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Direct and Inverse Estimates For Combinations of Bernstein Polynomials with Endpoint Singularities

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Abstract. We give direct and inverse theorems for the weighted approximation of functions with endpoint singularities by combinations of Bernstein polynomials by the *r*th Ditzian-Totik modulus of smoothness $\omega_\phi^r(f,t)_w$ where ϕ is an admissible step-weight function.

Key words and phrases. Bernstein polynomials; Endpoint singularities; Pointwise approximation; Direct and inverse theorems.

1. Introduction

The set of all continuous functions, defined on the interval *I*, is denoted by *C(I)*. For any $f \in C([0,1])$, the corresponding Bernstein operators are defined as follows:

$$B_n(f,x) := \sum_{k=0}^{n} f(\frac{k}{n})p_{n,k}(x),$$

Where

$$p_{n,k}(x) := \binom{n}{k} x^k(1-x)^{n-k},\ k=0,1,2,\ldots,n,\ x\in[0,1].$$

Approximation properties of Bernstein operators have been studied very well (see [2], [4], [5]-[9], [14]-[16], for example). In order to approximate the functions with singularities, Della Vecchia et al. [4] and Yu-Zhao [14] introduced some kinds of *modified Bernstein operators*. Throughout the paper, *C* denotes a positive constant independent of *n* and x, which may be different in different cases.

Ditzian and Totik [5] extended the method of combinations and defined the following combinations of Bernstein operators:

$$B_{n,r}(f,x) := \sum_{i=0}^{r-1} C_i(n)B_{n_i}(f,x),$$

with the conditions:

(a) $n = n_0 < n_1 < \cdots < n_{r-1} \leqslant Cn,$

(b) $\sum_{i=0}^{r-1} |C_i(n)| \leqslant C,$

(c) $\sum_{i=0}^{r-1} C_i(n) = 1,$

(d) $\sum_{i=0}^{r-1} C_i(n) n_i^{-k} = 0$, for $k = 1, \ldots, r-1$.

Now, we can define our new combinations of Bernstein operators as follows:

$$B_{n,r}^*(f,x) := B_{n,r}(F_n, x) = \sum_{i=0}^{r-1} C_i(n) B_{n_i}(F_n, x), \tag{1.1}$$

where $C_i(n)$ satisfy the conditions (a)-(d). For the details, it can be referred to [13].

Let $\varphi(x) = \sqrt{x(1-x)}$ and let $\phi : [0,1] \longrightarrow R, \ \phi \neq 0$ be an admissible step-weight function of the Ditzian-Totik modulus of smoothness, that is, ϕ satisfies the following conditions:

(I) For every proper subinterval $[a,b] \subseteq [0,1]$ there exists a constant $M_1 \equiv M(a,b) > 0$ Such that $M_1^{-1} \leqslant \phi(x) \leqslant M_1$ for $x \in [a,b]$.

(II) There are two numbers $\beta(0) \geqslant 0$ and $\beta(1) \geqslant 0$ for which

$$\phi(x) \sim \begin{cases} x^{\beta(0)}, & \text{as } x \to 0+, \\ (1-x)^{\beta(1)}, & \text{as } x \to 1-. \end{cases}$$

($X \sim Y$ which means $C^{-1}Y \leqslant X \leqslant CY$ for some C).

Combining condition (**I**) and (**II**) on ϕ; we can deduce that

$$M^{-1}\phi_2(x) \leqslant \phi(x) \leqslant M\phi_2(x), x \in [0,1],$$

Where $\phi_2(x) = x^{\beta(0)}(1-x)^{\beta(1)}$, and M is a positive constant independent of x.

Let

$$w(x) = x^\alpha (1-x)^\beta, \ \alpha, \ \beta \geqslant 0, \ \alpha + \beta > 0, \ 0 \leqslant x \leqslant 1.$$

and

$$C_w := \{f \in C((0,1)) : \lim_{x \longrightarrow 1} (wf)(x) = \lim_{x \longrightarrow 0} (wf)(x) = 0\}.$$

The norm in C_w is defined by $\|wf\|_{C_w} := \|wf\| = \sup_{0 \leqslant x \leqslant 1} |(wf)(x)|$. Define

$$W_\phi^r := \{f \in C_w : f^{(r-1)} \in A.C.((0,1)), \ \|w\phi^r f^{(r)}\| < \infty\},$$
$$W_{\varphi,\lambda}^r := \{f \in C_w : f^{(r-1)} \in A.C.((0,1)), \ \|w\varphi^{r\lambda} f^{(r)}\| < \infty\}.$$

