Journal of Progressive Research in Mathematics(JPRM)
ISSN: 2395-0218

SCITECH
RESEARCH ORGANISATION|

Volume 5, Issue 2
Published online: September 09, 2015|

Journal of Progressive Research in Mathematics
www.scitecresearch.com/journals

# Blow-up for Semidiscretizations of some Semilinear Parabolic Equations with a Convection Term

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## Abstract

This paper concerns the study of the numerical approximation for the following parabolic equations with a convection term

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) - u(x,t)u_x(x,t) + u^p(x,t), \quad 0 < x < 1, \quad t > 0, \\ u_x(0,t) = 0, \quad u_x(1,t) = 0, \quad t > 0, \\ u(x,0) = u_0(x) > 0, \quad 0 \le x \le 1, \end{cases}$$

where $p > 1$.

We obtain some conditions under which the solution of the semidiscrete form of the above problem blows up in a finite time and estimate its semidiscrete blow-up time. We also prove that the semidiscrete blow-up time converges to the real one, when the mesh size goes to zero. Finally, we give some numerical experiments to illustrate ours analysis.

**Keywords:** Burgers' equation; semidiscretizations; discretizations; parabolic equations; convection term; blow-up; blow-up time; convergence.

## 1. Introduction

Consider the following boundary value problem

$$u_t(x,t) = u_{xx}(x,t) - u(x,t)u_x(x,t) + u^p(x,t), \quad 0 < x < 1, \quad t > 0, \tag{1}$$

$$u_x(0,t) = 0, \quad u_x(1,t) = 0, \quad t > 0, \tag{2}$$

$$u(x,0) = u_0(x) > 0, \quad 0 \le x \le 1, \tag{3}$$

where $p > 1, u_0 \in C^2([0,1])$, $u_0$ is nondecreasing on the interval $(0,1)$ and verifies

$$u_0'(0) = 0, \ u_0'(1) = 0, \tag{4}$$

$$u_0''(x) - u_0(x)u_0'(x) + u_0^p(x) \ge 0, \quad 0 \le x \le 1, \tag{5}$$

$$u_0(x) > -p(p-1)\, u_0'(x), \quad 0 < x < 1, \tag{6}$$

### 1.1 Definition

We say that the solution $u$ of (1)-(3) blows up in a finite time if there exists a finite time $T_b$ such that $\|u(.,t)\|_\infty < \infty$ for $t \in [0, T_b)$ but

$$\lim_{t \to T_b} \|u(.,t)\|_\infty = \infty.$$

The time $T_b$ is called the blow-up time of the solution $u$.

The above problem arises in fluid mechanics and is called viscous Burgers' equation in one dimension with a reaction term. The solution *u(x,t)* represents the motion field of the fluid in space and time. Burgers' equation with a reaction term is a transport equation with a convection term. The term $uu_x$ is called convection term. It's a nonlinear term that ensures the movement, generates instability and also responsible for the turbulent appearance (here we'll refer to it as intermittent since we are in one dimension) when it happens. In the general case the term $u_{xx}$ is replaced by $\nu u_{xx}$ with $\nu > 0$. The term $\nu u_{xx}$ is the viscous term, which has the opposite effect of slicking and making it appear laminar that is ordered. The constant ν, coefficient of the viscous term, is called the kinematic viscosity (normalized by the density) of the fluid. The fluid's flow ability is inversely proportional to the size of the viscosity. The term $u^p$ (the reaction term) is the external force which is generally a white and Gaussian noise within the time scale which forces the fluid to flow faster, slower or make it mill around. It's the quantitative relation between the convection term and viscous, called Reynolds number that will condition the appearance of the flow in the case when there is no external force. The Burgers' equation occurs in various areas of applied mathematics such as modelling of gas dynamics and traffic flow. It was in 1939 that the Dutch scientist Johannes Martinus Burgers simplified the Navier-Stokes equation by just dropping the pressure term (see [2], [23]).

The theoretical study of blow-up solutions for the parabolic equations with a convection term has been the subject of investigations of many authors (see [3], [6], [7], [8], [9],[19], [20], [21] and the references cited therein). Local in time existence and uniqueness of the solution have been proved(see [4], [5], [24], [26] and the references cited therein). Here, we are interesting in the numerical study using a semidiscrete form of (1)-(3). We give some assumptions under which the solution of a semidiscrete form of (1)-(3) blows up in a finite time and estimate its semidiscrete blow-up time. We also show that the semidiscrete blow-up time converges to the theoretical one when the mesh size goes to zero. A similar study has been undertaken in [1] and [26].

The paper is organized as follows. In the next section, we present a semidiscrete scheme of (1)-(3) and give some lemmas which will be used throughout the paper. In section 3, under some conditions, we prove that the solution of the semidiscrete form of (1)-(3) blows up in a finite time. In section 4, we study the convergence of the semidiscrete blow-up time. Finally, in last section, taking some discrete forms of (1)-(3), we give some numerical

experiments.

## 2. Properties of the semidiscrete scheme

In this section, we give some lemmas which will be used later. We start by the construction of the semidiscrete scheme. Let $I$ be a positive integer and let $h=1/I$. Define the grid $x_i=ih$, $0 \le i \le I$ and approximate the solution $u$ of (1)-(3) by the solution $U_h(t) = (U_0(t), U_1(t), \cdots U_I(t))^T$ of the following semidiscrete equations

$$\frac{dU_i(t)}{dt} = \delta^2 U_i(t) - U_i(t)\delta^0 U_i(t) + U_i^p(t), \quad 1 \le i \le I-1, \quad t \in (0, T_b^h), \tag{7}$$

$$\frac{dU_0(t)}{dt} = \delta^2 U_0(t) + U_0^p(t), \quad t \in (0, T_b^h), \tag{8}$$

$$\frac{dU_I(t)}{dt} = \delta^2 U_I(t) + U_I^p(t), \quad t \in (0, T_b^h), \tag{9}$$

$$U_i(0) = \varphi_i > 0, \quad 0 \le i \le I, \tag{10}$$

where

$$\delta^2 U_i(t) = \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{h^2}, \qquad 1 \le i \le I-1,$$

$$\delta^2 U_0(t) = \frac{2U_1(t) - 2U_0(t)}{h^2}, \quad \delta^2 U_I(t) = \frac{2U_{I-1}(t) - 2U_I(t)}{h^2},$$

$$\delta^0 U_i(t) = \frac{U_{i+1}(t) - U_{i-1}(t)}{2h}, \qquad 1 \le i \le I-1,$$

$$\delta^0 U_0(t) = 0, \quad \delta^0 U_I(t) = 0,$$

$$\delta^+ \varphi_i = \frac{\varphi_{i+1} - \varphi_i}{h}, \ 0 \le i \le I-1,$$

$$\delta^+ \varphi_i \le 0, \ 0 \le i \le I-1,$$

$$\varphi_i^{p-1} > -p(p-1)h\varphi_{i-1}^{p-2}\delta^0\varphi_i, \ 1 \le i \le I-1, \ p \ge 2.$$

Here, $\left(0, T_b^h\right)$ is the maximal time interval on which $\|U_h(t)\|_\infty$ is finite, where

$$\|U_h(t)\|_\infty = \max_{0 \le i \le I} |U_i(t)|.$$

When the time $T_b^h$ is finite, we say that the solution $U_h(t)$ of (8)-(10) blows up in a finite time, and the time $T_b^h$ is called the blow-up time of the solution $U_h(t)$.

