SCITECH
RESEARCH ORGANISATION|

Volume 4, Issue 4

Published online: August 04, 2015|

Journal of Progressive Research in Mathematics
www.scitecresearch.com/journals

Approximation of Fourier Series of a function of Lipchitz class by Product Means

[1] Subrata K Sahu, [2] D. Acharya, [3] P.C.Nayak and [4] U.K.Misra*

[1] National Institute of Science and Technology, Ganjam, Odisha, India
E-mail: subrata_sai11@yahoo.com

[2] National Institute of Science and Technology, Ganjam, Odisha, India
E-mail: dpak.acharya888@gmail.com

[3] Bhadrak(Autonomous) College, Bhadrak, Odisha, India
E-mail: pcnayak02@gmail.com

[4] National Institute of Science and Technology, Ganjam, Odisha, India
E-mail: umakanta_misra@yahoo.com

*Correspondence author

Abstract: Lipchitz class of function had been introduced by McFadden [8]. Recently dealing with degree of approximation of Fourier series of a function of Lipchitz class Nigam [12] and Misra et al.[9,10,11] have established certain theorems. Extending their results, in this paper a theorem on degree of approximation of a function $f \in W(L^p, \xi(t))$ by product summability $(E,s)(N,p_n,q_n)$ has been established.

Keywords: Degree of Approximation; $W(L^p, \xi(t))$class of function; $(E,s)(N,p_n,q_n)$ product mea; Fourier series; Lebesgue integral.

2010-Mathematics subject classification: 42B05, 42B08.

1. Introduction:

Let $\sum a_n$ be a given infinite series with sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ and $\{q_n\}$ be sequences of positive real numbers such that

(1.1) $$P_n = \sum_{\upsilon=0}^{n} p_\upsilon \quad \text{and} \quad Q_n = \sum_{\upsilon=0}^{n} q_\upsilon .$$

Let

(1.2) $$t_n = \frac{1}{r_n} \sum_{\upsilon=0}^{n} p_{n-\upsilon} q_\upsilon s_\upsilon ,$$

where $r_n = p_0 q_n + p_1 q_{n-1} + \ldots + p_n q_0 (\neq 0)$, $p_{-1} = q_{-1} = r_{-1} = 0$.

Then $\{t_n\}$ is called the sequence of (N,p_n,q_n) mean of the sequence $\{s_n\}$. If

(1.3) $$t_n \to s \quad , \text{as} \quad n \to \infty ,$$

then the series $\sum a_n$ is said to be (N, p_n, q_n) summable to s.

The necessary and sufficient conditions for regularity of (N, p_n, q_n) method are[3]:

(1.4) (i) $\frac{p_{n-\nu} q_\nu}{r_n} \to 0$, as $n \to \infty$, for each integer $\nu \geq 0$

and

(1.5) (ii) $\sum_{\nu=0}^{n} |p_{n-\nu} q_\nu| < H |r_n|$,

where H is a positive number independent of n.

The sequence –to-sequence transformation [5],

(1.6) $$T_n = \frac{1}{(1+q)^n} \sum_{\upsilon=0}^{n} \binom{n}{\upsilon} q^{n-\upsilon} s_\upsilon \quad ,$$

defines the sequence $\{T_n\}$ of the (E, q) mean of the sequence $\{s_n\}$. If

(1.7) $$T_n \to s \text{ , as } n \to \infty,$$

then the series $\sum a_n$ is said to be (E, q) summable to s. Clearly (E, q) method is regular [5].

Further, the (E, q) transform of the (N, p_n, q_n) transform of $\{s_n\}$ is defined by

$$\tau_n = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} t_k$$

(1.8) $$= \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \frac{1}{r_k} \sum_{\upsilon=0}^{k} p_{k-\upsilon} q_\upsilon s_\upsilon \right\}$$

If

(1.9) $$\tau_n \to s \text{ , as } n \to \infty,$$

then $\sum a_n$ is said to be $(E, q)(N, p_n, q_n)$ - summable to s.