For $f \in C_w$, define the weighted modulus of smoothness by

$$\omega_\phi^r(f,t)_w := \sup_{0<h\leqslant t}\{\|w\Delta_{h\phi}^r f\|_{[16h^2,1-16h^2]} + \|w\overrightarrow{\Delta}_h^r f\|_{[0,16h^2]} + \|w\overleftarrow{\Delta}_h^r f\|_{[1-16h^2,1]}\},$$

where

$$\Delta_{h\phi}^r f(x) = \sum_{k=0}^{r}(-1)^k\binom{r}{k}f(x+(\frac{r}{2}-k)h\phi(x)),$$

$$\overrightarrow{\Delta}_h^r f(x) = \sum_{k=0}^{r}(-1)^k\binom{r}{k}f(x+(r-k)h),$$

$$\overleftarrow{\Delta}_h^r f(x) = \sum_{k=0}^{r}(-1)^k\binom{r}{k}f(x-kh).$$

Recently Felten showed the following two theorems in [6]:

Theorem A. Let $\varphi(x) = \sqrt{x(1-x)}$ and let $\phi : [0,1] \longrightarrow R,\ \phi \neq 0$ be an admissible step-weight function of the Ditzian-Totik modulus of smoothness([5]) such that ϕ^2 and φ^2/ϕ^2 are concave. Then, for $f \in C[0,1]$ and $0 < \alpha < 2,\ |B_n(f,x) - f(x)| \leqslant \omega_\phi^2(f, n^{-1/2}\frac{\varphi(x)}{\phi(x)})$.

Theorem B. Let $\varphi(x) = \sqrt{x(1-x)}$ and let $\phi : [0,1] \longrightarrow R,\ \phi \neq 0$ be an admissible step-weight function of the Ditzian-Totik modulus of smoothness([5]) such that ϕ^2 and φ^2/ϕ^2 are concave. Then, for $f \in C[0,1]$ and $0 < \alpha < 2,\ |B_n(f,x) - f(x)| = O((n^{-1/2}\frac{\varphi(x)}{\phi(x)})^\alpha)$ implies $\omega_\phi^2(f,t) = O(t^\alpha)$.

Our main results are the following:

Theorem 2. 1. *For any* $\alpha,\ \beta > 0,\ min\{\beta(0),\beta(1)\} \geqslant \frac{1}{2},\ f \in C_w$, *we have*

$$|w(x)\phi^r(x)B_{n,r-1}^{*(r)}(f,x)| \leqslant Cn^{\frac{r}{2}}\|wf\|. \tag{2.1}$$

Theorem 2.2. *For any* $\alpha,\ \beta > 0,\ f \in W_\phi^r$, *we have*

$$|w(x)\phi^r(x)B_{n,r-1}^{*(r)}(f,x)| \leqslant C\|w\phi^r f^{(r)}\|. \tag{2.2}$$

Theorem 2.3. *For* $f \in C_w,\ \alpha,\ \beta > 0,\ \min\{\beta(0),\beta(1)\} \geqslant \frac{1}{2}$, *we have*

$$w(x)|f(x) - B_{n,r-1}^*(f,x)| = O((n^{-\frac{1}{2}}\phi^{-1}(x)\delta_n(x))^{\alpha_0}) \Longleftrightarrow \omega_\phi^r(f,t)_w = O(t^{\alpha_0}), \tag{2.3}$$

where $\alpha_0 \in (0,r)$.

3. LEMMAS

Lemma 3.1. *(*[15]*) For any non-negative real* u *and* v, *we have*

$$\sum_{k=1}^{n-1}(\frac{k}{n})^{-u}(1-\frac{k}{n})^{-v}p_{n,k}(x) \leqslant Cx^{-u}(1-x)^{-v}. \tag{3.1}$$

Lemma 3.2. ([4])*If* $\gamma \in R$, *then*

$$\sum_{k=0}^{n} |k-nx|^{\gamma} p_{n,k}(x) \leqslant Cn^{\frac{\gamma}{2}} \varphi^{\gamma}(x). \tag{3.2}$$

Lemma 3.3. *For any* $f \in W_{\phi}^{r}$, $\alpha, \ \beta > 0$, *we have*

$$\|w\phi^r F_n^{(r)}\| \leqslant C\|w\phi^r f^{(r)}\|. \tag{3.3}$$

Proof. By symmetry, we only prove the above result when $x \in (0,1/2]$, the others can be done similarly. Obviously, when $x \in (0,1/n]$, by [5], we have

$$\begin{aligned} |L_r^{(r)}(f,x)| &\leqslant C|\overrightarrow{\Delta}_{\frac{1}{r}}^{r} f(0)| \leqslant Cn^{-\frac{r}{2}+1} \int_0^{\frac{r}{n}} u^{\frac{r}{2}} |f^{(r)}(u)| du \\ &\leqslant Cn^{-\frac{r}{2}+1} \|w\phi^r f^{(r)}\| \int_0^{\frac{r}{n}} u^{\frac{r}{2}} w^{-1}(u) \phi^{-r}(u) du. \end{aligned}$$

