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Modified Deficiencies of q-Difference Equations of Zero Order

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Abstract: In this paper, we deal with the modified deficiencies of q-difference equations and give some improvements for special types of meromorphic functions that would throw more light on the relative defects of difference polynomials, which extends the results of Toda [9], Sarangi and Patil [8], Bhoosnurmath S. S and Shankar. M. Patil [2].

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1. Introduction, Results and Definitions

For $a \in C$, Milloux [7] introduced the concept of absolute defect of α with respect to the derivative f'. This definition was further extended by Hiong [5]. He introduced

$$\delta_r(\alpha, f^{(k)}) = 1 - \limsup_{t\to\infty} \frac{N\left(t, \dfrac{1}{f^{(k)-\alpha}}\right)}{T(t, f)}$$

is called as the "Relative defect of the value α with respect to $f^{(k)}$", the suffix "r" in the left hand side of above is just to denote the "relative" defect in contrast to the usual defect

$$\delta(a, f) = 1 - \limsup_{t\to\infty} \frac{N\left(t, \dfrac{1}{f-\alpha}\right)}{T(t, f)}$$

and the "absolute" defect

$$\delta_a(a, f) = 1 - \limsup_{t\to\infty} \frac{N\left(t, \dfrac{1}{f-\alpha})\right)}{T(t, f)}$$

and found several other relations between the relative defect and the usual defect. We define similar difference analogue of relative defect and absolute defect with respect to difference polynomial $P(f)$ as follows.

$$\delta_r^q(\alpha, P(f)) = 1 - \limsup_{t\to\infty} \frac{N_q\left(t, \dfrac{1}{P(f)-\alpha}\right)}{T(t, f)}$$

is called "q" difference absolute defect of α with respect to $P(f)$ and

$$\delta_a^q(\alpha, P(f)) = 1 - \limsup_{t\to\infty} \frac{N_q\left(t, \dfrac{1}{P(f)-\alpha}\right)}{T(t, P(f))}$$

is called "q" difference absolute defect with respect to $P(f)$.

Also as a natural q-difference analogue of $\overline{N}(r,a)$ is $\tilde{\mathrm{N}}_q(r,a) = N(r,a) - N_q(r,a)$

as a result we have an analogue of $\Theta(a, f)$ as

$$\Pi_q(a, f) = 1 - \limsup_{r\to\infty} \frac{\tilde{\mathrm{N}}_q(r,a)}{T(r, f)}.$$

All notations and terminology of the work follows from Hayman [4](1964). By $S(t, f)$ we mean quanity satisfying $S(t, f) = o[T(r, f)]$ as $r \to \infty$ possibly outside a set r of finite linear measure.

Definition 1. [2]: A monomial in f is an expression of the form

$$M_j(f) = f^{n_{0j}} f(q_1 z)^{n_{1j}} f(q_2 z)^{n_{2j}} \dots f(q_k z)^{n_{kj}},$$

where $n_{oj}, n_{1j}, n_{2j}, \dots, n_{kj}$ are non-negative integers. $\gamma_{M_j} = n_{oj} + n_{1j} + n_{2j} + \dots + n_{kj}$ is called the degree of the monomial and $\Gamma_{M_j} = n_{oj} + 2n_{1j} + 3n_{2j} + \dots + (k+1)n_{k_j} = \sum_{i=0}^{k}(i+1)n_{1j}$ the weight of $M_j(f)$. $P_q(f) = \sum_{j=1}^{q} a_j M_j[f(qz)]$, where $a_i (i = 1,2,\dots,n)$ are constants, then $P(f)$ is called a differential polynomials in f of degree γ_{P_q} and the weight Γ_{P_q}, $P_q(f)$ are defined as follows, $\gamma_{P_q} = max_{1\le j\le q}\gamma_{Mj}$ and $\Gamma_{P_q} = max_{1\le j\le q}\Gamma_{Mj}$, also we call the number $\underline{\gamma_{P_q}} = min_{1\le j\le q}\gamma_{Mj}$ the lower degree of $P(f)$.