**Lemma 2.1** Let $a_h(t), b_h(t) \in C^0\left([0,T), \Re^{I+1}\right)$ and let $V_h(t) \in C^1\left([0,T), \Re^{I+1}\right)$ where $b_h(t)\delta^0 V_h(t) \le 0$, such that

$$\frac{dV_i(t)}{dt} - \delta^2 V_i(t) + b_i(t)\delta^0 V_i(t) + a_i(t)V_i(t) \ge 0, \quad 0 \le i \le I, \quad t \in (0,T), \tag{11}$$

$$V_i(0) \ge 0, \quad 0 \le i \le I. \tag{12}$$

Then we have

$$V_i(t) \ge 0, \ 0 \le i \le I, \ t \in (0,T). \tag{13}$$

*Proof.* Let $T_0$ be any quantity satisfying the inequality $T_0 < T$ and define the vector $Z_h(t) = e^{\lambda t} V_h(t)$ where λ is such that

$$a_i(t) - \lambda > 0 \ for \ \ 0 \le i \le I, \quad t \in [0, T_0].$$

Let $m = \min_{0 \le i \le I, 0 \le t \le T_0} Z_i(t)$. Since, $Z_i(t)$ is a continuous vector on the compact $[0, T_0]$, there exists $i_0 \in \{0, \cdots, I\}$ and $t_0 \in [0, T_0]$ such that $m = Z_{i_0}(t_0)$. We observe that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \le 0, \quad 0 \le i_0 \le I, \tag{14}$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \ge 0, \ 1 \le i_0 \le I-1, \tag{15}$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_1(t_0) - 2Z_0(t_0)}{h^2} \ge 0 \ \ if \ \ i_0 = 0, \tag{16}$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_{I-1}(t_0) - 2Z_I(t_0)}{h^2} \ge 0 \ \ if \ \ i_0 = I. \tag{17}$$

From (11), we obtain the following inequality

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + b_{i_0}(t_0)\delta^0 Z_{i_0}(t_0) + (a_{i_0}(t_0) - \lambda)Z_{i_0}(t_0) \geq 0. \tag{18}$$

It follows from (14)-(18) that

$$(a_{i_0}(t_0) - \lambda)Z_{i_0}(t_0) \geq 0, \tag{19}$$

which implies that $Z_{i_0}(t_0) \geq 0$ because $a_{i_0}(t_0) - \lambda > 0$. We deduce that $V_h(t) \geq 0$ for $t \in [0, T_0]$ and the proof is complete.

**Lemma 2.2** Let $V_h(t), W_h(t) \in C^1([0,T], \Re^{I+1})$ and $f \in C^1(\Re \times \Re, \Re)$ such that

$$\frac{dV_i(t)}{dt} - \delta^2 V_i(t) + V_i(t)\delta^0 V_i(t) + f(V_i(t), t) < \frac{dW_i(t)}{dt} -$$
$$\delta^2 W_i(t) + W_i(t)\delta^0 W_i(t) + f(W_i(t), t), \ 0 \leq i \leq I, \ t \in (0, T), \tag{20}$$

$$V_i(0) < W_i(0), \quad 0 \leq i \leq I. \tag{21}$$

Then we have

$$V_i(t) < W_i(t), \ 0 \leq i \leq I, \ t \in (0, T).$$

*Proof.* Define the vector $Z_h(t) = W_h(t) - V_h(t)$. Let $t_0$ be the first $t > 0$ such that $Z_i(t) > 0$ for $t \in [0, t_0), 0 \leq i \leq I$, but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, \cdots, I\}$. We remark that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \leq 0, \quad 0 \leq i_0 \leq I,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \geq 0, \ 1 \leq i_0 \leq I - 1,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_1(t_0) - 2Z_0(t_0)}{h^2} \geq 0 \ \ if \ \ i_0 = 0,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_{I-1}(t_0) - 2Z_I(t_0)}{h^2} \geq 0 \ \ if \ \ i_0 = I,$$

Therefore, we have

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + W_{i_0}(t_0)\delta^0 W_{i_0}(t_0) - V_{i_0}(t_0)\delta^0 V_{i_0}(t_0) + f(V_{i_0}(t_0), t_0) - f(W_{i_0}(t_0), t_0) \leq 0,$$

which contradicts the first strict inequality of the lemma and this end the proof.

**Lemma 2.3** Let $U_h(t)$ be the solution of (7)-(10). Then, we have

$$U_i(t) > 0 \ \ for \ \ 0 \leq i \leq I, t \in (0, T_b^h). \tag{22}$$

*Proof.* Assume that there exists a time $t_0 \in (0, T_b^h)$ such that $U_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, \cdots, I\}$. We observe that

$$\frac{dU_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{U_{i_0}(t_0) - U_{i_0}(t_0 - k)}{k} \leq 0, \quad 0 \leq i_0 \leq I,$$

$$\delta^2 U_{i_0}(t_0)=\frac{U_{i_0+1}(t_0)-2U_{i_0}(t_0)+U_{i_0-1}(t_0)}{h^2}>0,\ 1\le i_0\le I-1,$$

$$\delta^2 U_{i_0}(t_0)=\frac{2U_1(t_0)-2U_0(t_0)}{h^2}>0\ \ if\ \ i_0=0,$$

$$\delta^2 U_{i_0}(t_0)=\frac{2U_{I-1}(t_0)-2U_I(t_0)}{h^2}>0\ \ if\ \ i_0=I.$$

which implies that

$$\frac{dU_0(t_0)}{dt}-\delta^2 U_0(t_0)-U_0^p(t_0)<0,$$

$$\frac{dU_{i_0}(t_0)}{dt}-\delta^2 U_{i_0}(t_0)+U_{i_0}(t_0)\delta^0 U_{i_0}(t_0)-U_{i_0}^p(t_0)<0,\quad 1\le i_0\le I-1,$$

$$\frac{dU_I(t_0)}{dt}-\delta^2 U_I(t_0)-U_I^p(t_0)<0.$$

But these inequalities contradict (7)-(9) and we obtain the desired result.