So

$$|w(x)\phi^r(x)F_n^{(r)}(x)| \leqslant C\|w\phi^r f^{(r)}\|.$$

If $x \in [\frac{1}{n}, \frac{2}{n}]$, we have

$$\begin{aligned} |w(x)\phi^r(x)F_n^{(r)}(x)| &\leqslant |w(x)\phi^r(x)f^{(r)}(x)| + |w(x)\phi^r(x)(f(x)-F_n(x))^{(r)}| \\ &:= I_1 + I_2. \end{aligned}$$

For I_2, we have

$$\begin{aligned} f(x) - F_n(x) &= (\psi(nx-1)+1)(f(x) - L_r(f,x)). \\ w(x)\phi^r(x)|(f(x)-F_n(x))^{(r)}| &= w(x)\phi^r(x) \sum_{i=0}^{r} n^i |(f(x)-L_r(f,x))^{(r-i)}|. \end{aligned}$$

By [5], then

$$|(f(x)-L_r(f,x))^{(r-i)}|_{[\frac{1}{n},\frac{2}{n}]} \leqslant C(n^{r-i}\|f-L_r\|_{[\frac{1}{n},\frac{2}{n}]} + n^{-i}\|f^{(r)}\|_{[\frac{1}{n},\frac{2}{n}]}), \ 0<j<r.$$

Now, we estimate

$$I := w(x)\phi^r(x)|f(x)-L_r(x)|. \tag{3.4}$$

By Tailor expansion, we have

$$f(\frac{i}{n}) = \sum_{u=0}^{r-1} \frac{(\frac{i}{n}-x)^u}{u!} f^{(u)}(x) + \frac{1}{(r-1)!} \int_x^{\frac{i}{n}} (\frac{i}{n}-s)^{r-1} f^{(r)}(s) ds, \tag{3.5}$$

It follows from (3.5) and the identities

$$\sum_{i=1}^{r}(\frac{i}{n})^v l_i(x) = Cx^v,\ v = 0,1,\cdots,r.$$

We have

$$\begin{aligned} L_r(f,x) &= \sum_{i=1}^{r}\sum_{u=0}^{r-1}\frac{(\frac{i}{n}-x)^u}{u!}f^{(u)}(x)l_i(x) + \frac{1}{(r-1)!}\sum_{i=1}^{r}l_i(x)\int_x^{\frac{i}{n}}(\frac{i}{n}-s)^{r-1}f^{(r)}(s)ds \\ &= f(x) + \sum_{u=1}^{r-1}f^{(u)}(x)(\sum_{v=0}^{u}C_u^v(-x)^{u-v}\sum_{i=1}^{r}(\frac{i}{n})^v l_i(x)) \\ &+ \frac{1}{(r-1)!}\sum_{i=1}^{r}l_i(x)\int_x^{\frac{i}{n}}(\frac{i}{n}-s)^{r-1}f^{(r)}(s)ds, \end{aligned}$$

Which implies that

$$w(x)\phi^r(x)|f(x)-L_r(f,x)| = \frac{1}{(r-1)!}w(x)\phi^r(x)\sum_{i=1}^{r}l_i(x)\int_x^{\frac{i}{n}}(\frac{i}{n}-s)^{r-1}f^{(r)}(s)ds,$$

Since $|l_i(x)| \leqslant C$ for $x \in [0,\frac{2}{n}],\ i = 1,2,\cdots,r.$

It follows from $\frac{|\frac{i}{n}-s|^{r-1}}{w(s)} \leqslant \frac{|\frac{i}{n}-x|^{r-1}}{w(x)}$, s between $\frac{i}{n}$ and x, then

$$\begin{aligned} w(x)\phi^r(x)|f(x)-L_r(f,x)| &\leqslant Cw(x)\phi^r(x)\sum_{i=1}^{r}\int_x^{\frac{i}{n}}(\frac{i}{n}-s)^{r-1}|f^{(r)}(s)|ds \\ &\leqslant C\phi^r(x)\|w\phi^r f^{(r)}\|\sum_{i=1}^{r}\int_x^{\frac{i}{n}}(\frac{i}{n}-s)^{r-1}\phi^{-r}(s)ds \\ &\leqslant \frac{C}{n^r}\|w\phi^r f^{(r)}\|. \end{aligned}$$