If $\underline{\gamma_{P_q}} = \overline{\gamma_{P_q}} = \gamma_{P_q}$, $P_q(f)$ is called a Homogeneous polynomial in f, otherwise Non-homogeneous.

Definition 2.[1]: let $q \in C - \{0,1\}$ and $a \in C$. We define the counting function $n_q(r,a)$ to be the number of points z_0 in the disk of radius r centered at the origin such that $f(z_0) = a$, where the contribution to $n_q(r,a)$ is the number of equal terms in the beginning of Taylor series expansion of $f(z)$ and $f(qz)$ in a neighbourhood of z_0. We call such points q-separated a-points of f in the disc $\{z : |z| \le r\}$. The number of q-separated pole pairs $n_q(r,\infty)$ is the number of q-separated 0-pairs of $1/f$. This means that if f has a pole with multiplicity p at z_0 and another pole with multiplicity s at qz_0 then this pair is counted min $\{p,s\}+m$ times in $n_q(r,\infty)$, where m is the number of equal terms in the beginning of Laurent series expansion of $f(z)$ and $f(qz)$ in a neighbourhood of z_0.

Toda [9](1970) proved the following result.

Theorem A. *If $f(z)$ is a transcendental meromorphic function in $|z| < \infty$, then*

$$\sum_{a\neq\infty}\delta_\alpha(a) \le \liminf_{r\to\infty}\frac{T_\alpha(r, f')}{T_\alpha(r, f)} \le \limsup_{r\to\infty}\frac{T_\alpha(r, f')}{T_\alpha(r, f)} \le 2 - \Theta_\alpha(\infty, f).$$

Later, Sarangi and Patil [8] proved the following results.

Theorem B. *If* $f(z)$ *is a transcendental meromorphic function in* $|z|<\infty$, *then for any positive integer I,*

$$\sum_{a\neq\infty}\delta_\alpha(a)\leq\liminf_{r\to\infty}\frac{T_\alpha(r,f^{(I)})}{T_\alpha(r,f)}\leq\limsup_{r\to\infty}\frac{T_\alpha(r,f^{(I)})}{T_\alpha(r,f)}\leq(I+I)-I\Theta_\alpha(\infty,f).$$

Theorem C. *If* $f(z)$ *is a transcendental meromorphic function in* $|z|<\infty$, *then for any positive integer I,*

$$\frac{1}{(I+I)-I\Theta_\alpha(\infty)}\sum_{a\neq\infty}\delta_\alpha(a)\leq\delta_\alpha(0,f^{(I)}.$$

Again Shankar. M. Pawar and Bhoosnurmath S. S [2](2002)extended the above results for homogeneous differential polynomials and proved the following results.

Theorem D. *If* $f(z)$ *is a transcendental meromorphic function in* $|z|<\infty$, *then*

$$\sum_{a\neq\infty}\delta_\alpha(a)\leq\liminf_{r\to\infty}\frac{T_\alpha(r,P(f))}{T_\alpha(r,f)}\leq\limsup_{r\to\infty}\frac{T_\alpha(r,P(f))}{T_\alpha(r,f)}\leq[\Gamma_P-(\Gamma_P-\gamma_P)\Theta_\alpha(\infty,f)]$$

where $P(f)$ is a homogeneous differential polynomial, not involving the f term.

Theorem E. *Let* $f(z)$ *be a transcendental meromorphic function in the finite plane,* $P(f)$ *is a homogeneous differential polynomial, then*

$$\frac{\gamma_P}{\Gamma_P-(\Gamma_P-\gamma_P)\Theta_\alpha(\infty,f)}\sum_{a\in C}\delta_\alpha(a)\leq\delta_\alpha(0,P(f)).$$

We extend the above results to the difference polynomials and prove the following results.