**Lemma 2.4** Let $U_h(t)$ be the solution of (7)-(10). Then, we have

$$U_{i+1}(t)<U_i(t)\ \ for\ \ 0\le i\le I-1,\ t\in\left(0,T_b^h\right). \tag{23}$$

*Proof.* Introduce the vector $Z_h(t)$ defined as follows $Z_i(t)=U_{i+1}(t)-U_i(t)$ for $0\le i\le I-1$. Let $t_0$ be the first $t>0$ such that $Z_i(t)<0$ for $t\in[0,t_0)$ but $Z_{i_0}(t_0)=0$ for a certain $i_0\in\{0,\cdots,I-1\}$. Without loss of generality, we may suppose that $i_0$ is the smallest integer which satisfies the above equality. It follows that

$$\frac{dZ_{i_0}(t_0)}{dt}=\lim_{k\to 0}\frac{Z_{i_0}(t_0)-Z_{i_0}(t_0-k)}{k}\ge 0,\quad 0\le i_0\le I-1,$$

$$\delta^2 Z_{i_0}(t_0)=\frac{Z_{i_0+1}(t_0)-2Z_{i_0}(t_0)+Z_{i_0-1}(t_0)}{h^2}<0,\ 1\le i_0\le I-1,$$

$$\delta^2 Z_{i_0}(t_0)=\frac{2Z_1(t_0)-2Z_0(t_0)}{h^2}<0\ \ if\ \ i_0=0,$$

which implies that

$$\frac{dZ_{i_0}(t_0)}{dt}-\delta^2 Z_{i_0}(t_0)+U_{i_0+1}(t_0)\delta^0 U_{i_0+1}(t_0)-U_{i_0}(t_0)\delta^0 U_{i_0}(t_0)+U_{i_0}^p(t_0)-U_{i_0+1}^p(t_0)>0,\quad 1\le i_0\le I-1,$$

$$\frac{dZ_0(t_0)}{dt}-\delta^2 Z_0(t_0)+U_0^p(t_0)-U_1^p(t_0)>0.$$

Therefore, we have a contradiction because of (7)-(8). This ends the proof.

**Lemma 2.5** Let $U_h(t)$ be the solution of (7)-(10). Then, we have

$$\frac{dU_i(t)}{dt}>0\ \ for\ \ 0\le i\le I,\ t\in\left(0,T_b^h\right).$$

*Proof.* Consider the vector $Z_h(t)$ with $Z_i(t) = \frac{dU_i(t)}{dt}$, $0 \le i \le I$. Let $t_0$ be the first $t > 0$ such that $Z_i(t) > 0$ for $t \in [0, t_0)$ but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, \cdots, I\}$. Without loss of generality, we may suppose that $i_0$ is the smallest integer which satisfies the above equality. We get

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \le 0, \quad 0 \le i_0 \le I,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} > 0, \; 1 \le i_0 \le I-1,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_1(t_0) - 2Z_0(t_0)}{h^2} > 0 \;\; if \;\; i_0 = 0,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_{I-1}(t_0) - 2Z_0(t_0)}{h^2} > 0 \;\; if \;\; i_0 = I,$$

which implies that

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + U_{i_0}(t_0)\delta^0 Z_{i_0}(t_0) + (\delta^0 U_{i_0}(t_0) - pU_{i_0}^{p-1}(t_0))Z_{i_0}(t_0) < 0 \;\; if \;\; 1 \le i_0 \le I-1,$$

$$\frac{dZ_0(t_0)}{dt} - \delta^2 Z_0(t_0) - pU_0^{p-1}(t_0) < 0,$$

$$\frac{dZ_I(t_0)}{dt} - \delta^2 Z_I(t_0) - pU_I^{p-1}(t_0) < 0.$$

But these inequalities contradict (7)-(9) and leads to the desired result.

**Lemma 2.6** Let $U_h(t)$ be the solution of (7)-(10). Then, we have, $for \; p \ge 2,$

$$U_i^{p-1}(t) > -p(p-1)hU_{i-1}^{p-2}(t)\delta^0 U_i(t) \;\; for \;\; 1 \le i \le I-1, \; t \in (0, T_b^h).$$

*Proof.* Define the vectors $Z_h(t), K_h(t) \; and \; V_h(t)$ such that $Z_i(t) = K_i(t) - V_i(t)$ with $K_i(t) = U_i^{p-1}(t) \; and \; V_i(t) = -p(p-1)hU_{i-1}^{p-2}(t)\delta^0 U_i(t) \; for \; 1 \le i \le I-1$. Let $t_0$ be the first $t > 0$ such that $Z_i(t) > 0$ for $t \in [0, t_0)$ but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{1, \cdots, I-1\}$. We may suppose that $i_0$ is the smallest integer which satisfies the above equality. It follows that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \le 0, \quad 1 \le i_0 \le I-1,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} > 0, \; 1 \le i_0 \le I-1,$$

which implies that

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + K_{i_0}(t_0)\delta^0 K_{i_0}(t_0) - V_{i_0}(t_0)\delta^0 V_{i_0}(t_0) + V_{i_0}^p(t_0) - K_{i_0}^p(t_0) < 0, \; 1 \le i_0 \le I-1,$$

But this inequality contradicts (7) and we obtain the desired result.

**Lemma 2.7** Let $U_h \in C^1([0,T], \Re^{I+1}) \; such \; that \; U_h > 0.$ Then, we have,

$$\delta^2 U_i^p \geq p U_i^{p-1} \delta^2 U_i \quad for \quad 0 \leq i \leq I, \quad p \geq 2.$$

*Proof.* Using Taylor's expansion, we get

$$\delta^2 U_0^p = p U_0^{p-1} \delta^2 U_0 + (U_1 - U_0)^2 \frac{p(p-1)}{2h^2} \theta_0^{p-2},$$

$$\delta^2 U_i^p = p U_i^{p-1} \delta^2 U_i + (U_{i+1} - U_i)^2 \frac{p(p-1)}{2h^2} \xi_i^{p-2} + (U_{i-1} - U_i)^2 \frac{p(p-1)}{2h^2} \theta_i^{p-2}, \ 1 \leq i \leq I-1,$$

$$\delta^2 U_I^p = p U_I^{p-1} \delta^2 U_I + (U_{I-1} - U_I)^2 \frac{p(p-1)}{2h^2} \theta_I^{p-2},$$

Where $p \geq 2, \theta_0 \in (U_1, U_0), \theta_i \in (U_i, U_{i-1}), \xi_i \in (U_{i+1}, U_i)$ *and* $\theta_I \in (U_I, U_{I-1})$.

The result follows taking into account the fact that $U_h > 0$.