Thus $I \leqslant C\|w\phi^r f^{(r)}\|$. So we get $I_2 \leqslant C\|w\phi^r f^{(r)}\|$. Above all, we have

$$|w(x)\phi^r(x)F_n^{(r)}(x)| \leqslant C\|w\phi^r f^{(r)}\|.$$ □

Lemma 3.4. *If* $f \in W_\phi^r,\ \alpha,\ \beta > 0$, *then*

$$(3.6)\qquad |w(x)(f(x)-L_r(f,x))|_{[0,\frac{2}{n}]} \leqslant C(\frac{\delta_n(x)}{\sqrt{n}\phi(x)})^r\|w\phi^r f^{(r)}\|.$$

$$(3.7)\qquad |w(x)(f(x)-R_r(f,x))|_{[1-\frac{2}{n},1]} \leqslant C(\frac{\delta_n(x)}{\sqrt{n}\phi(x)})^r\|w\phi^r f^{(r)}\|.$$

Proof. By Taylor expansion, we have

$$(3.8)\quad f(\frac{i}{n}) = \sum_{u=0}^{r-1}\frac{(\frac{i}{n}-x)^u}{u!}f^{(u)}(x) + \frac{1}{(r-1)!}\int_x^{\frac{i}{n}}(\frac{i}{n}-s)^{r-1}f^{(r)}(s)ds,$$

It follows from (3.8) and the identities

$$\sum_{i=1}^{r-1}(\frac{i}{n})^v l_i(x) = Cx^v,\ v = 0, 1, \ldots, r.$$

we have

$$\begin{aligned} L_r(f,x) &= \sum_{i=1}^{r}\sum_{u=0}^{r-1}\frac{(\frac{i}{n}-x)^u}{u!}f^{(u)}(x)l_i(x) + \frac{1}{(r-1)!}\sum_{i=1}^{r} l_i(x)\int_x^{\frac{i}{n}}(\frac{i}{n}-s)^{r-1}f^{(r)}(s)ds \\ &= f(x) + \sum_{u=1}^{r-1} f^{(u)}(x)(\sum_{v=0}^{u} C_u^v(-x)^{u-v}\sum_{i=1}^{r}(\frac{i}{n})^v l_i(x)) \\ &+ \frac{1}{(r-1)!}\sum_{i=1}^{r} l_i(x)\int_x^{\frac{i}{n}}(\frac{i}{n}-s)^{r-1}f^{(r)}(s)ds, \end{aligned}$$

Which implies that

$$w(x)|f(x) - L_r(f,x)| = \frac{1}{(r-1)!}w(x)\sum_{i=1}^{r} l_i(x)\int_x^{\frac{i}{n}}(\frac{i}{n}-s)^{r-1}f^{(r)}(s)ds,$$

Since $|l_i(x)| \leqslant C$ for $x \in [0, \frac{2}{n}],\ i = 1, 2, \cdots, r.$

It follows from $\frac{|\frac{i}{n}-s|^{r-1}}{w(s)} \leqslant \frac{|\frac{i}{n}-x|^{r-1}}{w(x)},$ s between $\frac{i}{n}$ and x, then

$$\begin{aligned} w(x)|f(x) - L_r(f,x)| &\leqslant Cw(x)\sum_{i=1}^{r}\int_x^{\frac{i}{n}}(\frac{i}{n}-s)^{r-1}|f^{(r)}(s)|ds \\ &\leqslant C\frac{\varphi^r(x)}{\phi^r(x)}\|w\phi^r f^{(r)}\|\sum_{i=1}^{r}\int_x^{\frac{i}{n}}(\frac{i}{n}-s)^{r-1}\varphi^{-r}(s)ds \\ &\leqslant C\frac{\delta_n^r(x)}{\phi^r(x)}\|w\phi^r f^{(r)}\|\sum_{i=1}^{r}\int_x^{\frac{i}{n}}(\frac{i}{n}-s)^{r-1}\varphi^{-r}(s)ds \\ &\leqslant C(\frac{\delta_n(x)}{\sqrt{n}\phi(x)})^r\|w\phi^r f^{(r)}\|. \end{aligned}$$

The proof of (3.7) can be done similarly. □

Lemma 3.5. *(*[13]*) For every* $\alpha,\ \beta > 0$, *we have*

$$\|wB^*_{n,r-1}(f)\| \leqslant C\|wf\|. \tag{3.9}$$

Lemma 3.6. *(*[17]*) Let* $\min\{\beta(0), \beta(1)\} \geqslant \frac{1}{2}$, *then* $r \in N,\ 0 < t < \frac{1}{8r}$ *and* $\frac{rt}{2} < x < 1 - \frac{rt}{2}$, *we have*

$$\int_{-\frac{t}{2}}^{\frac{t}{2}}\cdots\int_{-\frac{t}{2}}^{\frac{t}{2}}\phi^{-r}(x + \sum_{k=1}^{r} u_k)du_1\cdots du_r \leqslant Ct^r\phi^{-r}(x). \tag{3.10}$$

