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On the Diophantine Equation $3^x + 5^y \cdot 19^z = u^2$

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Abstract

In this paper, we study the Diophantine equation $3^x + 5^y \cdot 19^z = u^2$. Using elementary methods we show that this Diophantine equation has exactly three solutions (*x,y,z,u*), namely: (1,0,0,2), (4,0,1,10) and (2,2,1,22).

Keywords: Exponential Diophantine equation; elementary methods; non-negative integer solutions.

MSC 2010 Classification: 11D61.

1. Introduction

Let *a*, *b* be positive integers. The Diophantine equation of type $a^x + b^y = z^2$ have been studied by some authors[1,3,4,6-9]. In 2012, Sroysang.B [6] showed that the only non-negative integer solution (*x,y,z*) to the Diophantine equation $3^x + 5^y = z^2$ is (1,0,2). In 2013, Rabago.J.F.T [3] proved that the only non-negative integer solutions (*x,y,z*) to the Diophantine equations $3^x + 19^y = z^2$ and $3^x + 91^y = z^2$ are (1,0,2), (4,1,10) and (1,0,2), (2,1,10) respectively. Let *a*, *b*, *c* be positive integers. In this paper, we study a special case of the Diophantine equation of type $a^x + b^y c^z = u^2$. Using elementary methods we show that Diophantine equation $3^x + 5^y 19^z = u^2$ has exactly three non-negative integer solutions (*x,y,z,u*), namely: (1,0,0,2), (4,0,1,10) and (2,2,1,22). Note that in the proofs of Theorems in [1] and [6-9], the result of Mihailescu.P [2] were used, while our proof is elementary.

2. Main result

Theorem 1. *The non-negative integer solutions* (*x*, *y*, *z*, *u*) *to the Diophantine equation*

$$3^x + 5^y 19^z = u^2 \qquad (2.1)$$

are (1,0,0,2), (4,0,1,10) *and* (2,2,1,22).

Proof. We consider four cases.

Case 1 *y* = *z* = 0. In this case we obtain a solution (*x,y,z,u*) = (1,0,0,2).

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Case 2 $y = 0$ and $z > 0$. Taking modulo 19 we have $3^x \equiv u^2 \pmod{19}$, which gives

$$(-1)^x = \left(\frac{3}{19}\right)^x = \left(\frac{u}{19}\right)^2 = 1$$

and we thus get $x = 2x_1$ .From $u^2 - 3^{2x_1} = 19^y$ we have

$$2 \cdot 3^{x_1} = 19^z - 1. \tag{2.2}$$

We consider two subcases .

Subcase 2.1 $z = 1$. In this subcase we obtain a solution $(x,y,z,u) = (4,0,1,10)$.

Subcase 2.2 $z > 1$. From $19^z - 1 = (19-1)(19^{z-1} + 19^{z-2} + \cdots + 19 + 1)$ we get

$$(19^{z-1} + 19^{z-2} + \cdots + 19 + 1) \equiv 1 + 1 + \cdots + 1 + 1 \equiv z \equiv 0 \pmod{3}$$

Let $z = 3z_1$ .Since

$$19^z - 1 = (19^3)^{z_1} - 1 = (19^3 - 1)\sum_{i=0}^{z_1-1}(19^3)^i = 2 \cdot 3^2 \cdot 127 \sum_{i=0}^{z_1-1}(19^3)^i$$

(2.2) has no solution.

Case 3 $y > 0$ and $z = 0$. In this case (2.1) becomes

$$3^x + 5^y = u^2. \tag{2.3}$$

Similar to the proof of Case 2, we get $x = 2x_1$. From $3^x + 5^y \equiv 2 \pmod 4$, $u^2 \equiv 0 \pmod 4$ it follows that (2.3) has no solution.

Case 4 $yz > 0$. In this case we have

$$(u - 3^{x_1})(u + 3^{x_1}) = 5^y \cdot 19^z.$$

We consider three subcases.

Subcase 4.1 $u - 3^{x_1} = 1, u + 3^{x_1} = 5^y 19^z$. In this subcase we have

$$2 \cdot 3^{x_1} = 5^y \cdot 19^z - 1. \tag{2.4}$$

Taking modulo 3, 4, 8 in turn (2.4) gives $y \equiv 0 \pmod 2$, $z \equiv 1 \pmod 2$ and $x_1 \equiv 0 \pmod 2$ respectively. Finally, from $2 \cdot 3^{x_1} \equiv 2,3 \pmod 5$, $5^y \cdot 19^z - 1 \equiv -1 \pmod 5$ it follows that (2.4) has no solution.

Subcase 4.2. $u - 3^{x_1} = 5^y, u + 3^{x_1} = 19^z$. In this subcase we have

$$2 \cdot 3^{x_1} = 19^z - 5^y. \tag{2.5}$$

Similar to the proof of subcase 4.1, taking (2.5) modulo 3, 4, 8, 5 in turn, we deduce that (2.5) has no solution.