**Lemma 2.8** Let $U_h \in C^1([0,T], \Re^{I+1})$ *such that* $U_h > 0$. Then, we have,

$$-U_i \delta^0 U_i^p \geq -p U_i^p \delta^0 U_i - p(p-1) h U_{i-1}^{p-2} (\delta^0 U_i)^2, \ 1 \leq i \leq I-1, \ p \geq 2.$$

*Proof.* Applying Taylor's expansion, we obtain

$$\delta^0 U_i^p = p U_{i-1}^{p-1} \delta^0 U_i + (U_{i+1} - U_{i-1})^2 \frac{p(p-1)}{4h} U_{i-1}^{p-2}, \ 1 \leq i \leq I-1, \ p = 2,$$

$$\delta^0 U_i^p = p U_{i-1}^{p-1} \delta^0 U_i + (U_{i+1} - U_{i-1})^2 \frac{p(p-1)}{4h} U_{i-1}^{p-2} + (U_{i+1} - U_{i-1})^3 \frac{p(p-1)(p-2)}{12h} \zeta_i^{p-3}, 1 \leq i \leq I-1, \ p \geq 3,$$

Where $\zeta_i \in (U_{i+1}, U_{i-1})$.

Using Lemma 2.4 and $U_h > 0$, we have the desired result.

## 3. Semidiscrete Blow-up solutions

In this section under some assumptions, we show that the solution $U_h$ of (7)-(10) blows up in a finite time and estimate its semidiscrete blow-up time.

**Theorem 3.1** Let $U_h$ be the solution of (7)-(10), then the solution $U_h$ blows up in a finite time $T_b^h$ with following estimate

$$T_b^h \leq \frac{1}{(p-1)} \frac{1}{(\min_{0 \leq i \leq I}(\varphi_i))^{p-1}}. \tag{24}$$

*Proof.* Consider the following differential equation

$$\dot{\alpha}(t) = \alpha^p(t), t \in (0, T_\alpha), \ p \geq 2, \tag{25}$$

$$\alpha(0) = \min_{0 \leq i \leq I}(\varphi_i), \tag{26}$$

with $T_\alpha = \dfrac{1}{(p-1)} \dfrac{1}{(\min_{0 \leq i \leq I}(\varphi_i))^{p-1}}$.

Introduce the vector $V_h(t)$ such that $V_i(t)=\alpha(t), 0\le i\le I,\ t\in(0,T_\alpha)$. Let the vector $Z_h(t)$ define as follow $Z_h(t)=U_h(t)-V_h(t)$. It not hard to see that

$$\frac{dZ_i(t)}{dt}-\delta^2 Z_i(t)+U_i(t)\delta^0 Z_i(t)+(\delta^0 V_i(t)-p\beta_i^{p-1}(t))Z_i(t)\ge 0\ if\quad 0\le t\le I, t\in(0,T_1),$$

$$Z_i(0)=0,$$

where $\beta_i(t)\in(V_i(t),U_i(t))\ and\ T_1=\min\{T_\alpha,T_b^h\}$.

Due to Lemma 2.2, we have $U_i(t)\ge V_i(t),\ \ 0\le t\le I,\ \ t\in(0,T_1)$. We deduce that

$$T_b^h\le T_\alpha\le\frac{1}{(p-1)}\frac{1}{(\min_{0\le i\le I}(\varphi_i))^{p-1}}.$$

The following theorem gives a best result than the previous.

**Theorem 3.2** Let $U_h$ be the solution of (7)-(10). Suppose that there exists a positive integer λ such that

$$\delta^2 U_i(0)-U_i(0)\delta^0 U_i(0)+U_i^p(0)\ge\lambda U_i^p(0),\quad 0\le i\le I. \tag{27}$$

Then, the solution $U_h$ blows up in a finite time $T_b^h$ and we have the following estimate

$$T_b^h\le\frac{1}{\lambda}\frac{\|U_h(0)\|_\infty^{1-p}}{(p-1)}.$$

*Proof.* Let $(0,T_p^h)$ be the maximal time interval on which $\|U_h(t)\|_\infty<\infty$. Our aim is to show that $T_b^h$ is finite and satisfies the above inequality. Introduce the vector $J_h(t)$ such that

$$J_i(t)=\frac{dU_i(t)}{dt}-\lambda U_i^p(t),\quad 0\le i\le I. \tag{28}$$

A straightforward calculation gives

$$\frac{dJ_i}{dt}-\delta^2 J_i+U_i\delta^0 J_i=\frac{d^2U_i}{dt^2}-\lambda pU_i^{p-1}\frac{dU_i}{dt}-\delta^2\frac{dU_i}{dt}+\lambda\delta^2 U_i^p+U_i\delta^0(\frac{dU_i}{dt})-\lambda U_i\delta^0 U_i^p, 1\le i\le I-1.$$

From Lemma 2.7, we have $\delta^2 U_i^p\ge pU_i^{p-1}\delta^2 U_i\quad for\quad 0\le i\le I,\ \ p\ge 2$, which implies that

$$\frac{dJ_i}{dt}-\delta^2 J_i+U_i\delta^0 J_i\ge\frac{d}{dt}(\frac{d}{dt}U_i-\delta^2 U_i)-\lambda pU_i^{p-1}(\frac{d}{dt}U_i-\delta^2 U_i)+U_i\delta^0(\frac{d}{dt}U_i)-\lambda U_i\delta^0 U_i^p, 1\le i\le I-1.$$

Using (7)-(9), we arrive at

$$\frac{dJ_i}{dt}-\delta^2 J_i+U_i\delta^0 J_i+(\delta^0 U_i-pU_i^{p-1})J_i\ge-\lambda U_i\delta^0 U_i^p+\lambda pU_i^p\delta^0 U_i-\lambda U_i^p\delta^0 U_i, 1\le i\le I-1.$$

$$\frac{dJ_0}{dt}-\delta^2 J_0-pU_0^{p-1}J_0\ge 0,$$

$$\frac{dJ_I}{dt}-\delta^2 J_I-pU_I^{p-1}J_I\ge 0.$$

Using Lemma 2.8 we get

$$\frac{dJ_i}{dt}-\delta^2 J_i+U_i\delta^0 J_i+(\delta^0 U_i-pU_i^{p-1})J_i\geq -\lambda U_i\delta^0 U_i(U_i^{p-1}+p(p-1)hU_{i-1}^{p-2}\delta^0 U_i),1\leq i\leq I-1,$$

$$\frac{dJ_0}{dt}-\delta^2 J_0-pU_0^{p-1}J_0\geq 0,$$

$$\frac{dJ_I}{dt}-\delta^2 J_I-pU_I^{p-1}J_I\geq 0.$$

From Lemma 2.6, we have $U_i^{p-1}>-p(p-1)hU_{i-1}^{p-2}\delta^0 U_i \quad for \quad 1\leq i\leq I-1$ and using the fact that $-\lambda U_i\delta^0 U_i\geq 0$, we get finally

$$\frac{dJ_i}{dt}-\delta^2 J_i+U_i\delta^0 J_i+(\delta^0 U_i-pU_i^{p-1})J_i\geq 0,\ 1\leq i\leq I-1,$$

$$\frac{dJ_0}{dt}-\frac{(2J_1-2J_0)}{h^2}-pU_0^{p-1}J_0\geq 0,$$

$$\frac{dJ_I}{dt}-\frac{(2J_{I-1}-2J_I)}{h^2}-pU_I^{p-1}J_I\geq 0.$$

From (27), we observe that

$$J_i(0)=\delta^2 U_i(0)-U_i(0)\delta^0 U_i(0)+U_i^p(0)\geq\lambda U_i^p(0),\quad 0\leq i\leq I.$$

