Journal of Progressive Research in Mathematics(JPRM)
ISSN: 2395-0218

SCITECH
RESEARCH ORGANISATION|

Volume 5, Issue 1
Published online: August 23, 2015|

Journal of Progressive Research in Mathematics
www.scitecresearch.com/journals

Generalization of a fixed point theorem of Suzuki type in complete metric space

K. C. Deshmukh, Rakesh Tiwari[1] and Savita Gupta.

Abstract. The aim of this paper is to generalize a fixed point result given by Popescu[17]. Our results complement and extend very recent results proved by Suzuki [T. Suzuki, A generalized Banach contraction principle that characterizes metric completeness, Proc. Amer. Math. Soc. 136 (2008) 1861 - 1869]. To validate our result an example is given.

Key Words and Phrases: Common fixed point; Complete metric Space.

2000 Mathematics Subject Classification: Primary: 47H10; Secondary: 54H25.

1. Introduction

Let (X, d) be a metric space, T a self-mapping on X and k a nonnegative real number such that the inequality $d(Tx, Ty) \leq kd(x, y)$ holds for any $x, y \in X$. If $k < 1$ then T is said to be a contractive mapping and if k = 1, then T is said to be a nonexpansive mapping. The Banach theorem states that if X is complete, then every contractive mapping has a unique fixed point. There exists a vast literature about contractive and nonexpansive type mappings, where the contractive and nonexpansive conditions are substituted with more general conditions (see, for instance [1 - 10]).

Bogin[1] proved the following result.

Theorem 1.1. Let (X, d) be a nonempty complete metric space and $T : X \to X$ a mapping satisfying

$$d(Tx,Ty) \leq ad(x,y) + b[d(x,Tx) + d(y,Ty)] + c[d(x,Ty) + d(y,Tx)], \tag{1}$$

here $a \geq 0$, $b > 0$, $c > 0$ and $a + 2b + 2c = 1$. Then T has a unique fixed point.

This result was generalized by Li[15] and Gregus[11] considered a class of self-mapping T on X which satisfy (1) with $c = 0$. He proved the following theorem.

Theorem 1.2 Let (X, d) be a complete metric space and $S : X \to X$. Define a non-increasing function θ from [0, 1) onto (1/2, 1] by

$$\theta(r) = \begin{cases} 1, & if \quad 0 \leq r \leq \frac{\sqrt{5}-1}{2}; \\ \frac{r-1}{r^2}, & if \quad \frac{\sqrt{5}-1}{2} \leq r \leq \frac{1}{\sqrt{2}}; \\ \frac{1}{1+r}, & if \quad \frac{1}{\sqrt{2}} \leq r \leq 1. \end{cases}$$

Assume that there exists $r \in [0,1)$ such that $\theta(r)d(x, Sx) \leq d(x, y)$ implies

$d(Sx, Sy) \leq rd(x, y)$, for all $x, y \in X$. Then S has a unique fixed point. Also, Kikkawa and Suzuki[14] proved Kannan, Meir and Keeler [13] versions of Theorem 1.2. Moreover, Suzuki studied a class of operators satisfying the following condition.

[1]Corresponding author

Theorem 1.3 Let T be a mapping on a subset C of a Banach space E. Then T is said to satisfy condition (C) if for all $x, y \in C$

(C) $1/2 \| Tx - Ty \| \leq \| x - y \|$.

In 2010, Tiwari et al. [18] proved a common fixed point theorem for weakly compatible mapping in symmetric spaces satisfying an integral type contractive condition. Recently Popescu [17] proved the following theorem.

Theorem 1.4 Let (X, d) be a nonempty complete metric space and $T : X \to X$ be a mapping satisfying $1/2\, d(x,Tx) \leq d(x,y)$ implies

$$d(Tx,Ty) \leq ad(x,y) + b[d(x,Tx) + d(y,Ty)] + c[d(x,Ty) + d(y,Tx)] \qquad (2)$$

where $a \geq 0$, $b > 0$, $c > 0$ and $a + 2b + 2c = 1$. Then T has a unique fixed point.