Theorem 1.1. *Let* $f(z)$ *be a transcendental meromorphic function of zero order with* $P(f)$ *as a homogeneous difference polynomial. For* $f^n(0)\neq\infty, a\neq 0,\infty$ *and for* $P[f(0)]\neq 0$. *We have*

$$\gamma_{P_q}\sum_{a\in C}\delta_\alpha(a_j)+2-[\Pi_q(\infty,f)+\delta_\alpha^q(\infty,f)]\leq\liminf_{r\to\infty}\frac{T_\alpha(r,P_q(f))}{T_\alpha(r,f)}$$

$$\leq\limsup_{r\to\infty}\frac{T_\alpha(r,P_q(f))}{T_\alpha(r,f)}\leq\Gamma_{P_q}-(\Gamma_{P_q}-\gamma_{P_q})\Theta_\alpha(\infty,f).$$

Theorem 1.2. *Let* $f(z)$ *be a transcendental meromorphic function of zero order in the finite plane with* $P(f)$ *as a homogeneous difference polynomial. For* $f^n(0)\neq\infty, a\neq 0,\infty$ *and for* $P[f(0)]\neq 0$. *We have*

$$\frac{\gamma_{P_q}}{\Gamma_{P_q}-(\Gamma_{P_q}-\gamma_{P_q})\Theta_\alpha(\infty,f)}\sum_{a\in C}\delta_\alpha(a)\leq\delta^q(0,P(f))+\Pi_q(\infty,P(f))-1.$$

2. Some Lemmas

To our main results we need the following Lemmas.

Lemma 2.1. *([2]). If* $f(z)$ *is a transcendental meromorphic function and* $a_1,a_2,\ldots,a_q$ *are distinct elements, then*

$$\gamma_P\sum_{j=1}^{q}m_\alpha(r,a_j,f)\leq T_\alpha(r,P(f))-N(r,0,P(f))+S_\alpha(r,f)$$

or

$$\underline{\gamma_P}\sum_{j=1}^{q} m_\alpha(r,a_j,f)\le m_\alpha\left(r,\frac{1}{P(f)}\right)+S_\alpha(r,f)$$

where $P(f)$ is a homogeneous differential polynomial of degree γ_P.

Lemma 2.2. *([3]) If $Q[f]$ is a differential polynomial in f with arbitrary meromorphic coefficients q_j $1\le j\le n$, then*

$$m(r,Q[f])\le \gamma_Q m(r,f)+\sum_{j=1}^{n} m(r,a_j)+S(r,f).$$

Lemma 2.3. *([6]) If $f(z)$ is a transcendental meromorphic function then*

$$N(r,P(f))\le \gamma_P N(r,f)+(\Gamma_P-\gamma_P)\overline{N}(r,f)+S(r,f).$$

3. Proofs of The Theorems.

In this section we present the proofs of the main results.

Proof of Theorem 1.1.

By Lemma 2.2 and Lema 2.3, we have

$$m_\alpha(r,P_q[f])\le \gamma_{P_q} m_\alpha(r,f)+S_\alpha(r,f)$$

and

$$N_\alpha(r,P_q[f])\le \gamma_{P_q} N_\alpha(r,f)+(\Gamma_{P_q}-\gamma_{P_q})\overline{N}_\alpha(r,f)+S_\alpha(r,f).$$

Then we get,

$$T_\alpha(r,P_q[f])\le \gamma_{P_q} T_\alpha(r,f)+(\Gamma_{P_q}-\gamma_{P_q})\overline{N}_\alpha(r,f)+S_\alpha(r,f).$$

Dividing by $T_\alpha(r,f)$ and taking limit superior both sides we get

$$\limsup_{r\to\infty}\frac{T_\alpha(r,P_q(f))}{T_\alpha(r,f)}=\gamma_{P_q}+(\Gamma_{P_q}-\gamma_{P_q})\limsup_{r\to\infty}\frac{\overline{N}_\alpha(r,f)}{T_\alpha(r,f)}$$

$$\begin{aligned}\limsup_{r\to\infty}\frac{T_\alpha(r,P_q(f))}{T_\alpha(r,f)} &\le \gamma_{P_q}+(\Gamma_{P_q}-\gamma_{P_q})[1-\Theta_\alpha(\infty,f)]\\ &\le \Gamma_{P_q}-(\Gamma_{P_q}-\gamma_{P_q})\Theta_\alpha(\infty,f).\end{aligned} \tag{3.1}$$