We deduce from Lemma 2.1 that $J_h(t)\geq 0 \quad for \quad t\in(0,T_b^h)$, which implies that

$$\frac{dU_i(t)}{dt}\geq\lambda U_i^p(t),\quad 0\leq i\leq I,\ t\in(0,T_b^h). \tag{29}$$

These estimates may be rewritten in the following form

$$U_i^{-p}dU_i\geq\lambda dt,\quad 0\leq i\leq I.$$

Integrating the above inequalities over $(t,T_b^h)$, we arrive at

$$T_b^h-t\leq\frac{1}{\lambda}\frac{(U_i(t))^{1-p}}{(p-1)}. \tag{30}$$

which implies that

$$T_b^h\leq\frac{1}{\lambda}\frac{\|U_h(0)\|_\infty^{1-p}}{(p-1)}.$$

**Remark 3.1** The inequalities (30) implies that

$$T_b^h-t_0\leq\frac{1}{\lambda}\frac{\|U_h(t_0)\|_\infty^{1-p}}{(p-1)}\quad if \quad 0<t_0<T_b^h.$$

## 4. Convergence of the semidiscrete blow-up time

In this section, under some assumptions, we show that the semidiscrete blow-up time converges to the real one when the mesh size goes to zero. In order to obtain the convergence of semidiscrete blow-up time, we firstly prove the following theorem about the convergence of the semidiscrete scheme.

**Theorem 4.1** Assume that (1)-(3) has a solution $u\in C^{4,1}([0,1]\times[0,T])$ and the initial condition at (10) satisfies

$$\left\|U_h^0 - u_h(0)\right\|_\infty = o(1) \text{ as } h \to 0, \tag{31}$$

Where $u_h(t) = (u(x_0,t),\cdots,u(x_I,t))^T$ . Then, for *h* sufficiently small, the problem (7)-(10) has a unique solution $U_h \in C^1\left(\left[0,T_b^h\right],\Re^{I+1}\right)$ such that

$$\max_{0\le t\le T}\left\|U_h(t) - u_h(t)\right\|_\infty = O(\left\|U_h^0 - u_h(0)\right\|_\infty + h^2) \text{ as } h \to 0. \tag{32}$$

*Proof.* Let $K > 0$ be such that

$$\|u\|_\infty \le K. \tag{33}$$

The problem (7)-(10) has for each *h*, a unique solution $U_h \in C^1\left(\left[0,T_b^h\right],\Re^{I+1}\right)$. Let *t(h)* the greatest value of *t>0* such that

$$\left\|U_h(t) - u_h(t)\right\|_\infty < 1 \text{ for } t \in (0,t(h)). \tag{34}$$

The relation (31) implies that *t(h)>0* for *h* sufficiently small. Let $t^*(h) = \min\{t(h),T\}$. By the triangular inequality, we obtain

$$\left\|U_h(t)\right\|_\infty \le \left\|u(.,t)\right\|_\infty + \left\|U_h(t) - u_h(t)\right\|_\infty \text{ for } t \in (0,t^*(h)),$$

which implies that

$$\left\|U_h(t)\right\|_\infty \le 1 + K \text{ for } t \in (0,t^*(h)). \tag{35}$$

Let $e_h(t) = U_h(t) - u_h(t)$ be the error of discretization. Using Taylor's expansion, we have for $t \in (0,t^*(h))$,

$$\frac{de_i(t)}{dt} - \delta^2 e_i(t) + u(x_i,t)\delta^0 e_i(t) = p\beta_i^{p-1}(t)e_i(t) - e_i(t)\delta^0 u(x_i,t) - \frac{h^2}{6}u(x_i,t)u_{xxx}(\tilde{x}_i,t),$$

$$\frac{de_0(t)}{dt} - \frac{(2e_1(t) - 2e_0(t))}{h^2} = p\beta_0^{p-1}(t)e_0(t) + \frac{h^2}{12}u_{xxxx}(\tilde{x}_0,t),$$

$$\frac{de_I(t)}{dt} - \frac{(2e_{I-1}(t) - 2e_I(t))}{h^2} = p\beta_I^{p-1}(t)e_I(t) + \frac{h^2}{12}u_{xxxx}(\tilde{x}_I,t),$$

where $\beta_i \in (U_i(t),u(x_i,t))$ *for* $i \in \{0,\ldots,I\}$.

Using (35), there exists a constant *M>0* such that

$$\frac{de_i(t)}{dt} - \delta^2 e_i(t) + u(x_i,t)\delta^0 e_i(t) \le M\left|e_i(t)\right| + Mh^2,\ 1 \le i \le I-1, \tag{36}$$

$$\frac{de_0(t)}{dt} - \frac{(2e_1(t) - 2e_0(t))}{h^2} \le M\left|e_0(t)\right| + Mh^2, \tag{37}$$

$$\frac{de_I(t)}{dt} - \frac{(2e_{I-1}(t) - 2e_I(t))}{h^2} \le M\left|e_I(t)\right| + Mh^2. \tag{38}$$

Consider the vector $W_h$ such that

$$W_i(t) = e^{(M+1)t}(\left\|U_h^0 - u_h(0)\right\|_\infty + Mh^2),\ 0 \le i \le I.$$

A direct calculation yields

$$\frac{dW_i(t)}{dt} - \delta^2 W_i(t) + u(x_i,t)\delta^0 W_i(t) > M|W_i(t)| + Mh^2,\ 1 \le i \le I-1, \tag{39}$$

$$\frac{dW_0(t)}{dt} - \frac{(2W_1(t) - 2W_0(t))}{h^2} > M|W_0(t)| + Mh^2, \tag{40}$$

$$\frac{dW_I(t)}{dt} - \frac{(2W_{I-1}(t) - 2W_I(t))}{h^2} > M|W_I(t)| + Mh^2, \tag{41}$$

$$W_i(0) > e_i(0),\ \ 0 \le i \le I. \tag{42}$$

It follows from Lemma 2.2 that

$$W_i(t) > e_i(t) \text{ for } t \in (0, t^*(h)),\ 0 \le i \le I.$$

By the same way, we also prove that

$$W_i(t) > -e_i(t) \text{ for } t \in (0, t^*(h)),\ 0 \le i \le I,$$

which implies that

$$W_i(t) > |e_i(t)| \text{ for } t \in (0, t^*(h)),\ 0 \le i \le I.$$

We deduce that

$$\|U_h(t) - u_h(t)\|_\infty \le e^{(M+1)T}(\|U_h^0 - u_h(0)\|_\infty + Mh^2),\ t \in (0, t^*(h)).$$

Let us show that $t^*(h) = T$. Suppose that $T > t^*(h)$. From (34), we obtain

$$1 = \|U_h(t(h)) - u_h(t(h))\|_\infty \le e^{(M+1)T}(\|U_h^0 - u_h(0)\|_\infty + Mh^2). \tag{43}$$

Since $e^{(M+1)T}(\|U_h^0 - u_h(0)\|_\infty + Mh^2) \to 0$ *when* $h \to 0$, we deduce from (43) that 1≤0, which is impossible. Consequently $t^*(h) = T$, and we conclude the proof.