Subcase 4.3. $u - 3^{x_1} = 19^z$, $u + 3^{x_1} = 5^y$. In this subcase we have

$$2 \cdot 3^{x_1} = 5^y - 19^z. \tag{2.6}$$

Taking modulo 3, 4, 8 in turn , (2.6) gives

$$y \equiv 0 \pmod 2,\ z \equiv 1 \pmod 2,\ x_1 \equiv 1 \pmod 2.$$

We show that $x_1 = 1$. Suppose $x_1 > 1$. If $y \equiv 0 \pmod 6$, let

$$x_1 \equiv 1,3,5 \pmod 6, z \equiv 1,3,5 \pmod 6,$$

taking modulo 7 (2.6) gives $x_1 \equiv 3 \pmod 6, z \equiv 5 \pmod 6$. Let $y \equiv 0,6 \pmod{12}, z \equiv 5,11 \pmod{12}$. From $2 \cdot 3^{x_1} \equiv 2 \pmod{13}$ and $5^y - 19^z \equiv 1,3,10,12 \pmod{13}$ we deduce that (2.6) has no solution. If $y \equiv 2,4 (\bmod 6)$, then from

$$5^y \equiv 7,4 (\bmod 9), 19^z \equiv 1 \pmod 9$$

we get $5^y - 19^z \equiv 6,3 \pmod 9$. But $2 \cdot 3^{x_1} \equiv 0 \pmod 9$, this is a contradiction. Thus $x_1 = 1$, and (2.6) becomes

$$5^y - 19^z = 6. \qquad (2.7)$$

Since $5^{6k} \equiv 1 \pmod 9, 19^z \equiv 1 \pmod 9$, taking modulo 9 (2.7) gives $y \equiv 2 \pmod 6$. Then taking modulo 7 (2.7) gives $z \equiv 1 (\bmod 6)$. Let

$$y \equiv 2,8 \ (\bmod 12),\ z \equiv 1,7 \pmod{12}.$$

Since

$$5^y \equiv 12,1 \pmod{13}, 19^z \equiv 6,7 \pmod{13}, 5^y - 19^z \equiv 6 \pmod{13},$$

taking (2.7) modulo 13 we obtain $y \equiv 2 \pmod{12}, z \equiv 1 \pmod{12}$.

Let

$$y \equiv 2,14,26,38,50 \pmod{60}, z \equiv 1,13,25,37,49 \pmod{60}.$$

Because

$$5^y \equiv 3,9,5,4,1 \pmod{11}, 19^z \equiv 8,6,10,2,7 \pmod{11},$$

taking modulo 11 (2.7) gives

$$y \equiv 2 \pmod{60}, z \equiv 1 \pmod{60};$$

or

$$y \equiv 26 \pmod{60}, z \equiv 25 \pmod{60};$$

or

$$y \equiv 50 \pmod{60}, z \equiv 13 \pmod{60}.$$

Taking (2.7) modulo 31 we obtain that $y \equiv 2 \pmod{60}, z \equiv 1 \pmod{60}$.Let

$$y \equiv 2,62,122,182,242 \pmod{300}, z \equiv 1,61,121,181,241 \pmod{300}.$$

Because

$$5^y \equiv 25,92,80,52,54 \pmod{101}, 19^z \equiv 19,37,88,81,78 \pmod{101},$$

taking (2.7) modulo 101 we obtain that $y \equiv 2 \pmod{300}$, $z \equiv 1 \pmod{300}$. Suppose $y = 2+300k$, $z = 1+300l$.

From 6=25-19 and (2.7) we have

$$25\left(5^{300 \cdot k} - 1\right) = 19\left(19^{300 \cdot l} - 1\right).$$

Then from $19^{300\cdot l}-1\equiv 0(\text{mod}125)$ we get k = 0, and thereby l = 0. Thus the only solution to (2.7) is (y, z) = (2,1), which implies that the only positive integer solution to (2.1) is (2,2,1,22). This completes the proof of Theorem 1.

Corollary 1. *The only nonnegative integer solution (x,y,z) to the Diophantine equation* $3^x+5^y=z^2$ *is* (1,0,2).

Corollary 2. *The only non-negative integer solutions (x,y,z) to the Diophantine equations* $3^x+19^y=z^2$ *are* (1,0,2), (4,1,10).

Remark. In 1993, Scott.R [5] showed that if p is prime, $b>1$ and c are positive integers, then except for five cases, the Diophantine equations $p^x-b^y=c$ has at most one solution. Using this result we can show that the only solution to (2.7) is (y,z) = (2,1). But the method used in [5] is not elementary.

Conclusions

In this paper, using elementary method we show that the Diophantine equation has exactly three solutions (x,y,z,u), namely: (1,0,0,2), (4,0,1,10) and (2,2,1,22). As corollaries, we get the result of Sroysang.B [6] and part of the results of Rabago.J.F.T [3].

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