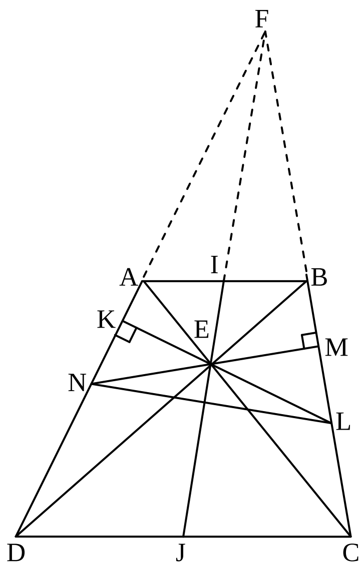

Figure 4

Prove that $NL \perp IJ$.

Proof

The continuations of the sides of the trapezoid AD and BC are concurrent at the point F.

From Steiner's theorem for the trapezoid [1], the points F, I, E, J are located on a straight line. In the triangle $\triangle NFL$ the point E is the point of intersection of the altitudes NM and KL. Therefore the straight line that passes through the points F and E is the third altitude of the triangle, or in other words: $FE \perp NL$ and therefore $NL \perp IJ$.

The converse is also true, and the proof shall be given in the following theorem.

6. Theorem 4 (Indicator for identifying a trapezoid)

Let ABCD be a convex quadrilateral in which $AD \nparallel BC$.

The point E is the point of intersection of its diagonals,

The segment MN passes through the point E and is perpendicular to the side BC,

The segment KL passes through the point E and is perpendicular to the side AD.

The points I and J are the midpoints of the sides AB and CD, respectively.

Given is: $IJ \perp NL$, as shown in Figure 5.

Prove that the quadrilateral ABCD is a trapezoid.

To prove Theorem 4 we use the following lemma:

Lemma

Given is a quadrilateral ABCD whose opposing sides are not parallel ($AB \nparallel CD$, $BC \nparallel AD$).

The point E is the point of intersection of the diagonals.

The point F is the point of intersection of the continuations of the sides BC and AD.

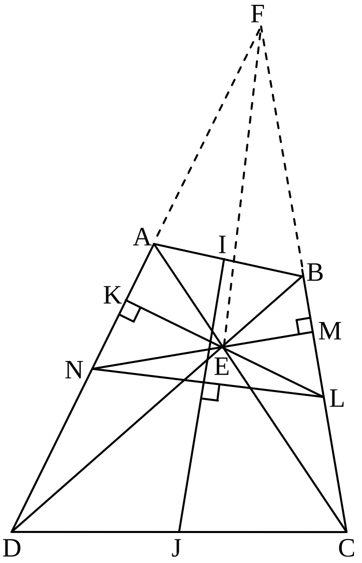

Figure 5

The points I and J are the midpoints of the sides AB and CD, respectively, as shown in figure 6.

Prove that the points E and F lie on both sides of the straight line that connects the points I and J.

The proof of the lemma is based on a theorem attributed to Menelaus [6]:

In a complete quadrilateral the midpoints of the diagonals are located on a straight line. The complete quadrilateral is ABCDFG, and the points L, M, N are the midpoints of the diagonals FG, AC, BD, and are located on a straight line, as shown in Figure 7. As shown in Figure 6, the complete quadrilateral is AFBECD, his diagonals are AB, DC, EF, and the midpoints of the diagonals are I, J, H. Therefore it is clear that the continuation of the straight line IJ intersects the diagonal EF, and the points E and F are on different sides of the straight line.

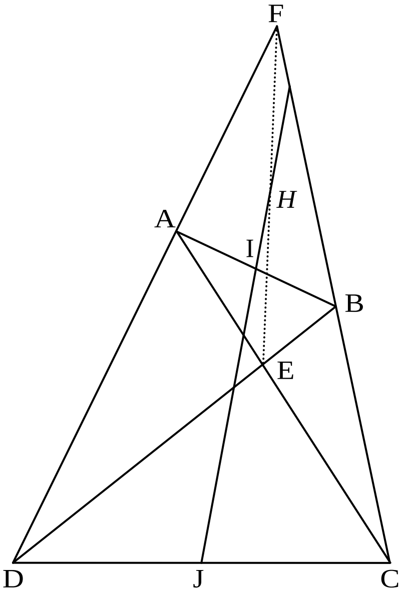

Figure 6

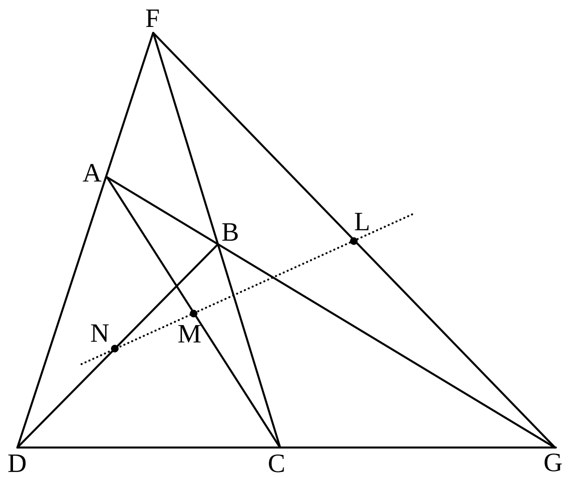

Figure 7

Proof of Theorem 4

We shall prove that when the rest of the data are taken into account, $AB \nparallel CD$ is not possible.

The point E is the point of intersection of the altitudes NM and KL in the triangle $\triangle FLN$, in which the vertex F is the point of intersection of the continuations of the sides BC and AD (Figure 5). Therefore the third altitude of the triangle lies on the straight line FE, and is therefore perpendicular to NL.

From the data, the straight line IJ is also perpendicular to NL, therefore $EF \parallel IJ$.

From the aforementioned lemma, the points E and F lie on different sides of the straight line IJ. Therefore EF and IJ intersect.

A contradiction has been obtained and therefore $AB \parallel CD$, and the quadrilateral ABCD is a trapezoid.

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