**Theorem 4.2** Suppose that the solution u of (1)-(3) blows up in a finite time $T_b$ such that $u \in C^{4,1}([0,1] \times [0,T_b])$ and the initial condition at (10) satisfies

$$\|U_h^0 - u_h(0)\|_\infty = o(1) \text{ as } h \to 0. \tag{44}$$

Assume that there exists a constant λ >0 such that

$$\delta^2 U_i(0) - U_i(0)\delta^0 U_i(0) + U_i^p(0) \ge \lambda U_i^p(0),\quad 0 \le i \le I. \tag{45}$$

Then the solution $U_h$ of (7)-(10) blows up in a finite time $T_b^h$ and

$$\lim_{h\to 0} T_b^h = T_b. \tag{46}$$

*Proof*. Let $\varepsilon > 0$. There exists N such that

$$\frac{1}{\lambda}\frac{y^{1-p}}{(p-1)} \le \frac{\varepsilon}{2} < \infty \ \ for \quad y \in [N, +\infty[. \tag{47}$$

Since $\lim_{t\to T_b} \max_{x\in[0,1]} |u(x,t)| = +\infty$, then, there exists $T_1$ such that

$|T_1 - T_b| \le \frac{\varepsilon}{2}$ *and* $\|u(x,t)\|_\infty \ge 2N$ *for* $t \in [T_1, T_b]$. Let $T_2 = \frac{T_1 + T_b}{2}$, then $\sup_{t\in[0,T_2]}|u(x,t)| < +\infty$.

It follows from Theorem 4.1 that $\sup_{t\in[0,T_2]}|U_h(t) - u_h(t)| < N$. Applying the triangular inequality, we get

$\|U_h(t)\|_\infty \ge \|u(.,t)\|_\infty - \|U_h(t) - u_h(t)\|_\infty$, which leads to $\|U_h(t)\|_\infty \ge N$ for $t \in [0, T_2]$.

From Theorem 3.2, $U_h(t)$ blows up at the time $T_b^h$. We deduce from Remark 3.1 and (47) that

$$|T_b - T_b^h| \le |T_b - T_2| + |T_b^h - T_2| \le \frac{\varepsilon}{2} + \frac{1}{\lambda}\frac{\|U_h(T_2)\|_\infty^{1-p}}{(p-1)} \le \varepsilon,$$

which leads us to the desired result.

## 5. Numerical results

In this section, we present some numerical approximations to the blow-up time of (1)-(3). We use the following explicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2} - U_i^{(n)}\frac{U_{i+1}^{(n)} - U_{i-1}^{(n)}}{2h} + (U_i^{(n)})^p, \quad 1 \le i \le I-1,$$

$$\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} = \frac{2U_1^{(n)} - 2U_0^{(n)}}{h^2} + (U_0^{(n)})^p,$$

$$\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n} = \frac{2U_{I-1}^{(n)} - 2U_I^{(n)}}{h^2} + (U_I^{(n)})^p,$$

where $n \ge 0$, $p \ge 2$, $\Delta t_n = \min\left\{\frac{h^2}{2}, \tau\|U_h^{(n)}\|_\infty^{1-p}\right\}$ *with* $\tau = cont \in (0,1)$.

Also we use the implicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \frac{U_{i+1}^{(n+1)} - 2U_i^{(n+1)} + U_{i-1}^{(n+1)}}{h^2} - U_i^{(n)}\frac{U_{i+1}^{(n+1)} - U_{i-1}^{(n+1)}}{2h} + (U_i^{(n)})^p, \quad 1 \le i \le I-1,$$

$$\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} = \frac{2U_1^{(n+1)} - 2U_0^{(n+1)}}{h^2} + (U_0^{(n)})^p,$$

$$\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n} = \frac{2U_{I-1}^{(n+1)} - 2U_I^{(n+1)}}{h^2} + (U_I^{(n)})^p,$$

where $n \ge 0$, $p \ge 2$, $\Delta t_n = \tau\|U_h^{(n)}\|_\infty^{1-p}$ *with* $\tau = cont \in (0,1)$.

In the tables 1-8, in rows, we present the numerical blow-up times, numbers of iterations, the CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128, 256, 512, 1024. The numerical blow-up time $T^n = \sum_{j=0}^{n-1}\Delta t_j$ is computed at the first time when $\Delta t_n = |T^{n+1} - T^n| \le 10^{-16}$. The order(s) of the method is computed from $s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}$.

**First case:** $U_i^{(0)} = \frac{1}{2},\ p = 2\ and\ \tau = \frac{h^2}{2}.$

**Table 1**: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

| I | $T^n$ | n | CPUtime | s |
|---|---|---|---|---|
| 16 | 2.003306 | 15788 | - | - |
| 32 | 2.000827 | 60265 | - | - |
| 64 | 2.000207 | 229656 | 2 | 1.99 |
| 128 | 2.000052 | 873150 | 9 | 1.99 |
| 256 | 2.000013 | 3310849 | 63 | 2.00 |
| 512 | 2.000003 | 12516533 | 464 | 2.00 |
| 1024 | 2.000001 | 47158825 | 3458 | 2.00 |

**Table 2**: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

| I | $T^n$ | n | CPUtime | s |
|---|---|---|---|---|
| 16 | 2.003906 | 15631 | - | - |
| 32 | 2.000977 | 59637 | - | - |
| 64 | 2.000244 | 227142 | 2 | 2.00 |
| 128 | 2.000061 | 863093 | 12 | 2.00 |
| 256 | 2.000015 | 3270629 | 89 | 2.00 |
| 512 | 2.000004 | 12355655 | 672 | 2.00 |
| 1024 | 2.000001 | 46515309 | 5159 | 2.00 |

**Second case:** $U_i^{(0)} = 2,\ p = 2\ and\ \tau = \frac{h^2}{2}.$

**Table 3**: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

| I | $T^n$ | n | CPUtime | s |
|---|---|---|---|---|
| 16 | 0.500977 | 15631 | - | - |
| 32 | 0.500244 | 59637 | - | - |
| 64 | 0.500061 | 227142 | 2 | 2.00 |
| 128 | 0.500015 | 863093 | 9 | 2.00 |
| 256 | 0.500004 | 3270629 | 62 | 2.00 |
| 512 | 0.500001 | 12355655 | 457 | 2.00 |
| 1024 | 0.500000 | 46515309 | 3407 | 2.00 |