Let $f(t)$ be a periodic function with period 2π and L- integrable over $(-\pi,\pi)$, The Fourier series associated with f at any point x is defined by

(1.10) $$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x).$$

The L_∞- norm of a function $f : R \to R$ is defined by

(1.11) $$\|f\|_\infty = \sup\{ |f(x)| : x \in R \}$$

and the L_υ- norm is defined by

(1.12) $$\|f\|_\upsilon = \left(\int_0^{2\pi} |f(x)|^\upsilon\right)^{\frac{1}{\upsilon}} \quad , \upsilon \ge 1.$$

The degree of approximation of a function $f : R \to R$ by a trigonometric polynomial $P_n(x)$ of degree n under norm $\|\ .\ \|_\infty$ is defined by

(1.13) $$\|P_n - f\|_\infty = \sup\{|P_n(x) - f(x)| : x \in R\}$$

and the degree of approximation $E_n(f)$ of a function $f \in L_\upsilon$ is given by [17]

(1.14) $$E_n(f) = \min_{P_n} \|P_n - f\|_\nu .$$

This method of approximation is called Trigonometric Fourier approximation.

A function $f \in Lip\alpha$ if [8]

(1.15) $$|f(x+t) - f(x)| = O\left(|t|^\alpha\right) \ , \ 0 < \alpha \le 1,$$

and $f \in Lip(\alpha, r)$, for $0 \le x \le 2\pi$, if [8]

(1.16) $$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r \, dx\right)^{\frac{1}{r}} = O\left(|t|^\alpha\right) \ , \ 0 < \alpha \le 1, \ r \ge 1, \ t > 0.$$

For a positive increasing function $\xi(t)$ and an integer $r > 1$, $f \in Lip(\xi(t), r)$ if [15]

(1.17) $$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r \, dx\right)^{\frac{1}{r}} = O(\xi(t)).$$

For a given positive increasing function $\xi(t)$ and an integer $p > 1$the function $f(x) \in W(L^p, \xi(t))$, if [7]

(1.18) $$\left(\int_0^{2\pi} |f(x+t) - f(x)|^p \ (\sin x)^{p\beta} \, dx\right)^{\frac{1}{p}} = O(\xi(t)), \ \beta \ge 0.$$

We use the following notation throughout this paper:

(1.19) $$\phi(t) = f(x+t) + f(x-t) - 2f(x),$$

(1.20) $s_n(f; x)$: nth partial sum of the Fourier series given by (1.10)

and

(1.21) $$K_n(t) = \frac{1}{2\pi(1+s)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} \left\{ \frac{1}{r_k} \sum_{\upsilon=0}^{k} p_{k-\upsilon} q_\nu \frac{\sin\left(\upsilon + \frac{1}{2}\right)t}{\sin\frac{t}{2}} \right\}.$$

Further, the method $(E, q)(N, p_n, q_n)$ is assumed to be regular and this case is supposed throughout the paper.

2. Known Theorems:

Bernestein[2], Alexits[1], Sahney and Goel [13], Chandra [4] and several others have determined the degree of approximation of the Fourier series of the function $f \in Lip\alpha$ by $(C,1)$, (C,δ), (N,p_n) and $(\bar{N},p_n)$ means. Subsequently, working on the same direction Sahney and Rao[14], and Khan[6] have established results on the degree of approximation of the function belonging to the class $Lip\alpha$ and $Lip(\alpha,r)$ by (N,p_n) and (N,p_n,q_n) means respectively. However, dealing with product summability Nigam et al [12] proved the following theorem on the degree of approximation by the product $(E,q)(C,1)$- mean of Fourier series.