On the other hand by Lemma 2.1, we have

$$\begin{aligned}\gamma_{P_q}\sum_{j=1}^{q} m_\alpha(r,a_j,f) &\le m_\alpha\left(r,\frac{1}{P_q(f)}\right)+S_\alpha(r,f)\\ &\le T_\alpha\left(r,\frac{1}{P_q(f)}\right)-N_\alpha\left(r,\frac{1}{P_q(f)}\right)+S_\alpha(r,f)\end{aligned}$$

or

$$T_\alpha\left(r,\frac{1}{P_q(f)}\right)\geq \gamma_{P_q}\sum_{j=1}^{q}m_\alpha(r,a_j,f)+N_\alpha\left(r,\frac{1}{P_q(f)}\right)+S_\alpha(r,f)$$

$$\geq \gamma_{P_q}\sum_{j=1}^{q}m_\alpha(r,a_j,f)+\tilde{N}_q\left(r,\frac{1}{P_q(f)}\right)+N_q\left(r,\frac{1}{P_q(f)}\right)+S_\alpha(r,f)$$

$$\limsup_{r\to\infty}\frac{T_\alpha(r,P_q(f))}{T_\alpha(r,f)}\geq \gamma_{P_q}\sum_{j=1}^{q}\delta_\alpha(a_j)+[1-\Pi_q(\infty,f)]+[1-\delta_\alpha^q(\infty,f)]+S_\alpha(r,f)$$

$$\geq \gamma_{P_q}\sum_{j=1}^{q}\delta_\alpha(a_j)+2-[\Pi_q(\infty,f)+\delta_\alpha^q(\infty,f)]+S_\alpha(r,f).$$

Therefore,

$$\gamma_{P_q}\sum_{a\in C}\delta_\alpha(a_j)+2-[\Pi_q(\infty,f)+\delta_\alpha^q(\infty,f)]\leq \liminf_{r\to\infty}\frac{T_\alpha(r,P_q(f))}{T_\alpha(r,f)}$$

$$\leq \limsup_{r\to\infty}\frac{T_\alpha(r,P_q(f))}{T_\alpha(r,f)}\leq \Gamma_{P_q}-(\Gamma_{P_q}-\gamma_{P_q})\Theta_\alpha(\infty,f).$$

Corollay 1. If $f(z)$ is a transcendental meromorphic function with $\sum_{a\in C}\delta_\alpha(a)=1,\Theta_\alpha(\infty)=1$ and $\Pi_q(\infty,f)+\delta_\alpha^q(\infty,f)=2$ then $T_\alpha(r,P_q(f))\sim\gamma_{P_q}T_\alpha(r,f)$.

Corollay 2 If $f(z)$ is a transcendental meromorphic function in $|z|<\infty$, then if $\Gamma_{P_q}=2,\ \gamma_{P_q}=1,\ \sum_{a\in C}\delta_\alpha(a,f)=2$ and $\Pi_q(\infty,f)+\delta_\alpha^q(\infty,f)=2$ then

$$\lim_{r\to\infty}\frac{T_\alpha(r,P_q(f))}{T_\alpha(r,f)}\leq 2-\Theta_\alpha(\infty,f).$$

Proof of Theorem 1.2. By Lemma 2.1

$$\gamma_{P_q}\sum_{j=1}^{q}m_\alpha(r,a_j,f)\leq m_\alpha\left(r,\frac{1}{P_q(f)}\right)+S_\alpha(r,f)$$

$$\leq T_\alpha\left(r,\frac{1}{P_q(f)}\right)-N_\alpha\left(r,\frac{1}{P_q(f)}\right)+S_\alpha(r,f)$$

$$\leq T_\alpha(r,P_q(f))-\tilde{N}_q\left(r,\frac{1}{P_q(f)}\right)-N_q\left(r,\frac{1}{P_q(f)}\right)+S_\alpha(r,f)$$