Lemma 3.7. *(*[10]*) Let* $\alpha,\ \beta > 0,$ *for any* $f \in C_w$, *we have*

$$\|wB^{*(r)}_{n,r-1}(f)\| \leqslant Cn^r\|wf\|. \tag{3.11}$$

4. Proof of Theorems

4.1. Proof of Theorem 2.1. When $f \in C_w$,min$\{\beta(0),\beta(1)\} \geqslant \frac{1}{2}$, we discuss it as follows:

Case 1. If $0 \leqslant \varphi(x) \leqslant \frac{1}{\sqrt{n}}$, by (3.11), we have

$$\begin{aligned}|w(x)\phi^r(x)B^{*(r)}_{n,r-1}(f,x)| &= C\varphi^r(x)\frac{\phi^r(x)}{\varphi^r(x)}|w(x)B^{*(r)}_{n,r-1}(f,x)| \\ &\leqslant Cn^{\frac{r}{2}}\|wf\|. \end{aligned}\tag{4.1}$$

Case 2. If $\varphi(x) > \frac{1}{\sqrt{n}}$, we have

$$\begin{aligned}|B^{*(r)}_{n,r-1}(f,x)| &= |B^{(r)}_{n,r-1}(F_n,x)| \\ \leqslant (\varphi^2(x))^{-r}\sum_{i=0}^{r-2}\sum_{j=0}^{r}Q_j(x,n_i)C_i(n)n_i^j&\sum_{k=0}^{n_i}|(x-\frac{k}{n_i})^jF_n(\frac{k}{n_i})|p_{n_i,k}(x),\end{aligned}$$

By [5], we have

$$Q_j(x,n_i) = (n_ix(1-x))^{[\frac{r-j}{2}]}, \text{ and } (\varphi^2(x))^{-r}Q_j(x,n_i)n_i^j \leqslant C(n_i/\varphi^2(x))^{\frac{r+j}{2}}.$$

So

$$\begin{aligned}&|w(x)\phi^r(x)B^{*(r)}_{n,r-1}(f,x)| \\ \leqslant Cw(x)\phi^r(x)\sum_{i=0}^{r-2}\sum_{j=0}^{r}&(\frac{n_i}{\varphi^2(x)})^{\frac{r+j}{2}}\sum_{k=0}^{n_i}|(x-\frac{k}{n_i})^jF_n(\frac{k}{n_i})|p_{n_i,k}(x) \\ \leqslant Cw(x)\phi^r(x)\|wf\|&\sum_{i=0}^{r-2}\sum_{j=0}^{r}(\frac{n_i}{\varphi^2(x)})^{\frac{r+j}{2}}\{\sum_{k=0}^{n_i}(x-\frac{k}{n_i})^{2j}\}^{\frac{1}{2}}\cdot \\ &\{\sum_{k=0}^{n_i}w^{-2}(\frac{k}{n_i})p_{n_i,k}(x)\}^{\frac{1}{2}} \\ &\leqslant Cn^{\frac{r}{2}}\|wf\|.\end{aligned}\tag{4.2}$$

It follows from combining with (4.1) and (4.2) that the theorem is proved. □

4.2. Proof of Theorem 2.2. When $f \in W^r_\phi$, by [5], we have

$$B^{(r)}_{n,r-1}(F_n,x) = \sum_{i=0}^{r-2}C_i(n)n_i^r\sum_{k=0}^{n_i-r}\overrightarrow{\Delta}^r_{\frac{1}{n_i}}F_n(\frac{k}{n_i})p_{n_i-r,k}(x). \tag{4.3}$$

If $0 < k < n_i - r$, we have

$$(4.4) \qquad |\overrightarrow{\Delta}^r_{\frac{1}{n_i}} F_n(\frac{k}{n_i})| \leqslant Cn_i^{-r+1} \int_0^{\frac{r}{n_i}} |F_n^{(r)}(\frac{k}{n_i} + u)|du,$$

If $k = 0$, we have

$$(4.5) \qquad |\overrightarrow{\Delta}^r_{\frac{1}{n_i}} F_n(0)| \leqslant C \int_0^{\frac{r}{n_i}} u^{r-1} |F_n^{(r)}(u)|du,$$

Similarly

$$(4.6) \qquad |\overrightarrow{\Delta}^r_{\frac{1}{n_i}} F_n(\frac{n_i - r}{n_i})| \leqslant Cn_i^{-r+1} \int_{1-\frac{r}{n_i}}^1 (1-u)^{\frac{r}{2}} |F_n^{(r)}(u)|du.$$

By (4.3)-(4.6), we have

$$(4.7) \qquad \begin{aligned} &|w(x)\phi^r(x) B^{*(r)}_{n,r-1}(f,x)| \\ \leqslant\ & Cw(x)\phi^r(x)\|w\phi^r F_n^{(r)}\| \sum_{i=0}^{r-2} \sum_{k=0}^{n_i-r} (w\phi^r)^{-1}(\frac{k^*}{n_i}) p_{n_i-r,k}(x), \end{aligned}$$

If $k^* = 1$ for $k = 0$, $k^* = n_i - r - 1$ for $k = n_i - r$ and $k^* = k$ or $1 < k < n_i - r$.