Inspired by this theorem, we present a common fixed point result of Suzuki type in complete metric space in this paper.

2. Main results

The following theorem generalizes result of Popescu[17].

Theorem 2.1. Let (X, d) be a nonempty complete metric space and $T : X \to X$ be a mapping satisfying $\frac{1}{2}d(x,Tx) \leq d(x,y)$ implies

$$d(Tx,Ty) + p\, max[d(x,y), d(x,Tx) + d(y,Ty), d(x,Ty) + d(y,Tx)]$$
$$\leq ad(x,y) + b[d(x,Tx) + d(y,Ty)] + c[d(x,Ty) + d(y,Tx)] \qquad (3)$$

where $a \geq 0$, $b > 0$, $c > 0$, $p \geq 0$ and $a + 2b + 2c - 2p = 1$. Then T has a unique fixed point.

Proof. Let $x \in X$ be arbitrary. we have

$$d(Tx,T^2x) + p\ max\ \{d(x,Tx), [d(x,Tx) + d(Tx,T^2x)], [d(x,T^2x) + d(Tx,Tx)]\}$$
$$\leq a\ d(x,Tx) + b\ [d(x,Tx) + d(Tx,T^2x)] + c\ [d(x,T^2x) + d(Tx,Tx)].$$

Hence

$$d(Tx,T^2x) + p\ max\{d(x,Tx), d(x,Tx) + d(Tx,T^2x), d(x,Tx) + d(Tx,T^2x)\}$$
$$\leq (a+b)d(x,Tx) + b\ d(Tx,T^2x) + c\ [d(x,Tx) + d(Tx,T^2x)].$$

Therefore we obtain,

$$d(Tx,T^2x) \leq (a+b+c-p)/(1-b-c-p)\ d(x, Tx)$$
$$= d(x,Tx).$$

This implies that the sequence $\{d_n\}_{n=0}^{\infty}$ is a decreasing one, where

$d_n := d(T^n x, T^{n+1}x)$ and $T^0 x = x$.

Next, we will show that there exists a nonnegative number $m < 2$ such that $d(Tx, T^3x) \leq md_0$. First, we suppose that $d(x,T^2x) \geq d(x,Tx)$. Then $1/2d(x,Tx) \leq d(x,T^2x)$ and we have

$$d(Tx,T^3x) + p\ max\{d(x,T^2x), d(x,Tx) + d(T^2x,T^3x), d(x,T^3x) + d(Tx,Tx)\}$$
$$\leq a\ d(x,T^2x) + b\ d(x,Tx) + b\ d(T^2x,T^3x) + c\ d(x,T^3x) + cd(Tx,T^2x).$$

Thus,

$d(Tx, T^3x) + p \max\{d(x, Tx), d(Tx, T^3x)\} \leq a(d_0+d_1)+ bd_0+ bd_2 + c[d_0+ d(Tx, T^3x)]\, d_1$.

Setting $m = (1 + a)/(1-c-p)$, we have $m_1 < 2$ and $d(Tx, T^3x) \leq m_1 d_0$. Now, we assume that $d(x, T^2x) < d(x, Tx)$. Since

$$d(Tx, T^2x) + p\max\{d(x, Tx), d(x, Tx) + d(Tx, T^2x), d(x, T^2x) + d(Tx, Tx)\}$$

$$\leq a\, d(x, Tx) + b\, d(x, Tx) + b\, d(Tx, T^2x) + c\, d(x, T^2x) + c\, d(Tx, Tx),$$

we get $d_1 < (a+b)d_0 + bd_1 + cd_0 + p(d_0 + d_1)$. Hence, $d_1 < (a+b+c-p) / (1-b-p)$ and then

$d(Tx, T^3x) \leq d(Tx, T^2x) + d(T^2x, T^3x) \leq 2d(Tx, T^2x) \leq (2a + 2b+ 2c - 2p)/(1-b +c) . d_0 = (1+a)/ (1-b+p)$.

