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# A Study on Sum Formulas for Generalized Tribonacci Numbers: Closed Forms of the Sum Formulas $\sum_{k=0}^{n} kx^{k}W_{k}$, $\sum_{k=1}^{n} kx^{k}W_{-k}$

Yüksel Soykan
Department of Mathematics,
Art and Science Faculty,
Zonguldak Bülent Ecevit University,
67100, Zonguldak, Turkey
e-mail: yuksel_soykan@hotmail.com

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**Abstract.** In this paper, closed forms of the sum formulas $\sum_{k=0}^{n} kx^{k}W_{k}$, $\sum_{k=1}^{n} kx^{k}W_{-k}$ for generalized Tribonacci numbers are presented. As special cases, we give summation formulas of Tribonacci, Tribonacci-Lucas, Padovan, Perrin, Narayana and some other third order linear recurrance sequences.

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**Keywords.** Tribonacci numbers, Padovan numbers, Perrin numbers, sum formulas.

## 1. Introduction

In this paper, we consider linear summation formulas of generalized Tribonacci numbers. Summing formulas of the Pell and Pell-Lucas numbers are well known and given in [8, 9], see also [6]. For linear sums of Fibonacci, Tribonacci, Tetranacci, Pentanacci and Hexanacci numbers, see [7,16], [5,11], [19, 43], [21], and [23] respectively. First, in this section, we present some background about generalized Tribonacci numbers. The generalized Tribonacci sequence $\{W_n(W_0, W_1, W_2; r, s, t)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, \qquad W_0 = a, W_1 = b, W_2 = c, \quad n \geq 3 \tag{1.1}$$

where $W_0, W_1, W_2$ are arbitrary complex numbers and $r, s, t$ are real numbers. The generalized Tribonacci sequence has been studied by many authors, see for example [1,2,3,4,10,12,13,14,15,38,39,40,41,42].

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

for $n = 1, 2, 3, \ldots$ when $t \neq 0$. Therefore, recurrence (1.1) holds for all integers $n$.

If we set $r = s = t = 1$ and $W_0 = 0, W_1 = 1, W_2 = 1$ then $\{W_n\}$ is the well-known Tribonacci sequence and if we set $r = s = t = 1$ and $W_0 = 3, W_1 = 1, W_2 = 3$ then $\{W_n\}$ is the well-known Tribonacci-Lucas sequence.

In fact, the generalized Tribonacci sequence is the generalization of the well-known sequences like Tribonacci, Tribonacci-Lucas, Padovan (Cordonnier), Perrin, Padovan-Perrin, Narayana, third order Jacobsthal and third order Jacobsthal-Lucas. In literature, for example, the following names and notations (see Table 1) are used for the special case of $r, s, t$ and initial values.