**Table 4**: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

| I | $T^n$ | n | CPUtime | s |
|---|---|---|---|---|
| 16 | 0.500977 | 15631 | - | - |
| 32 | 0.500244 | 59637 | - | - |
| 64 | 0.500061 | 227142 | 2 | 2.00 |
| 128 | 0.500015 | 863093 | 12 | 2.00 |
| 256 | 0.500004 | 3270629 | 90 | 2.00 |
| 512 | 0.500001 | 12355655 | 674 | 2.00 |
| 1024 | 0.500000 | 46515309 | 5070 | 2.00 |

**Third case:** $U_i^{(0)} = \left(\frac{1}{2}\right)^{3-p} + \left(1-(ih)^2\right)^2,\ p = 2\ and\ \tau = \frac{h^2}{2}.$

**Table 5**: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

| I | $T^n$ | n | CPUtime | s |
|---|---|---|---|---|
| 16 | 0.861878 | 16031 | - | - |
| 32 | 0.860194 | 60831 | - | - |
| 64 | 0.859773 | 231919 | 1 | 2.00 |
| 128 | 0.859668 | 882201 | 9 | 2.00 |
| 256 | 0.859641 | 3347050 | 66 | 2.00 |
| 512 | 0.859635 | 12661342 | 488 | 2.00 |
| 1024 | 0.859633 | 47738061 | 3670 | 2.00 |

**Table 6**: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

| I | $T^n$ | n | CPUtime | s |
|---|---|---|---|---|
| 16 | 0.861680 | 15961 | - | - |
| 32 | 0.860144 | 60832 | - | - |
| 64 | 0.859761 | 231919 | 2 | 2.00 |
| 128 | 0.859665 | 882201 | 13 | 2.00 |
| 256 | 0.859641 | 3347052 | 93 | 2.00 |
| 512 | 0.859635 | 12661342 | 699 | 2.00 |
| 1024 | 0.859633 | 47738061 | 5294 | 2.00 |

**Fourth case:** $U_i^{(0)} = \left(\frac{1}{2}\right)^{3-p} + \left(1-(ih)^2\right)^2,\ p = 4\ and\ \tau = \frac{h^2}{2}.$

**Table 7**: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

| I | $T^n$ | n | CPUtime | s |
|---|---|---|---|---|
| 16 | 0.012839 | 4693 | - | - |
| 32 | 0.012798 | 17825 | - | - |
| 64 | 0.012788 | 67544 | - | 1.98 |
| 128 | 0.012786 | 255190 | 3 | 1.99 |
| 256 | 0.012785 | 960804 | 18 | 1.99 |
| 512 | 0.012785 | 3603268 | 135 | 1.99 |
| 1024 | 0.012785 | 13452596 | 1002 | 1.99 |

**Table 8**: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

| I | $T^n$ | n | CPUtime | s |
|---|---|---|---|---|
| 16 | 0.012841 | 4694 | - | - |
| 32 | 0.012799 | 17825 | - | - |
| 64 | 0.012788 | 67545 | 1 | 1.98 |
| 128 | 0.012786 | 255190 | 4 | 1.99 |
| 256 | 0.012785 | 960804 | 27 | 1.99 |
| 512 | 0.012785 | 3603268 | 197 | 1.99 |
| 1024 | 0.012785 | 13452596 | 1490 | 1.99 |

In the following, we also give some plots to illustrate our analysis. For the different plots, we used both explicit and implicit schemes in the case where *I=16* and *p=2*. In Figures 1 and 2, we can appreciate that the discrete solution blows up in a finite time where the initial data is a constant. In Figures 3 and 4, we see that the blow-up is faster when the initial data is not a constant. The Figures 5, 6, 7 and 8 show the effect of the convection term on the evolution of the solution. In Figures 9, 10, 11 and 12, we observe that the solution of our problem blows up in a finite time $t \approx 2$ when the initial data is $\frac{1}{2}$ and $t \approx 0.86$ when the initial data is