Theorem 2.1:

If a function f is 2π- periodic and of class $Lip\alpha$, then its degree of approximation by $(E,q)(C,1)$ summability mean on its Fourier series $\sum_{n=0}^{\infty} A_n(t)$ is given by $\left\|E_n^q C_n^1 - f\right\|_\infty = O\left(\frac{1}{(n+1)^\alpha}\right)$, $0<\alpha<1$, where $E_n^q C_n^1$ represents the (E,q) transform of $(C,1)$ transform of $s_n(f;x)$.

Subsequently Misra et al [9] have established the following theorem on degree of approximation by the product mean $(E,q)(N,p_n)$ of the Fourier series:

Theorem 2.2:

If f is a $2\pi-$ Periodic function of class $Lip(\alpha,r)$, then degree of approximation by the product $(E,q)(N,p_n)$ summability means on its Fourier series (defined above) is given by

$$\left\|\tau_n - f\right\|_\infty = O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right), 0<\alpha<1,\ r\geq 1,$$

where τ_n as defined in (1.8) .

Further, Misra et al [10] have established the following theorem on degree of approximation by the product mean $(E,s)(N,p_n,q_n)$ of the Fourier series:

Theorem 2.3:

If f is a $2\pi-$ Periodic function of the class $Lip(\alpha,l)$, then degree of approximation by the product $(E,s)(N,p_n,q_n)$ summability means on its Fourier series (1.10) is given by

$$\left\|\tau_n - f\right\|_\infty = O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{l}}}\right), \ 0<\alpha<1, \quad l\geq 1.,$$

where τ_n is as defined in (1.8).

Recently, Misra et al [11] proved the following Theorem

Theorem -2.4 :

For a positive increasing function $\xi(t)$ and an integer $l>1$, if f is a $2\pi-$ Periodic function of the class $Lip(\xi(t),l)$, then degree of approximation by the product $(E,s)(N,p_n,q_n)$ summability on its Fourier series (1.10) is given by $\left\|\tau_n - f\right\|_\infty = O\left((n+1)^{\frac{1}{l}}\xi\left(\frac{1}{n+1}\right)\right)$, $l\geq 1$, where τ_n is as defined in (1.8).

In this paper, we have established a theorem on degree of approximation by the product mean $(E,s)(N,p_n,q_n)$ of the Fourier series of a function of class $W(L^p,\xi(t))$. We prove:

3. Main Theorem

Let $\xi(t)$ be a positive increasing function and f a $2\pi-$ Periodic function of the class $W\left(L^p,\xi(t)\right),\ p>1,t>0$. Then degree of approximation by the product $(E,s)(N,p_n,q_n)$ summability means on the conjugate series (1.10) of the Fourier series (1.9) is given by

$$\|\tau_n - f\|_r = O\left((n+1)^{\beta+\frac{1}{l}}\xi\left(\frac{1}{n+1}\right)\right) \quad ,l\geq 1 , \tag{3.1.1}$$

provided

$$\left(\int_0^{\frac{1}{n+1}}\left(\frac{t\,\phi(t)\sin^\beta t}{\xi(t)}\right)^l dt\right)^{\frac{1}{l}} = O\left(\frac{1}{n+1}\right) \tag{3.1.2}$$

and

$$\left(\int_{\frac{1}{n+1}}^{\pi}\left(\frac{t^{-\delta}|\phi(t)|}{\xi(t)}\right)^l dt\right)^{\frac{1}{l}} = O\left((n+1)^\delta\right) \tag{3.1.3}$$

hold uniformly in x , where δ is an arbitrary number such that $m(1-\delta)-1>0$ and τ_n is as defined in (1.7).

4. Required Lemmas:

We require the following Lemmas for the proof the theorem.