Table 1 A few special case of generalized Tribonacci sequences

| No | Sequences (Numbers) | Notation | OEIS [37] | References |
|---|---|---|---|---|
| 1 | Tribonacci | $\{T_n\} = \{W_n(0,1,1;1,1,1)\}$ | A000073, A057597 | [26] |
| 2 | Tribonacci-Lucas | $\{K_n\} = \{W_n(3,1,3;1,1,1)\}$ | A001644, A073145 | [26] |
| 3 | Tribonacci-Perrin | $\{M_n\} = \{W_n(3,0,2;1,1,1)\}$ | | [26] |
| 4 | modified Tribonacci | $\{U_n\} = \{W_n(1,1,1;1,1,1)\}$ | | [26] |
| 5 | modified Tribonacci-Lucas | $\{G_n\} = \{W_n(4,4,10;1,1,1)\}$ | | [26] |
| 6 | adjusted Tribonacci-Lucas | $\{H_n\} = \{W_n(4,2,0;1,1,1)\}$ | | [26] |
| 7 | third order Pell | $\{P_n^{(3)}\} = \{W_n(0,1,2;2,1,1)\}$ | A077939, A077978 | [27] |
| 8 | third order Pell-Lucas | $\{Q_n^{(3)}\} = \{W_n(3,2,6;2,1,1)\}$ | A276225, A276228 | [27] |
| 9 | third order modified Pell | $\{E_n^{(3)}\} = \{W_n(0,1,1;2,1,1)\}$ | A077997, A078049 | [27] |
| 10 | third order Pell-Perrin | $\{R_n^{(3)}\} = \{W_n(3,0,2;2,1,1)\}$ | | |
| 11 | Padovan (Cordonnier) | $\{P_n\} = \{W_n(1,1,1;0,1,1)\}$ | A000931 | [28] |
| 12 | Perrin (Padovan-Lucas) | $\{E_n\} = \{W_n(3,0,2;0,1,1)\}$ | A001608, A078712 | [28] |
| 13 | Padovan-Perrin | $\{S_n\} = \{W_n(0,0,1;0,1,1)\}$ | A000931, A176971 | [28] |
| 14 | modified Padovan | $\{A_n\} = \{W_n(3,1,3;0,1,1)\}$ | | [28] |
| 15 | Pell-Padovan | $\{R_n\} = \{W_n(1,1,1;0,2,1)\}$ | A066983, A128587 | [29] |
| 16 | Pell-Perrin | $\{C_n\} = \{W_n(3,0,2;0,2,1)\}$ | | [29] |
| 17 | third order Fibonacci-Pell | $\{G_n\} = \{W_n(1,0,2;0,2,1)\}$ | | [29] |
| 18 | third order Lucas-Pell | $\{B_n\} = \{W_n(3,0,4;0,2,1)\}$ | | [29] |
| 19 | Jacobsthal-Padovan | $\{Q_n\} = \{W_n(1,1,1;0,1,2)\}$ | A159284 | [31] |
| 20 | Jacobsthal-Perrin (-Lucas) | $\{L_n\} = \{W_n(3,0,2;0,1,2)\}$ | A072328 | [31] |
| 21 | adjusted Jacobsthal-Padovan | $\{K_n\} = \{W_n(0,1,0;0,1,2)\}$ | | [31] |
| 22 | modified Jacobsthal-Padovan | $\{M_n\} = \{W_n(3,1,3;0,1,2)\}$ | | [31] |
| 23 | Narayana | $\{N_n\} = \{W_n(0,1,1;1,0,1)\}$ | A078012 | [30] |
| 24 | Narayana-Lucas | $\{U_n\} = \{W_n(3,1,1;1,0,1)\}$ | A001609 | [30] |
| 25 | Narayana-Perrin | $\{H_n\} = \{W_n(3,0,2;1,0,1)\}$ | | [30] |
| 26 | third order Jacobsthal | $\{J_n^{(3)}\} = \{W_n(0,1,1;1,1,2)\}$ | A077947 | [32] |
| 27 | third order Jacobsthal-Lucas | $\{j_n^{(3)}\} = \{W_n(2,1,5;1,1,2)\}$ | A226308 | [32] |
| 28 | modified third order Jacobsthal-Lucas | $\{K_n^{(3)}\} = \{W_n(3,1,3;1,1,2)\}$ | | [32] |
| 29 | third order Jacobsthal-Perrin | $\{Q_n^{(3)}\} = \{W_n(3,0,2;1,1,2)\}$ | | [32] |
| 30 | 3-primes | $\{G_n\} = \{W_n(0,1,2;2,3,5)\}$ | | [33] |
| 31 | Lucas 3-primes | $\{H_n\} = \{W_n(3,2,10;2,3,5)\}$ | | [33] |
| 32 | modified 3-primes | $\{E_n\} = \{W_n(0,1,1;2,3,5)\}$ | | [33] |
| 33 | reverse 3-primes | $\{N_n\} = \{W_n(0,1,5;5,3,2)\}$ | | [34] |
| 34 | reverse Lucas 3-primes | $\{S_n\} = \{W_n(3,5,31;5,3,2)\}$ | | [34] |
| 35 | reverse modified 3-primes | $\{U_n\} = \{W_n(0,1,4;5,3,2)\}$ | | [34] |

Here, OEIS stands for On-line Encyclopedia of Integer Sequences.

The evaluation of sums of powers of these sequences is a challenging issue. Two pretty examples are

$\sum_{k=0}^{n} k(-1)^k T_k = \frac{1}{2}(-1)^n((n+2)T_{n+3} - (2n+3)T_{n+2} + (n-1)T_{n+1})$

and

$\sum_{k=1}^{n} k(-1)^k T_{-k} = \frac{1}{2}(-1)^n(nT_{-n-1} + T_{-n-2} + (n+1)T_{-n-3})$.

In this work, we derive expressions for sums of generalized Tribonacci numbers. We present some works on sum formulas of the numbers in the following Table 2.