Setting $m_2 = (1+a)/ (1-b-p)$, we have, $m_2 < 2$ and $d(Tx, T^3x) \leq m_2 d_0$.

Taking $m = max\,\{m_1, m_2\}$, we get $0 < m < 2$ and $d(Tx, T^3x) \leq md(x, Tx)$, for all $x \in X$.

Since $1/2\, d(Tx, T^2x) \leq d(Tx, T^2x)$ we have,

$$d(T^2x, T^3x) + p \max\{d(Tx, T^2x), [d(Tx, T^2x) + d(T^2x, T^3x)], d(Tx, T^3x)\}$$

$$\leq a\, d(Tx, T^2x) + b[d(Tx, T^2x) + d(T^2x, T^3x)] + cd(Tx, T^3x)).$$

Thus,

$$d_2 \leq (a + 2b - 2p)d_0 + mcd_0 = (a + 2b + 2p + mc)d_0$$

Setting $k = a + 2b - 2p + mc$, we have $k < 1$ and $d_2 \leq kd_0$ for all $x \in X$.

Let $x_0 \in X$ and $u_n = T^n x_0$. Then $d_{n+2} \leq kd_n$ for all $n \geq 0$, where $d_n = d(u_n, u_{n+1})$. Therefore, for any even integer $n \geq 0$ we have by induction $d_n \leq k^{n/2} d_0 \leq k^{(n-1)/2} d_0$ and for every odd integer $n \geq 1$ we have also by induction $d_n \leq k^{(n-1)/2} d_1 \leq k^{(n-1)/2} d_0$. Hence, for all $n \geq 0$ we get $d_n \leq k^{(n-1)/2} d_0$. Since $k \in (0,1)$ we obtain that u_n is a Cauchy sequence and by completeness of X there exists $z \in X$ such that the sequence $\{u_n\}$ converges to z as $n \to \infty$.

Next, we will show that z is a fixed point of T. Assuming that there exists n such that $d(z, u_n) < 1/2 d(u_n, u_{n+1})$ and $d(z, u_{n+1}) < 1/2 d(u_{n+1}, u_{n+2})$ we obtain

$$d_n = d(u_n, u_{n+1}) \leq d(z, u_n) + d(z, u_{n+1}) < 1/2(d_n + d_{n+1}) \leq d_n.$$

This is a contradiction, so for all $n \geq 0$ we have either $d(z, u_n) \geq 1/2 d(u_n, u_{n+1})$ or $d(z, u_{n+1}) \geq 1/2 d(u_{n+1}, u_{n+2})$. Thus, there exists a subsequence $\{n_j\}$ of n such that $d(u_{n_j}, z) \leq 1/2 d(u_{n_{j+1}}, u_{n_j})$ for every integer $j \geq 0$. Then, we have

$$d(Tz, u_{nj+1}) + p \max\{d(z, u_{nj}), d(z, Tz) + d(u_{nj}, u_{nj+1}), d(z, u_{nj+1}) + d(Tz, u_{nj})\}$$

$$\leq ad(z, u_{nj}) + bd(z, Tz) + bd(u_{nj}, u_{nj+1}) + cd(z, u_{nj+1}) + cd(Tz, u_{nj}).$$

Taking $j \to \infty$ we get $d(Tz, z) \leq (b + c)\, d(Tz, z)$. This implies $d(Tz, z) = 0$ and so, $Tz = z$. If z' is another fixed point T then $d(z', z) \leq 1/2d(z, Tz) = 0$ and then

$$d(z', z) = d(Tz', Tz) + p\max\{d(z', z), [d(z', z') + d(z, z)], [d(z', z) + d(z', z)]\}$$

$$\leq a\, d(z', z) + b\, [d(z', z') + d(z, z)] + c\, [d(z', z) + d(z', z)].$$

Hence,

$$d(z', z) \leq (a + 2c - 2p)d(z', z).$$

This implies $d(z', z) = 0$, which is a contradiction. So, T has a unique fixed point.