Lemma -4.1:

$$|K_n(t)| = O(n) \quad ,0\leq t\leq \frac{1}{n+1}.$$

Proof of Lemma-4.1:

For $0\leq t\leq \frac{1}{n+1}$, we have $\sin nt \leq n\sin t.$

then

$$|K_n(t)| = \frac{1}{2\pi(1+s)^n}\left|\sum_{k=0}^{n}\binom{n}{k}s^{n-k}\left\{\frac{1}{r_k}\sum_{\upsilon=0}^{k}p_{k-\upsilon}q_\upsilon\frac{\sin\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right|$$

$$\leq \frac{1}{2\pi(1+s)^n}\left|\sum_{k=0}^{n}\binom{n}{k}s^{n-k}\left\{\frac{1}{r_k}\sum_{\upsilon=0}^{k}p_{k-\upsilon}q_\upsilon\frac{(2\upsilon+1)\sin\frac{t}{2}}{\sin\frac{t}{2}}\right\}\right|$$

$$\leq \frac{1}{2\pi(1+s)^n}\left|\sum_{k=0}^{n}\binom{n}{k}s^{n-k}(2k+1)\left\{\frac{1}{r_k}\sum_{\upsilon=0}^{k}p_{k-\upsilon}q_\upsilon\right\}\right|$$

$$\leq \frac{(2n+1)}{2\pi(1+s)^n}\left|\sum_{k=0}^{n}\binom{n}{k}s^{n-k}\right|$$

$$= O(n).$$

This proves the lemma.

Lemma-4.2:

$$|K_n(t)| = O\left(\frac{1}{t}\right),\ for\ \frac{1}{n+1} \leq t \leq \pi .$$

Proof of Lemma-4.2:

For $\frac{1}{n+1} \leq t \leq \pi$, we have by Jordan's lemma, $\sin\left(\frac{t}{2}\right) \geq \frac{t}{\pi}$, $\sin nt \leq 1$.

Then

$$|K_n(t)| = \frac{1}{2\pi(1+s)^n}\left|\sum_{k=0}^{n}\binom{n}{k}s^{n-k}\left\{\frac{1}{r_k}\sum_{\upsilon=0}^{k}p_{k-\upsilon}q_{\upsilon}\frac{\sin\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right|$$

$$\leq \frac{1}{2\pi(1+s)^n}\left|\sum_{k=0}^{n}\binom{n}{k}s^{n-k}\left\{\frac{1}{r_k}\sum_{\upsilon=0}^{k}\frac{\pi\ p_{k-\upsilon}q_{\upsilon}}{t}\right\}\right|$$

$$= \frac{1}{2(1+s)^n t}\left|\sum_{k=0}^{n}\binom{n}{k}s^{n-k}\left\{\frac{1}{r_k}\sum_{\upsilon=0}^{k}p_{k-\upsilon}q_{\upsilon}\right\}\right|.$$

$$= \frac{1}{2(1+s)^n t}\left|\sum_{k=0}^{n}\binom{n}{k}s^{n-k}\right|$$

$$= O\left(\frac{1}{t}\right).$$

This proves the lemma.

5. Proof of Theorem 3.1:

Using Riemann –Lebesgue theorem, for the n-th partial sum $s_n(f;x)$ of the Fourier series (1.10) of $f(x)$ and following Titchmarch [16], we have

$$s_n(f;x) - f(x) = \frac{1}{2\pi}\int_0^{\pi}\phi(t)\,\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\,dt \quad .$$

Using (1.2), the (N, p_n, q_n) transform of $s_n(f;x)$ is given by

$$t_n - f(x) = \frac{1}{2\pi r_n} \int_0^{\pi} \varphi(t) \sum_{k=0}^{n} p_{n-k} q_{\upsilon} \frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)} \, dt \, .$$

Denoting the $(E,q)(N,p,q)$ transform of $s_n(f;x)$ by τ_n, we have

$$\|\tau_n - f\| = \frac{1}{2\pi(1+s)^n} \int_0^{\pi} \varphi(t) \sum_{k=0}^{n} \binom{n}{k} s^{n-k} \left\{ \frac{1}{r_k} \sum_{\upsilon=0}^{k} p_{k-\upsilon} q_{\upsilon} \frac{\sin\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}} \right\} dt$$

$$= \int_0^{\pi} \phi(t) \, K_n(t) \, dt$$

$$= \left\{ \int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi} \right\} \phi(t) \, K_n(t) \, dt$$