Table 2. A few special study of sum formulas.

| Name of sequence | Papers which deal with summing formulas |
|---|---|
| Pell and Pell-Lucas | [6],[8,9] |
| Generalized Fibonacci | [7,16,17,24,25] |
| Generalized Tribonacci | [5,11,18,35,36] |
| Generalized Tetranacci | [19,20,43] |
| Generalized Pentanacci | [21,22] |
| Generalized Hexanacci | [23] |

The following theorem presents some summing formulas of generalized Tribonacci numbers with positive subscripts.

THEOREM 1.1. *Let $x$ be a complex number. For $n \geq 0$, we have the following formulas:*

**(a):** *If $tx^3 + sx^2 + rx - 1 \neq 0$ then*

$$\sum_{k=0}^{n} x^k W_k = \frac{\Omega_1}{tx^3 + sx^2 + rx - 1}$$

*where*

$$\begin{aligned}\Omega_1 &= x^{n+3}W_{n+3} - (rx-1)x^{n+2}W_{n+2} - (sx^2 + rx - 1)x^{n+1}W_{n+1} \\ &\quad -x^2W_2 + x(rx-1)W_1 + (sx^2+rx-1)W_0.\end{aligned}$$

**(b):** *If $r^2x - s^2x^2 + t^2x^3 + 2sx + 2rtx^2 - 1 \neq 0$ then*

$$\sum_{k=0}^{n} x^k W_{2k} = \frac{\Omega_2}{r^2x - s^2x^2 + t^2x^3 + 2sx + 2rtx^2 - 1}$$

*where*

$$\begin{aligned}\Omega_2 &= -(sx-1)x^{n+1}W_{2n+2} + (t+rs)x^{n+2}W_{2n+1} + t(r+tx)x^{n+2}W_{2n} \\ &\quad +x(sx-1)W_2 - (t+rs)x^2W_1 + \left(r^2x - s^2x^2 + 2sx + rtx^2 - 1\right)W_0.\end{aligned}$$

**(c):** *If $r^2x - s^2x^2 + t^2x^3 + 2sx + 2rtx^2 - 1 \neq 0$ then*

$$\sum_{k=0}^{n} x^k W_{2k+1} = \frac{\Omega_3}{r^2x - s^2x^2 + t^2x^3 + 2sx + 2rtx^2 - 1}$$

*where*

$$\begin{aligned}\Omega_3 &= (r+tx)\,x^{n+1}W_{2n+2}+\left(s-s^2x+t^2x^2+rtx\right)x^{n+1}W_{2n+1}\\ &\quad -t\,(sx-1)\,x^{n+1}W_{2n}-x\,(r+tx)\,W_2+\left(r^2x+sx+rtx^2-1\right)W_1+tx\,(sx-1)\,W_0.\end{aligned}$$

*Proof.* It is given in [35]. □

The following theorem presents some summing formulas (identities) of generalized Tribonacci numbers with negative subscripts.

THEOREM 1.2. *Let $x$ be a complex number. For $n \geq 1$, we have the following formulas:*

**(a):** *If $t+rx^2+sx-x^3 \neq 0$, then*

$$\sum_{k=1}^{n} x^k W_{-k} = \frac{\Omega_4}{t+rx^2+sx-x^3}$$

*where*

$$\begin{aligned}\Omega_4 &= -\left(t+rx^2+sx\right)x^{n+1}W_{-n-1}-(t+sx)\,x^{n+2}W_{-n-2}-tx^{n+3}W_{-n-3}\\ &\quad +xW_2-x\,(r-x)\,W_1+x\left(-s-rx+x^2\right)W_0.\end{aligned}$$

**(b):** *If $2sx^2-s^2x+r^2x^2+t^2-x^3+2rtx \neq 0$ then*

$$\sum_{k=1}^{n} x^k W_{-2k} = \frac{\Omega_5}{2sx^2-s^2x+r^2x^2+t^2-x^3+2rtx}$$

*where*

$$\begin{aligned}\Omega_5 &= -(t+rx)\,x^{n+1}W_{-2n+1}+\left(r^2x+rt+sx-x^2\right)x^{n+1}W_{-2n}+t\,(s-x)\,x^{n+1}W_{-2n-1}\\ &\quad -x\,(s-x)\,W_2+x\,(t+rs)\,W_1+x\left(-r^2x-rt-2sx+s^2+x^2\right)W_0.\end{aligned}$$