By (3.1), we have

$$\sum_{k=0}^{n_i-r} (w\phi^r)^{-1}(\frac{k^*}{n_i}) p_{n_i-r,k}(x) \leqslant C(w\phi^r)^{-1}(x).$$

which combining with (4.7) give

$$|w(x)\phi^r(x) B^{*(r)}_{n,r-1}(f,x)| \leqslant C\|w\phi^r f^{(r)}\|. \Box$$

Combining with the theorem 2.1 and theorem 2.2, we can obtain

Corollary. *For any* $\alpha,\ \beta > 0,\ 0 \leqslant \lambda \leqslant 1$, we have

$$(4.8) \qquad |w(x)\varphi^{r\lambda}(x) B^{*(r)}_{n,r-1}(f,x)| \leqslant \begin{cases} Cn^{r/2}\{max\{n^{r(1-\lambda)/2}, \varphi^{r(\lambda-1)}(x)\}\}\|wf\|, & f \in C_w, \\ C\|w\varphi^{r\lambda} f^{(r)}\|, & f \in W^r_{w,\lambda}. \end{cases}$$

4.3 Proof of Theorem 2.3.

4.3.1. *The direct theorem.* We know

$$(4.9) \qquad F_n(t) = F_n(x) + F_n'(t)(t-x) + \cdots + \frac{1}{(r-1)!}\int_x^t (t-u)^{r-1} f^{(r)}(u)du,$$

$$(4.10) \qquad B_{n,r-1}((\cdot - x)^k, x) = 0,\ k = 1, 2, \cdots, r-1.$$

According to the definition of W_ϕ^r, for any $g \in W_\phi^r$, we have $B_{n,r-1}^*(g,x) = B_{n,r-1}(G_n(g),x)$,

and $w(x)|G_n(x) - B_{n,r-1}(G_n,x)| = w(x)|B_{n,r-1}(R_r(G_n,t,x),x)|$,

thereof $R_r(G_n,t,x) = \int_x^t (t-u)^{r-1} G_n^{(r)}(u)du$, we have

$$w(x)|G_n(x) - B_{n,r-1}(G_n,x)| \leqslant C\|w\phi^r G_n^{(r)}\| w(x) B_{n,r-1}(\int_x^t \frac{|t-u|^{r-1}}{w(u)\phi^r(u)}du, x)$$
$$\leqslant C\|w\phi^r G_n^{(r)}\| w(x)(B_{n,r-1}(\int_x^t \frac{|t-u|^{r-1}}{\phi^{2r}(u)}du, x))^{\frac{1}{2}}\cdot$$
$$(B_{n,r-1}(\int_x^t \frac{|t-u|^{r-1}}{w^2(u)}du, x))^{\frac{1}{2}}. \tag{4.11}$$

also

$$\int_x^t \frac{|t-u|^{r-1}}{\phi^{2r}(u)}du \leqslant C\frac{|t-x|^r}{\phi^{2r}(x)}, \int_x^t \frac{|t-u|^{r-1}}{w^2(u)}du \leqslant \frac{|t-x|^r}{w^2(x)}. \tag{4.12}$$

By (3.2), (3.3) and (4.12), we have

$$w(x)|G_n(x) - B_{n,r-1}(G_n,x)| \leqslant C\|w\phi^r G_n^{(r)}\| \phi^{-r}(x) B_{n,r-1}(|t-x|^r, x)$$
$$\leqslant Cn^{-\frac{r}{2}}\frac{\varphi^r(x)}{\phi^r(x)}\|w\phi^r G_n^{(r)}\|$$
$$\leqslant Cn^{-\frac{r}{2}}\frac{\delta_n^r(x)}{\phi^r(x)}\|w\phi^r G_n^{(r)}\|$$
$$= C(\frac{\delta_n(x)}{\sqrt{n}\phi(x)})^r\|w\phi^r G_n^{(r)}\|. \tag{4.13}$$

By (3.6), (3.7) and (4.13), when $g \in W_\phi^r$, then

$$w(x)|g(x) - B_{n,r-1}^*(g,x)| \leqslant w(x)|g(x) - G_n(g,x)| + w(x)|G_n(g,x) - B_{n,r-1}^*(g,x)|$$
$$\leqslant |w(x)(g(x) - L_r(g,x))|_{[0,\frac{2}{n}]} + |w(x)(g(x) - R_r(g,x))|_{[1-\frac{2}{n},1]}$$
$$+ \ C(\frac{\delta_n(x)}{\sqrt{n}\phi(x)})^r\|w\phi^r G_n^{(r)}\|$$
$$\leqslant C(\frac{\delta_n(x)}{\sqrt{n}\phi(x)})^r\|w\phi^r g^{(r)}\|. \tag{4.14}$$