Remark 2.2 2 If we put $p = 0$ in above, we get Theorem 1.4 of Popescu[17].

Now we present the following example to validate our result.

Example 2.3. Let $X = [-1,1]$ with the usual metric and let $T : X \to X$ be given as

$$Tx = \begin{cases} -x, & if \quad x \in [0, 1/2) \bigcup (1/2, 1] = U; \\ \frac{x}{4}, & if \quad x \in [-1, 0) = V; \\ 0, & if \quad x = \frac{1}{2}. \end{cases}$$

We will prove that:

1. T has a unique fixed point.

2. T satisfies condition (3) with a = 1/3, b = c = 1/4, p = 1/6

i.e. 1/2 d(x, Tx) ≤ d(x, y) ⇒ d(Tx, Ty) ≤ m(x, y) where

m(x, y) = 1/3d(x, y) + 1/4 [d(x, Tx) +d(y, Ty) + d(x, Ty) + d(y, Tx)] − p max{d(x, y), d(x, Tx) + d(y, Ty), d(x, Ty) + d(y, Tx)}.

3. T does not satisfy Suzuki condition of Theorem 1.2.

4. T does not satisfy Popescu condition of Theorem 1.4 with a = 1/3, b = c = ¼ and p = 1/6.

Proof. 1 is obvious. Secondly we consider the following.

(i) For x, y ∈ U then

m(x, y) =1/3 |y −x|+(1/4 + 1/4)|2y +2x|− 1/6 |2y +2x| = 1/3 |y −x|+1/2 |2y +2x|− 1/6 |2y +2x|, or

m(x, y) = (1/3 |y − x| + 2/3 |y + x| ≥ |y − x| = d(Tx, Ty) and (2) holds.

(ii) If x, y ∈ V , then m(x, y) = (1/3 + 3/4 (1/4 + 1/4 + 1/6))|y − x| = 5/6 |y − x| ≥ 1/4 |y − x| = d(Tx, Ty) so (2) holds.

(iii) If x ∈ U, y ϵV , then m(x, y) = 1/3 |(x−y)|+(1/4 + 1/4 + 1/6 |2x− 3y/4 |) = 13x/12 – 5y/24 + 1/4 |y +x| ≥x + y/4 = d(Tx, Ty) so (2) holds.

(iv) If x ∈ V , y ϵ U, then m(x, y) ≥ d(Tx, Ty) like in (iii).

(v) For x ∈ U, y = 1/2 , then m(x, y) = 1/3 |x− 1/2 |+(1/4 |4x+1|− 1/6 |2x| = x+ 1/12 ≥ x = d(Tx, Ty)

and (2) holds.

(vi) For x ∈ V , y = 1/2 , then m(x, y) = 1/3 |x− 1/2 |+ 1/4 | 3x/2 +1|− 1/6 | 3x/4 + 1/2 | = 7x/12 ≥x/4 = d(Tx, Ty) and (2) holds.

(vii) If x = 1/2 , y ∈ U, then m(x, y) = 1/3 |y – 1/2 | + 1/4 |1 + 4y| − 1/6 |2y + 1/2 | = y + 5/6 ≥ d(Tx, Ty) and (2) holds.

(viii) If x = 1/2 , y ∈ V , then

m(x, y) = 1/3 | 1/2 −y|+ 1/4 | 3y/2 +1|− 1/12 – y/8 = 1/3 + y/6 ≥ 1/2 and d(Tx, Ty) = 0−y/4 ,−y/4 ≤ 1/4 (y/4 ∈ [−1/4 , 0)) Hence (2) holds.

(xi) If x = y then (2) is obvious.