**(c):** *If $2sx^2-s^2x+r^2x^2+t^2-x^3+2rtx \neq 0$ then*

$$\sum_{k=1}^{n} x^k W_{-2k+1} = \frac{\Omega_6}{2sx^2-s^2x+r^2x^2+t^2-x^3+2rtx}$$

*where*

$$\begin{aligned}\Omega_6 &= (s-x)\,x^{n+2}W_{-2n+1}-(t+rs)\,x^{n+2}W_{-2n}-t\,(t+rx)\,x^{n+1}W_{-2n-1}\\ &\quad +x\,(t+rx)\,W_2+x\left(-r^2x-rt-sx+x^2\right)W_1-tx\,(s-x)\,W_0.\end{aligned}$$

*Proof.* It is given in [35]. □

## 2. Sum Formulas of Generalized Tribonacci Numbers with Positive Subscripts

The following Theorem presents some sum formulas of generalized Tribonacci numbers with positive subscripts.

THEOREM 2.1. *Let $x$ be a complex number. For $n \geq 0$, we have the following formulas:*

**(a):** *If* $(sx^2 + tx^3 + rx - 1) \neq 0$ *then*

$$\sum_{k=0}^{n} kx^k W_k = \frac{\Delta_1}{(sx^2 + tx^3 + rx - 1)^2}$$

**(b):** *If* $(r^2x - s^2x^2 + t^2x^3 + 2sx + 2rtx^2 - 1) \neq 0$ *then*

$$\sum_{k=0}^{n} kx^k W_{2k} = \frac{\Delta_2}{(r^2x - s^2x^2 + t^2x^3 + 2sx + 2rtx^2 - 1)^2}$$

**(c):** *If* $(r^2x - s^2x^2 + t^2x^3 + 2sx + 2rtx^2 - 1) \neq 0$ *then*

$$\sum_{k=0}^{n} kx^k W_{2k+1} = \frac{\Delta_3}{(r^2x - s^2x^2 + t^2x^3 + 2sx + 2rtx^2 - 1)^2}$$

*where*

$$\Delta_1 = \sum_{k=1}^{6} \Gamma_k, \ \Delta_2 = \sum_{k=1}^{6} \Theta_k, \ \Delta_3 = \sum_{k=1}^{6} \Phi_k,$$

*with*

$\Gamma_1 = x^{n+3}(n(sx^2 + tx^3 + rx - 1) + sx^2 + 2rx - 3)W_{n+3},$

$\Gamma_2 = -x^{n+2}(n(rx - 1)(sx^2 + tx^3 + rx - 1) + rsx^3 + tx^3 + 2r^2x^2 - 4rx + 2)W_{n+2},$

$\Gamma_3 = -x^{n+1}(n(sx^2+rx-1)(sx^2+tx^3+rx-1)-2sx^2+2tx^3+r^2x^2+s^2x^4+2rsx^3-rtx^4-2rx+1)$ $W_{n+1},$

$\Gamma_4 = +x^2(tx^3 - rx + 2)W_2,$

$\Gamma_5 = +x(-rtx^4 + 2tx^3 + sx^2 + r^2x^2 - 2rx + 1)W_1,$

$\Gamma_6 = -tx^3(sx^2 + 2rx - 3)W_0,$

$\Theta_1 = -x^{n+1}(n(sx - 1)(r^2x - s^2x^2 + t^2x^3 + 2sx + 2rtx^2 - 1) + s^2x^2 + 2t^2x^3 + r^2sx^2 - st^2x^4 + 2rtx^2 - 2sx + 1)W_{2n+2},$

$\Theta_2 = +x^{n+2}(t + rs)(n(r^2x - s^2x^2 + t^2x^3 + 2sx + 2rtx^2 - 1) + r^2x - t^2x^3 + 2sx - 2)W_{2n+1},$

$\Theta_3 = +tx^{n+2}(n(r + tx)(r^2x - s^2x^2 + t^2x^3 + 2sx + 2rtx^2 - 1) + r^3x + 2r^2tx^2 + rt^2x^3 - 3tx - s^2tx^3 + 4stx^2 + 2rsx - 2r)W_{2n},$