For $f \in C_w$, we choose proper $g \in W_\phi^r$, by (3.9) and (4.14), then

$$w(x)|f(x) - B_{n,r-1}^*(f,x)| \leqslant w(x)|f(x) - g(x)| + w(x)|B_{n,r-1}^*(f-g,x)|$$
$$+w(x)|g(x) - B_{n,r-1}^*(g,x)|$$
$$\leqslant C(\|w(f-g)\| + (\frac{\delta_n(x)}{\sqrt{n}\phi(x)})^r\|w\phi^r g^{(r)}\|)$$
$$\leqslant C\omega_\phi^r(f, \frac{\delta_n(x)}{\sqrt{n}\phi(x)})_w.\Box$$

4.3.2. The inverse theorem. We define the weighted main-part modulus fo $D = R_+$ by(see [5])

$$\Omega_\phi^r(C,f,t)_w = \sup_{0<h\leqslant t} \|w\Delta_{h\phi}^r f\|_{[Ch^*,\infty]},$$
$$\Omega_\phi^r(1,f,t)_w = \Omega_\phi^r(f,t)_w.$$

The main-part *K*-functional is given by

$$K_{r,\phi}(f,t^r)_w = \sup_{0<h\leqslant t} \inf_g \{\|w(f-g)\|_{[Ch^*,\infty]} + t^r\|w\phi^r g^{(r)}\|_{[Ch^*,\infty]},\ g^{(r-1)} \in A.C.((Ch^*,\infty))\}.$$

By [5], we have

$$C^{-1}\Omega_\phi^r(f,t)_w \leqslant \omega_\phi^r(f,t)_w \leqslant C\int_0^t \frac{\Omega_\phi^r(f,\tau)_w}{\tau}d\tau, \tag{4.15}$$

$$C^{-1}K_{r,\phi}(f,t^r)_w \leqslant \Omega_\phi^r(f,t)_w \leqslant CK_{r,\phi}(f,t^r)_w. \tag{4.16}$$

Proof. Let $\delta > 0$, we choose proper g so that

$$\|w(f-g)\| \leqslant C\Omega_\phi^r(f,\delta)_w,\ \|w\phi^r g^{(r)}\| \leqslant C\delta^{-r}\Omega_\phi^r(f,\delta)_w. \tag{4.17}$$

For $r \in N,\ 0 < t < \frac{1}{8r}$ and $\frac{rt}{2} < x < 1 - \frac{rt}{2}$, we have

$$\begin{aligned}
|w(x)\Delta_{h\phi}^r f(x)| &\leqslant |w(x)\Delta_{h\phi}^r(f(x) - B_{n,r-1}^*(f,x))| + |w(x)\Delta_{h\phi}^r B_{n,r-1}^*(f-g,x)| \\
&\quad + |w(x)\Delta_{h\phi}^r B_{n,r-1}^*(g,x)| \\
&\leqslant \sum_{j=0}^r C_r^j \Big(n^{-\frac{1}{2}}\frac{\delta_n(x+(\frac{r}{2}-j)h\phi(x))}{\phi(x+(\frac{r}{2}-j)h\phi(x))}\Big)^{\alpha_0} \\
&\quad + \int_{-\frac{h\phi(x)}{2}}^{\frac{h\phi(x)}{2}} \cdots \int_{-\frac{h\phi(x)}{2}}^{\frac{h\phi(x)}{2}} w(x)B_{n,r-1}^{*(r)}(f-g, x+\sum_{k=1}^r u_k)du_1\cdots du_r \\
&\quad + \int_{-\frac{h\phi(x)}{2}}^{\frac{h\phi(x)}{2}} \cdots \int_{-\frac{h\phi(x)}{2}}^{\frac{h\phi(x)}{2}} w(x)B_{n,r-1}^{*(r)}(g, x+\sum_{k=1}^r u_k)du_1\cdots du_r \\
&:= J_1 + J_2 + J_3.
\end{aligned} \tag{4.18}$$

Obviously

$$J_1 \leqslant C(n^{-\frac{1}{2}}\phi^{-1}(x)\delta_n(x))^{\alpha_0}. \tag{4.19}$$