(3) If x = 0, y = 1, then θ (r)d(x, Tx) = 0 < 1 = d(x, y) and d(Tx, Ty) = 1, so condition

from Theorem 1.2. does not hold.

(4) If x = 1/2 , y = 1 we have d(Tx, Ty) = 1 and m(x, y) = 1/3 |1|+1/4 | 1/2−0|+1/4 |1| = 1/3+1/8+1/4+1/4 = 23/24 so d(Tx, Ty) > m(x, y). Therefore Popescu's condition Theorem 1.4 does not hold.

References

[1] J. Bogin, A generalization of a fixed point theorem of Goebel, Kirk and Shimi, Canad.Math. Bull. 19 (1976) 7 - 12.

[2] Lj.B. Ciric, On some nonexpansive type mappings and fixed point, Indian J. Pure Appl.Math. 24 (3) (1993) 145 - 149.

[3] Lj.B. Ciric, On a common fixed point theorem of a Gregus type, Publ. Inst. Math. 49 (1991) 174 - 178.

[4] Lj.B. Ciric, Diviccaro, Fisher and Sessa Open questions, Arch. Math. 29 (1993) 145 - 152.

[5] Lj.B. Ciric, On a generalization of a Gregus fixed point theorem, Czechoslovak Math. J. 50(2000) 449 - 458.

[6] Lj.B.Ciric, A new class of nonexpansive type mappings and fixed points, Czechoslovak Math. J. 49 (124) (1999) 891 - 899.

[7] D. Delbosco, O. Ferrero, F. Rossati, Teoremi di punto fisso per applicazioni negli spazi di Banach, Boll. Un. Mat. Ital. Sez. A (6) 2 (1993) 297 - 303.

[8] S. Dhompongsa and H. Yingtaweesittikul, Fixed points for multivalued mappings and the metric completeness, Fixed Point Theory Appl. (2009) 1 - 15. Article ID 972395.

[9] M.L. Diviccaro, B. Fisher, S. Sessa, A common fixed point theorem of Gregu type, Publ. Math. Debrecen 34 (1997) 83 - 89.

[10] B. Fisher, Common fixed points on a Banach space, Chung Yuan J. 11 (1982) 19 - 26.

[11] M. Gregus, A fixed point theorem in Banach spaces, Boll. Unione Mat. Ital. Sez. A (5) 17(1980) 193 - 198.

[12] G. Jungck, On a fixed point theorem of Fisher and Sessa, Int. J. Math. Math. Sci. 13 (1990) 497 - 500.

[13] R. Kannan, Some results on fixed point theory-II, Amer. Math. Monthly 76 (1969) 405 - 408.

[14] M. Kikkawa, T. Suzuki, Three fixed point theorems for generalized contractions with constants in complete metric spaces, Nonlinear Anal. 69 (2008) 2942 - 2949.

[15] B.I. Li, Fixed point theorems of nonexpansive mappings in convex metric spaces, Appl.Math. Mech. 10 (1989) 183 - 188.

[16] O. Popescu, Two fixed point theorems for generalized contractions with constants in complete metric space, Cent. Eur. J. Math. 7 (3) (2009) 529 - 538.

[17] O. Popescu, Two generalizations of some fixed point theorems, Computer and mathematics with applications 62(2011) 3912 - 3919.

[18] Rakesh Tiwari, S. KShrivastava, V. K. Pathak, A common fixed point theorem for weak-compatible mappings in symmetric spaces satisfying an integral type contractive condition, Hecettepe Journal of mathematics and statistics 39(2), (2010) 151 - 158.

K. C. Deshmukh,
Department Of Mathematics
R. T. M. Nagpur University Nagpur
(Maharashtra),440013 India.

Rakesh Tiwari,
Department Of Mathematics
Govt. V.Y.T. PG. Autonomous College Durg
(C.G.),491001. India.

Savita Gupta,
Department Of Mathematics
Shri Shankaracharya Institute Of Technology And Management
Bhilai(C.G.),492001, India.