$\Theta_4 = +x(s^2x^2 + 2t^2x^3 - 2sx + 2rtx^2 + r^2sx^2 - st^2x^4 + 1)W_2,$

$\Theta_5 = -x^2(t + rs)(r^2x - t^2x^3 + 2sx - 2)W_1,$

$\Theta_6 = +tx^2(2r - r^3x + 3tx - 4stx^2 - 2r^2tx^2 - rt^2x^3 + s^2tx^3 - 2rsx)W_0,$

$\Phi_1 = x^{n+1}(n(r + tx)(r^2x - s^2x^2 + t^2x^3 + 2sx + 2rtx^2 - 1) + rs^2x^2 - r^2tx^2 - 2rt^2x^3 + 2stx^2 - 2tx - r - t^3x^4)W_{2n+2},$

$\Phi_2 = +x^{n+1}(n(s-s^2x+t^2x^2+rtx)(r^2x-s^2x^2+t^2x^3+2sx+2rtx^2-1)+r^3tx^2+rt^3x^4+2st^2x^3-2rtx-r^2s^2x^2+2r^2t^2x^3-s+2s^2x-s^3x^2-3t^2x^2)W_{2n+1},$

$\Phi_3 = -tx^{n+1}(n(sx-1)(r^2x-s^2x^2+t^2x^3+2sx+2rtx^2-1)+s^2x^2+2rtx^2+2t^2x^3+r^2sx^2-st^2x^4-2sx+1)W_{2n},$

$\Phi_4 = +x(r+t^3x^4+2tx-2stx^2-rs^2x^2+r^2tx^2+2rt^2x^3)W_2,$

$\Phi_5 = +x(s-2s^2x+s^3x^2+3t^2x^2+r^2s^2x^2-2r^2t^2x^3-r^3tx^2-rt^3x^4-2st^2x^3+2rtx)W_1,$

$\Phi_6 = +tx(s^2x^2+2t^2x^3-2sx+2rtx^2+r^2sx^2-st^2x^4+1)W_0.$

*Proof.*

**(a):** Using the recurrence relation

$$W_n = rW_{n-1}+sW_{n-2}+tW_{n-3}$$

i.e.

$$tW_{n-3} = W_n - rW_{n-1} - sW_{n-2}$$

we obtain

$$\begin{array}{rcl}
tnx^nW_n &=& nx^nW_{n+3}-rnx^nW_{n+2}-snx^nW_{n+1}\\
t(n-1)x^{n-1}W_{n-1} &=& (n-1)x^{n-1}W_{n+2}-r(n-1)x^{n-1}W_{n+1}-s(n-1)x^{n-1}W_n\\
t(n-2)x^{n-2}W_{n-2} &=& (n-2)x^{n-2}W_{n+1}-r(n-2)x^{n-2}W_n-s(n-2)x^{n-2}W_{n-1}\\
t(n-3)x^{n-3}W_{n-3} &=& (n-3)x^{n-3}W_n-r(n-3)x^{n-3}W_{n-1}-s(n-3)x^{n-3}W_{n-2}\\
&& \vdots\\
t\times 3\times x^3W_3 &=& 3\times x^3W_6-r\times 3\times x^3W_5-s\times 3\times x^3W_4\\
t\times 2\times x^2W_2 &=& 2\times x^2W_5-r\times 2\times x^2W_4-s\times 2\times x^2W_3\\
t\times 1\times x^1W_1 &=& 1\times x^1W_4-r\times 1\times x^1W_3-s\times 1\times x^1W_2\\
t\times 0\times x^0W_0 &=& 0\times x^0W_3-r\times 0\times x^0W_2-s\times 0\times x^0W_1
\end{array}$$

If we add the equations side by side, we get

$$\begin{array}{rcl}
t\sum_{k=0}^{n}kx^kW_k &=& (nx^nW_{n+3}+(n-1)x^{n-1}W_{n+2}+(n-2)x^{n-2}W_{n+1}\\
&& +x^{-1}W_2+2x^{-2}W_1+3x^{-3}W_0+\sum_{k=0}^{n}(k-3)x^{k-3}W_k)\\
&& -r(nx^nW_{n+2}+(n-1)x^{n-1}W_{n+1}+x^{-1}W_1+2x^{-2}W_0+\sum_{k=0}^{n}(k-2)x^{k-2}W_k)\\
&& -s(nx^nW_{n+1}+x^{-1}W_0+\sum_{k=0}^{n}(k-1)x^{k-1}W_k)
\end{array} \tag{2.1}$$

Then, using Theorem 1.1 (a) and solving (2.1), the required result of (a) follows.