By (3.11) and (4.17), we have

$$\begin{aligned}
J_2 &\leqslant Cn^r\|w(f-g)\| \int_{-\frac{h\phi(x)}{2}}^{\frac{h\phi(x)}{2}} \cdots \int_{-\frac{h\phi(x)}{2}}^{\frac{h\phi(x)}{2}} du_1\cdots du_r \\
&\leqslant Cn^r h^r\phi^r(x)\|w(f-g)\| \\
&\leqslant Cn^r h^r \phi^r(x)\Omega_\phi^r(f,\delta)_w.
\end{aligned} \tag{4.20}$$

By the first inequality of (4.8), we let $\lambda = 1$, and (3.10) as well as (4.17), then

$$\begin{aligned} J_2 &\leqslant Cn^{\frac{r}{2}}\|w(f-g)\| \int_{-\frac{h\phi(x)}{2}}^{\frac{h\phi(x)}{2}} \cdots \int_{-\frac{h\phi(x)}{2}}^{\frac{h\phi(x)}{2}} \varphi^{-r}(x+\sum_{k=1}^{r} u_k) du_1 \cdots du_r \\ &\leqslant Cn^{\frac{r}{2}} h^r \phi^r(x) \varphi^{-r}(x) \|w(f-g)\| \\ &\leqslant Cn^{\frac{r}{2}} h^r \phi^r(x) \varphi^{-r}(x) \Omega_\phi^r(f,\delta)_w. \end{aligned} \tag{4.21}$$

By the second inequality of (3.10) and (4.17), we have

$$\begin{aligned} J_3 &\leqslant C\|w\phi^r g^{(r)}\| w(x) \int_{-\frac{h\phi(x)}{2}}^{\frac{h\phi(x)}{2}} \cdots \int_{-\frac{h\phi(x)}{2}}^{\frac{h\phi(x)}{2}} w^{-1}(x+\sum_{k=1}^{r} u_k) \phi^{-r}(x+\sum_{k=1}^{r} u_k) du_1 \cdots du_r \\ &\leqslant Ch^r \|w\phi^r g^{(r)}\| \\ &\leqslant Ch^r \delta^{-r} \Omega_\phi^r(f,\delta)_w. \end{aligned} \tag{4.22}$$

Now, by (4.18)-(4.22), there exists $M > 0$ so that

$$\begin{aligned} |w(x)\Delta_{h\phi}^r f(x)| &\leqslant C((n^{-\frac{1}{2}}\frac{\delta_n(x)}{\phi(x)})^{\alpha_0} \\ &+ \min\{n^{\frac{r}{2}}\frac{\phi^r(x)}{\varphi^r(x)}, n^r\phi^r(x)\} h^r \Omega_\phi^r(f,\delta)_w + h^r\delta^{-r}\Omega_\phi^r(f,\delta)_w) \\ &\leqslant C((n^{-\frac{1}{2}}\frac{\delta_n(x)}{\phi(x)})^{\alpha_0} \\ &+ h^r M^r (n^{-\frac{1}{2}}\frac{\varphi(x)}{\phi(x)} + n^{-\frac{1}{2}}\frac{n^{-1/2}}{\phi(x)})^{-r}\Omega_\phi^r(f,\delta)_w + h^r\delta^{-r}\Omega_\phi^r(f,\delta)_w) \\ &\leqslant C((n^{-\frac{1}{2}}\frac{\delta_n(x)}{\phi(x)})^{\alpha_0} \\ &+ h^r M^r (n^{-\frac{1}{2}}\frac{\delta_n(x)}{\phi(x)})^{-r}\Omega_\phi^r(f,\delta)_w + h^r\delta^{-r}\Omega_\phi^r(f,\delta)_w). \end{aligned}$$

When $n \geqslant 2$, we have

$$n^{-\frac{1}{2}}\delta_n(x) < (n-1)^{-\frac{1}{2}}\delta_{n-1}(x) \leqslant \sqrt{2}n^{-\frac{1}{2}}\delta_n(x),$$

Choosing proper $x,\ \delta,\ n \in N$, so that

$$n^{-\frac{1}{2}}\frac{\delta_n(x)}{\phi(x)} \leqslant \delta < (n-1)^{-\frac{1}{2}}\frac{\delta_{n-1}(x)}{\phi(x)},$$

Therefore

$$|w(x)\Delta_{h\phi}^r f(x)| \leqslant C\{\delta^{\alpha_0} + h^r\delta^{-r}\Omega_\phi^r(f,\delta)_w\}.$$

By Borens-Lorentz lemma, we get

$$\Omega^r_\phi(f,t)_w \leqslant Ct^{\alpha_0}. \tag{4.23}$$

So, by (4.15) and (4.23), we get

$$\omega^r_\phi(f,t)_w \leqslant C\int_0^t \frac{\Omega^r_\phi(f,\tau)_w}{\tau}d\tau = C\int_0^t \tau^{\alpha_0-1}d\tau = Ct^{\alpha_0}.$$

□

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