(5.1) $$= I_1 + I_2 \text{, say.}$$

Now

$$|I_1| = \frac{1}{2\pi(1+s)^n} \left| \int_0^{1/n+1} \varphi(t) \sum_{k=0}^{n} \binom{n}{k} s^{n-k} \left\{ \frac{1}{r_k} \sum_{\upsilon=0}^{k} p_{k-\upsilon} q_{\upsilon} \frac{\sin\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}} \right\} dt \right|$$

$$\leq \left| \int_0^{\frac{1}{n+1}} \varphi(t) K_n(t) dt \right|$$

$$\leq \left(\int_0^{\frac{1}{n+1}} \left| \frac{t\,\phi(t) \sin^{\beta} t}{\xi(t)} \right|^{l} dt \right)^{\frac{1}{l}} \left(\int_0^{\frac{1}{n+1}} \left| \frac{\xi(t) \bar{K}_n(t)}{t \sin^{\beta} t} \right|^{m} dt \right)^{\frac{1}{m}} \text{, where } \frac{1}{l} + \frac{1}{m} = 1 \text{ , using}$$

Hölder's inequality

$$= O(1) \left(\int_0^{\frac{1}{n+1}} \left(\frac{\xi(t)}{t^{1+\beta}} \right)^{m} dt \right)^{\frac{1}{m}} \text{, using Lemma 4.1 and (3.1.2)}$$

$$= O(\xi\left(\frac{1}{n+1}\right))\left(\int_0^{\frac{1}{n+1}} \frac{dt}{t^{(1+\beta)m}}\right)^{\frac{1}{m}}$$

$$= O\left(\xi\left(\frac{1}{n+1}\right)\right)O\left((n+1)^{-\frac{1}{m}+1+\beta}\right).$$

(5.2) $$= O\left(\xi\left(\frac{1}{n+1}\right)(n+1)^{\beta+\frac{1}{l}}\right).$$

Next

$$|I_2| \le \left(\int_{\frac{1}{n+1}}^{\pi} \left|\frac{t^{-\delta}\phi(t)\sin^{\beta} t}{\xi(t)}\right|^l dt\right)^{\frac{1}{l}} \left(\int_{\frac{1}{n+1}}^{\pi} \left|\frac{\xi(t)\bar{K}_n(t)}{t^{-\delta}\sin^{\beta} t}\right|^m dt\right)^{\frac{1}{m}},$$

where $\frac{1}{l}+\frac{1}{m}=1$, using Hölder's inequality

$$= O((n+1)^{\delta})\left(\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\xi(t)}{t^{\beta+1-\delta}}\right)^m dt\right)^{\frac{1}{m}}$$, using Lemma 4.2 and (3.1.3)

$$= O((n+1)^{\delta})\left(\int_{\frac{1}{\pi}}^{n+1}\left(\frac{\xi\left(\frac{1}{y}\right)}{y^{\delta-\beta-1}}\right)^m \frac{dy}{y^2}\right)^{\frac{1}{m}},$$

since $\xi(t)$ is a positive increasing function, so is $\xi(1/y)/(1/y)$. Using second mean value theorem we get

$$= O((n+1)^{1+\delta}\,\xi\left(\frac{1}{n+1}\right))\left(\int_{\varepsilon}^{n+1}\frac{dy}{y^{m(\delta-\beta-1)+2}}\right)^{\frac{1}{m}}, \text{ for some } \frac{1}{\pi}\le\varepsilon\le n+1$$

$$= O\left((n+1)^{1+\delta}\,\xi\left(\frac{1}{n+1}\right)\right)O\left((n+1)^{\beta+1-\delta-\frac{1}{m}}\right)$$

(5.3) $$= O\left((n+1)^{\beta+\frac{1}{l}}\,\xi\left(\frac{1}{n+1}\right)\right)$$