**(b) and (c):** Using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}$$

i.e.

$$rW_{n-1} = W_n - sW_{n-2} - tW_{n-3}$$

we obtain

$$\begin{aligned}
rnx^n W_{2n+1} &= nx^n W_{2n+2} - snx^n W_{2n} - tnx^n W_{2n-1} \\
r(n-1)x^{n-1} W_{2n-1} &= (n-1)x^{n-1} W_{2n} - s(n-1)x^{n-1} W_{2n-2} - t(n-1)x^{n-1} W_{2n-3} \\
&\vdots \\
r \times 2 \times x^2 W_5 &= 2 \times x^2 W_6 - s \times 2 \times x^2 W_4 - t \times 2 \times x^2 W_3 \\
r \times 1 \times x^1 W_3 &= 1 \times x^1 W_4 - s \times 1 \times x^1 W_2 - t \times 1 \times x^1 W_1 \\
r \times 0 \times x^0 W_1 &= 0 \times x^0 W_2 - s \times 0 \times x^0 W_0 - t \times 0 \times x^0 W_{-1}
\end{aligned}$$

Now, if we add the above equations side by side, we get

$$\begin{aligned}
r\sum_{k=0}^{n} kx^k W_{2k+1} &= (nx^n W_{2n+2} - (-1)x^{-1} W_0 + \sum_{k=0}^{n} (k-1)x^{k-1} W_{2k}) - s\sum_{k=0}^{n} kx^k W_{2k} \\
&\quad -t(-(n+1)x^{n+1} W_{2n+1} + 0 \times x^0 W_{-1} + \sum_{k=0}^{n} (k+1)x^{k+1} W_{2k+1}).
\end{aligned} \tag{2.2}$$

Similarly, using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}$$

i.e.

$$rW_{n-1} = W_n - sW_{n-2} - tW_{n-3}$$

we write the following obvious equations;

$$\begin{aligned}
r(n+1)x^{n+1} W_{2n+2} &= (n+1)x^{n+1} W_{2n+3} - s(n+1)x^{n+1} W_{2n+1} - t(n+1)x^{n+1} W_{2n} \\
rnx^n W_{2n} &= nx^n W_{2n+1} - snx^n W_{2n-1} - tnx^n W_{2n-2} \\
r(n-1)x^{n-1} W_{2n-2} &= (n-1)x^{n-1} W_{2n-1} - s(n-1)x^{n-1} W_{2n-3} - t(n-1)x^{n-1} W_{2n-4} \\
&\vdots \\
r \times 3 \times x^3 W_6 &= 3 \times x^3 W_7 - s \times 3 \times x^3 W_5 - t \times 3 \times x^3 W_4 \\
r \times 2 \times x^2 W_4 &= 2 \times x^2 W_5 - s \times 2 \times x^2 W_3 - t \times 2 \times x^2 W_2 \\
r \times 1 \times x^1 W_2 &= 1 \times x^1 W_3 - s \times 1 \times x^1 W_1 - t \times 1 \times x^1 W_0 \\
r \times 0 \times x^0 W_0 &= 0 \times W_1 - s \times 0 \times x^0 W_{-1} - t \times 0 \times x^0 W_{-2}
\end{aligned}$$

Now, if we add the above equations side by side, we obtain

$$\begin{aligned} r\sum_{k=0}^{n}kx^{k}W_{2k} &= \sum_{k=0}^{n}kx^{k}W_{2k+1}-s(-(n+1)x^{n+1}W_{2n+1}+\sum_{k=0}^{n}(k+1)x^{k+1}W_{2k+1}) \\ &\quad -t(-(n+1)x^{n+1}W_{2n}+\sum_{k=0}^{n}(k+1)x^{k+1}W_{2k}). \end{aligned} \tag{2.3}$$

Then, using Theorem 1.1 (b) and (c) and solving the system (2.2)-(2.3), the required result of (b) and (c) follow. □

**2.1. Special Cases.** In this section, we present the closed form solutions (identities) of the sums $\sum_{k=0}^{n}kx^{k}W_{k}$, $\sum_{k=0}^{n}kx^{k}W_{2k}$ and $\sum_{k=0}^{n}kx^{k}W_{2k+1}$ for the specific case of sequence $\{W_{n}\}$.