Dividing by $T_\alpha(r,f)$ and taking limsup on both sides, we get

$$\gamma_{P_q}\sum_{i=1}^{q}\delta_\alpha(a_i,f)\le \limsup_{r\to\infty}\frac{T_\alpha(r,P_q(f))}{T_\alpha(r,f)}-\limsup_{r\to\infty}\left[\frac{\tilde{N}_q\left(r,\frac{1}{P_q(f)}\right)+N_q\left(r,\frac{1}{P_q(f)}\right)}{T_\alpha(r,P_q(f))}\right]$$

$$.\limsup_{r\to\infty}\frac{T_\alpha(r,P_q(f))}{T_\alpha(r,f)}+\limsup_{r\to\infty}\frac{S_\alpha(r,f)}{T_\alpha(r,f)}$$

$$\le \limsup_{r\to\infty}\left[1-\frac{\tilde{N}_q\left(r,\frac{1}{P_q(f)}\right)+N_q\left(r,\frac{1}{P_q(f)}\right)}{T_\alpha(r,P_q(f))}\right].\limsup_{r\to\infty}\frac{T_\alpha(r,P_q(f))}{T_\alpha(r,f)}$$

$$+\limsup_{r\to\infty}\frac{S_\alpha(r,f)}{T_\alpha(r,f)}$$

$$\le\left[\delta^q(0,P(f))+\Pi_q(\infty,P(f))-1\right].\limsup_{r\to\infty}\frac{T_\alpha(r,P_q(f))}{T_\alpha(r,f)}+o(1).$$

By Theorem 2.1, we have

$$\gamma_{P_q}\sum_{i=1}^{q}\delta_\alpha(a_i,f)\le\left[\delta^q(0,P(f))+\Pi_q(\infty,P(f))-1\right].\limsup_{r\to\infty}\frac{T_\alpha(r,P_q(f))}{T_\alpha(r,f)}$$

$$\le\left[\delta^q(0,P(f))+\Pi_q(\infty,P(f))-1\right]\Gamma_{P_q}-(\Gamma_{P_q}-\gamma_{P_q})\Theta_\alpha(\infty,f).$$

Therefore,

$$\frac{\gamma_{P_q}}{\Gamma_{P_q}-(\Gamma_{P_q}-\gamma_{P_q})\Theta_\alpha(\infty,f)}\sum_{a\in C}\delta_\alpha(a)\le\delta^q(0,P(f))+\Pi_q(\infty,P(f))-1.$$

Corollary 3. If $f(z)$ is a meromorphic function with $\Theta_\alpha(\infty,f)=1, \Pi_q(\infty,P(f))=1$ then

$$\sum_{a\in C}\delta_\alpha(a)\le\delta^q(0,P(f)).$$

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References

[1] Barnett D. C., Halburd R. G., Korhonen R. J., and Morgan W., *Nevanlinna theory for the q difference operator and meromorphic solutions of q-difference equations*, Preprint submitted to Elsevier Science, February 6 (2006). J. Inequal. Pure Appl. Math.,**6** (116)(2005).

[2] Bhoosnurmath S. S. and Shankar M. Pawar *On modified deficiencies of meromorphic functions and exceptional values of differential polynomials*, Ph. D. thesis submittedb to Karnatak University, Dharwad, 2002.

[3] Doeringer W., *Exceptional values of differential polynomials*, Pacific J. Math.,**98** (1982) 55-62.

[4] Hayman W. K., *Meromorphic Functions* , Clarendon Press, Oxford (1964).

[5] Hiong K. L., *Chinesse Mathematic*, **9** (1): 146. (1967).

[6] Indrajit Lahari, *Deficiencies of differential polynomials*, Indian J. Pure Appl. Math., **30** (5)(1999) 435-447.

[7] Milloux, *Annales Ecole Normale Superior*, **63** (3): 289 (1946).

[8] S. M. Sarangi and S. J. patil, *On modified deficiencies of Meromorphic functions* , J. Inequal. Pure Appl. Math., **10** (1), 6-13 (1979).

[9] Toda N., *On a modified deficiencies of Meromorphic functions* , Tohoku Math. J., **22**, 635-658 (1970).