$$\left(\frac{1}{2}\right)^{3-p} + \left(1-(ih)^2\right)^2.$$

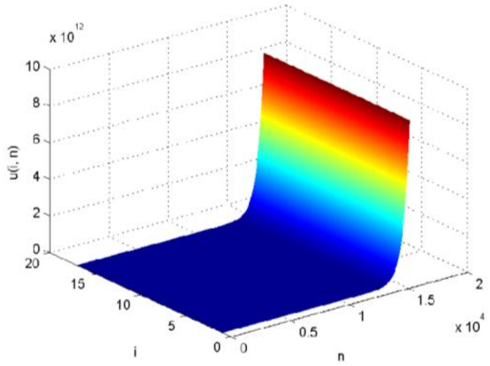


**Figure 1**: Evolution of the discrete solution(Explicit scheme)

$$U_i^{(0)} = 2,\ p = 2$$

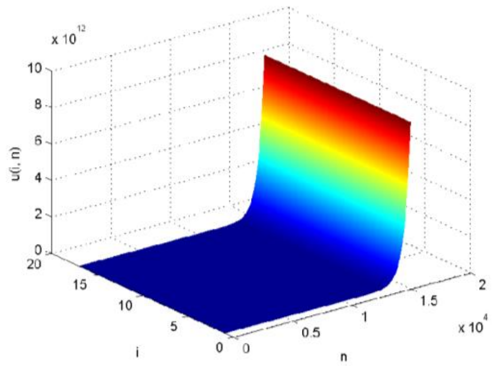


**Figure 2**: Evolution of the discrete solution(Implicit scheme)

$$U_i^{(0)} = 2,\ p = 2$$

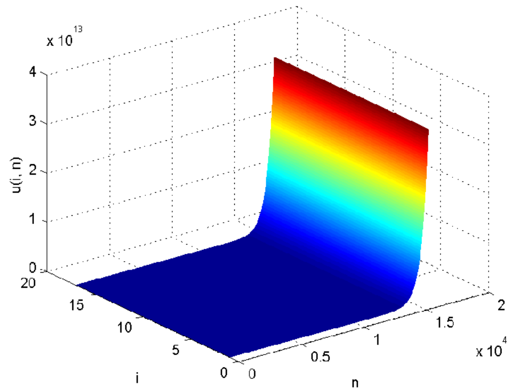


**Figure 3**: Evolution of the discrete solution(Explicit scheme)

$$U_i^{(0)} = \left(\frac{1}{2}\right)^{3-p} + \left(1-(ih)^2\right)^2,\ p = 2$$

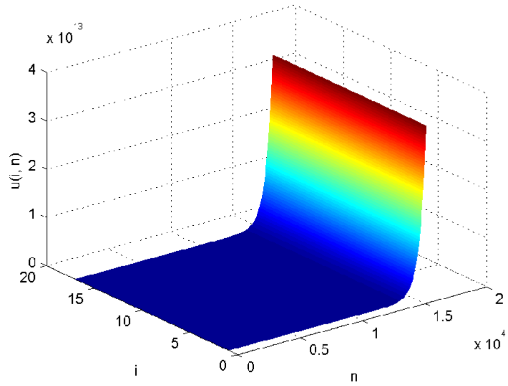


**Figure 4**: Evolution of the discrete solution(Implicit scheme)

$$U_i^{(0)} = \left(\frac{1}{2}\right)^{3-p} + \left(1-(ih)^2\right)^2,\ p = 2$$

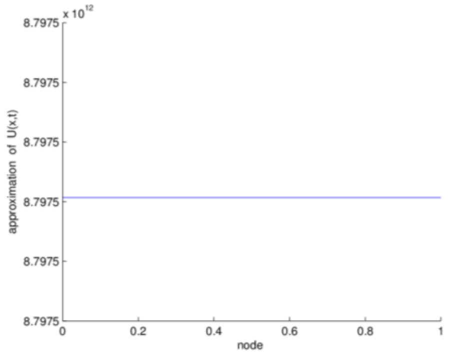


**Figure 5**: Evolution of U(x,t) according to the node (explicit scheme) , $U_i^{(0)} = 2,\ p = 2$

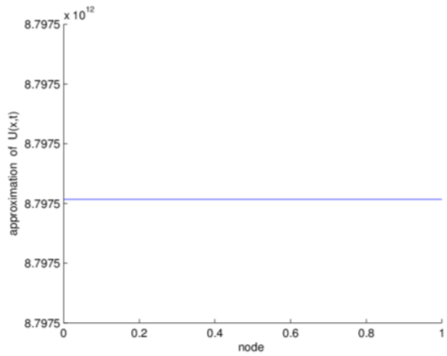


**Figure 6**: Evolution of U(x,t) according to the node (implicit scheme), $U_i^{(0)} = 2,\ p = 2$

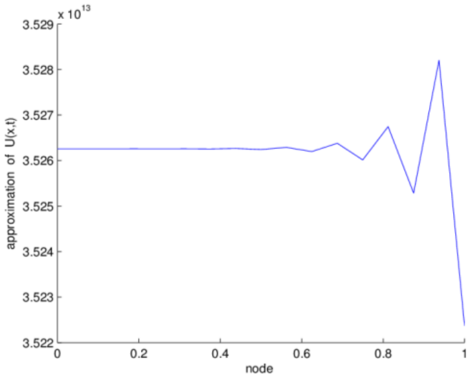


**Figure 7**: Evolution of U(x,t) according to the node (explicit scheme) ,

$$U_i^{(0)} = \left(\frac{1}{2}\right)^{3-p} + \left(1-(ih)^2\right)^2, \; p = 2$$

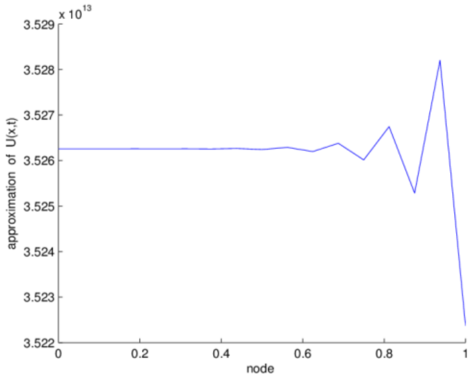


**Figure 8**: Evolution of U(x,t) according to the node (implicit scheme),

$$U_i^{(0)} = \left(\frac{1}{2}\right)^{3-p} + \left(1-(ih)^2\right)^2, \; p = 2$$

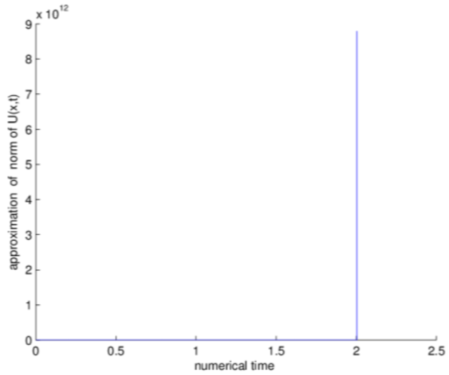


**Figure 9**: Evolution of norm of U(x,t) according to the time (explicit scheme), $U_i^{(0)} = 2, \; p = 2$

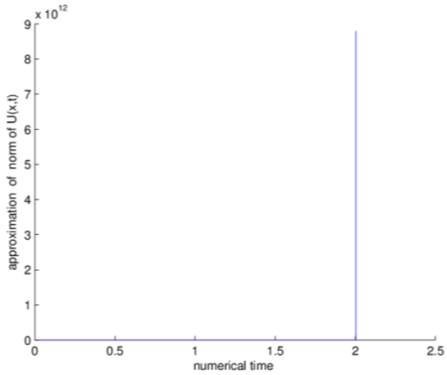


**Figure 10**: Evolution of norm of U(x,t) according to the time (implicit scheme), $U_i^{(0)} = 2, \; p = 2$

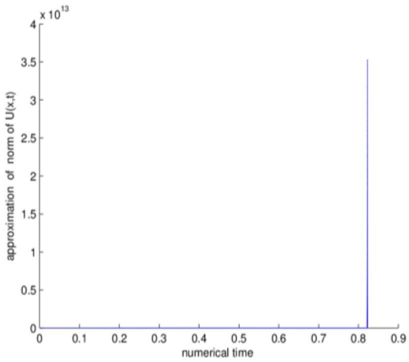


**Figure 11**: Evolution of norm of U(x,t) according to the time (explicit scheme),

$$U_i^{(0)} = \left(\frac{1}{2}\right)^{3-p} + \left(1-(ih)^2\right)^2, \; p = 2$$

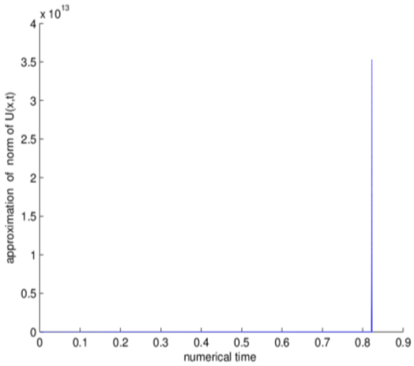


**Figure 12** Evolution of norm of U(x,t) according to the time (implicit scheme),

$$U_i^{(0)} = \left(\frac{1}{2}\right)^{3-p} + \left(1-(ih)^2\right)^2, \; p = 2$$

**Remark 5.1**

We observe that the blow-up phenomenon occurs faster for the large values of the initial data and the exponent $p$. In the case where the initial data is a constant, the solution of our problem blows up in a finite time for all $p \geq 2$, but slowly. This slowness is due to the absence of the turbulence effect, generated by the convection term. Therefore the blow-up only depends on the reaction term. When the initial data is not a constant, the convection term, head of turbulence, accelerates the blow-up created by the reaction term.

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