**2.2. The Case $x=1$.** We now consider the case $x=1$ in Theorem 2.1. In this subsection, we only consider the case $x=1$, $r=0,s=2,t=1$ (this special case was not given in [36] because we can not use Theorem 2.1 directly). Observe that setting $x=1$, $r=0,s=2,t=1$ (i.e. for the generalized Pell-Padovan case) in Theorem 2.1, (b) and (c) make the right hand side of the sum formulas to be an indeterminate form. Application of L'Hospital rule (using twice) however provides the evaluation of the sum formulas. If $x=1$, $r=0,s=2,t=1$ then we have the following theorem (in fact taking $r=0,s=2,t=1$ in Theorem 2.1 and then using L'Hospital rule twice for $x=1$ we obtain the following theorem).

THEOREM 2.2. *If $r=0,s=2,t=1$ then for $n\geq 0$, we have the following formulas:*

- **(a):** $\sum_{k=0}^{n}kW_{k}=\frac{1}{4}((2n-1)W_{n+3}+(2n-3)W_{n+2}-(2n+3)W_{n+1}+3W_{2}+5W_{1}+W_{0})$.
- **(b):** $\sum_{k=0}^{n}kW_{2k}=\frac{1}{2}(n(n+3)W_{2n+2}-n(n+1)W_{2n+1}-(n+2)(n+1)W_{2n}+2W_{0})$.
- **(c):** $\sum_{k=0}^{n}kW_{2k+1}=\frac{1}{2}(-n(n+1)W_{2n+2}+(n^{2}+3n-2)W_{2n+1}+n(n+3)W_{2n}+2W_{1})$.

Proof

- **(a):** Taking $x=1,r=0,s=2,t=1$ in Theorem 2.1 (a), we obtain (a).
- **(b):** We use Theorem 2.1 (b). If we set $r=0,s=2,t=1$ in Theorem 2.1 (b) then we have

$$\sum_{k=0}^{n}kx^{k}W_{2k}=\frac{g_{1}(x)}{(-x^{3}+4x^{2}-4x+1)^{2}}$$

where

$g_{1}(x)=-x^{n+1}(4x^{2}-4x+2x^{3}-2x^{4}-n(2x-1)(-x^{3}+4x^{2}-4x+1)+1)W_{2n+2}-x^{n+2}(n(-x^{3}+4x^{2}-4x+1)-4x+x^{3}+2)W_{2n+1}-x^{n+2}(3x-8x^{2}+4x^{3}+nx(-x^{3}+4x^{2}-4x+1))W_{2n}+x(-2x^{4}+2x^{3}+4x^{2}-4x+1)W_{2}+x^{2}(x^{3}-4x+2)W_{1}+x^{2}(4x^{3}-8x^{2}+3x)W_{0}$.

For $x=1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get

$$\begin{aligned}\sum_{k=0}^{n} kW_{2k} &= \left.\frac{\frac{d}{dx}(g_1(x))}{\frac{d}{dx}((-x^3+4x^2-4x+1)^2)}\right|_{x=1} \\ &= \left.\frac{\frac{d^2}{dx^2}(g_1(x))}{\frac{d^2}{dx^2}((-x^3+4x^2-4x+1)^2)}\right|_{x=1} \\ &= \frac{1}{2}(n(n+3)W_{2n+2}-n(n+1)W_{2n+1}-(n+2)(n+1)W_{2n}+2W_0).\end{aligned}$$

**(c):** We use Theorem 2.1 (c). If we set $r=0, s=2, t=1$ in Theorem 2.1 (c) then we have

$$\sum_{k=0}^{n} kx^k W_{2k+1} = \frac{g_2(x)}{(-x^3+4x^2-4x+1)^2}$$

where

$g_2(x) = -x^{n+1}(2x-4x^2+x^4+nx(-x^3+4x^2-4x+1))W_{2n+2} - x^{n+1}(11x^2-8x-4x^3+n(x^2-4x+2)(-x^3+4x^2-4x+1)+2)W_{2n+1} - x^{n+1}(4x^2-4x+2x^3-2x^4-n(2x-1)(-x^3+4x^2-4x+1)+1)W_{2n} + x(x^4-4x^2+2x)W_2 - x(4x^3-11x^2+8x-2)W_1 + x(-2x^4+2x^3+4x^2-4x+1)W_0.$