Then from (5.2) and (5.3), we have

$$|\tau_n - f(x)| = O\left((n+1)^{\beta+\frac{1}{l}}\,\xi\left(\frac{1}{n+1}\right)\right), \text{ for } r\ge 1.$$

$$\left\|\tau_n - f(x)\right\|_r = \left(\int_0^{2\pi} O\left((n+1)^{\beta+\frac{1}{l}}\xi\left(\frac{1}{n+1}\right)\right)^l dx\right)^{\frac{1}{l}}, \quad l \geq 1.$$

$$= O\left((n+1)^{\beta+\frac{1}{l}}\xi\left(\frac{1}{n+1}\right)\right)\left(\int_0^{2\pi} dx\right)^{\frac{1}{l}}$$

$$= O\left((n+1)^{\beta+\frac{1}{l}}\xi\left(\frac{1}{n+1}\right)\right).$$

This completes the proof of the theorem.

References

[1] G.Alexits. (1928) *Ü ber die Annanherung einer stetigen function durch die Cesarochen Mittel in hrer Fourier reihe,* Math. Annal 100, 264-277.

[2] S.Bernstein**.** (1912) *Sur l' order de la Melleure approximation des function continue par des polynomes de degree' donne'e*, Memories Acad. Roy-Belyique 4, 1-104.

[3] D.Borwein**.** (1958) *On product of sequences,* Journal of London Mathematical Society, 33, 352-357.Sd

[4] P.Chandra**.** (1970) *On degree of approximation of functions belonging to Lipchitz class,* Nanta Math. 80, 88-89.Df

[5] G.H. Hardy. (1949) *Divergent series, First edition,* Oxford University press.Mm

[6] Huzoor H. Khan. (1982)*On degree of approximation of function belonging to the class* , Indian Journal of pure and applied Mathematics, 13, 132-136.

[7] Shyam Lal. (1999) on degree of approximation of Fourier series of function belonging to the Neighbourhood $W(L^p, \xi(t))$ class by (C,1)(E,1) means, Tamkang Journal of mathematics 30,47-52Jh

[8] L. McFadden. (1942) *Absolute* $\ddot{N}$ *orlund summabilty, D*uke Maths. Journal, 9,168-207.Ss

[9] U.K.Misra, M. Misra, B.P. Padhy and M.K. Muduli. (2011) On degree of approximation by product mean $(E,q)(N,p_n)$ of Fourier series**,** Gen. Math. Notes ISSN 2219 – 7184, Vol.6, No.2**,**

[10] U.K.Misra, M. Misra, B.P. Padhy and P.C.Das. "Degree of approximation of the Fourier Series of a function of Lipchitz class by Product Means", Communicated to Journal of Mathematical Modeling, SciKnow PublicationSs

[11] U.K.Misra, M. Misra, B.P. Padhy ,P.palo and P.Samanta. (2014) **A**approximation of the Fourier Series of a function of Lipchitz class by Product Means", Journal of Advanced in Mathematics , Vol.9,No.4,2475-2484.

[12] H.K. Nigam and Ajay Sharma. (2010) On degree of Approximation by product means, Ultra Scientist of Physical Sciences, Vol.22 (3) M, 889-894,.

[13] B .N. Sahney and D.S.Goel**.** (1973) *On degree of approximation of continuous functions,* Ranchi University Mathematical Journal, 4, 50-53.

[14] B .N. Sahney and G.Rao**.** (1972) *Errors bound in the approximation function,* Bulletin of Australian Mathematical Society, 6

[15] A.H.Sidiqui **:** *Ph.D. Thesis*, Aligarh Muslim University, Aligarh, a967

[16] E.C. Titchmarch. (1939) *The theory of functions, oxford university press,* p.p402-403.

[17] A . Zygmund**.** (1959) *Trigonometric Series , second Edition ,Vol.I* , Cambridge University press.