For $x=1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule. Then we get

$$\begin{aligned}\sum_{k=0}^{n} kW_{2k+1} &= \left.\frac{\frac{d}{dx}(g_2(x))}{\frac{d}{dx}((-x^3+4x^2-4x+1)^2)}\right|_{x=1} \\ &= \left.\frac{\frac{d^2}{dx^2}(g_2(x))}{\frac{d^2}{dx^2}((-x^3+4x^2-4x+1)^2)}\right|_{x=1} \\ &= \frac{1}{2}(-n(n+1)W_{2n+2}+(n^2+3n-2)W_{2n+1}+n(n+3)W_{2n}+2W_1).\end{aligned}$$

□

From the last theorem, we have the following corollary which gives sum formulas of Pell-Padovan numbers (take $W_n = R_n$ with $R_0 = 1, R_1 = 1, R_2 = 1$).

Corollary 2.3. *For $n \geq 0$, Pell-Padovan numbers have the following properties:*

**(a):** $\sum_{k=0}^{n} kR_k = \frac{1}{4}((2n-1)R_{n+3}+(2n-3)R_{n+2}-(2n+3)R_{n+1}+9)$.

**(b):** $\sum_{k=0}^{n} kR_{2k} = \frac{1}{2}(n(n+3)R_{2n+2}-n(n+1)R_{2n+1}-(n+2)(n+1)R_{2n}+2)$.

**(c):** $\sum_{k=0}^{n} kR_{2k+1} = \frac{1}{2}(-n(n+1)R_{2n+2}+(n^2+3n-2)R_{2n+1}+n(n+3)R_{2n}+2)$.

Taking $W_n = C_n$ with $C_0 = 3, C_1 = 0, C_2 = 2$ in the last theorem, we have the following corollary which presents sum formulas of Pell-Perrin numbers.

Corollary 2.4. *For $n \geq 0$, Pell-Perrin numbers have the following properties:*

**(a):** $\sum_{k=0}^{n} kC_k = \frac{1}{4}((2n-1)C_{n+3}+(2n-3)C_{n+2}-(2n+3)C_{n+1}+9)$.

**(b):** $\sum_{k=0}^{n} kC_{2k} = \frac{1}{2}(n(n+3)C_{2n+2}-n(n+1)C_{2n+1}-(n+2)(n+1)C_{2n}+6)$.

**(c):** $\sum_{k=0}^{n} kC_{2k+1} = \frac{1}{2}(-n(n+1)C_{2n+2} + (n^2+3n-2)C_{2n+1} + n(n+3)C_{2n})$.

From the last theorem, we have the following corollary which gives sum formulas of third order Fibonacci-Pell numbers (take $W_n = G_n$ with $G_0 = 1, G_1 = 0, G_2 = 2$).

Corollary 2.5. *For $n \geq 0$, third order Fibonacci-Pell numbers have the following properties:*

- **(a):** $\sum_{k=0}^{n} kG_k = \frac{1}{4}((2n-1)G_{n+3} + (2n-3)G_{n+2} - (2n+3)G_{n+1} + 7)$.
- **(b):** $\sum_{k=0}^{n} kG_{2k} = \frac{1}{2}(n(n+3)G_{2n+2} - n(n+1)G_{2n+1} - (n+2)(n+1)G_{2n} + 2)$.
- **(c):** $\sum_{k=0}^{n} kG_{2k+1} = \frac{1}{2}(-n(n+1)G_{2n+2} + (n^2+3n-2)G_{2n+1} + n(n+3)G_{2n})$.

Taking $W_n = B_n$ with $B_0 = 3, B_1 = 0, B_2 = 4$ in the last theorem, we have the following corollary which presents sum formulas of third order Lucas-Pell numbers.

Corollary 2.6. *For $n \geq 0$, third order Lucas-Pell numbers have the following properties:*

- **(a):** $\sum_{k=0}^{n} kB_k = \frac{1}{4}((2n-1)B_{n+3} + (2n-3)B_{n+2} - (2n+3)B_{n+1} + 15)$.
- **(b):** $\sum_{k=0}^{n} kB_{2k} = \frac{1}{2}(n(n+3)B_{2n+2} - n(n+1)B_{2n+1} - (n+2)(n+1)B_{2n} + 6)$.
- **(c):** $\sum_{k=0}^{n} kB_{2k+1} = \frac{1}{2}(-n(n+1)B_{2n+2} + (n^2+3n-2)B_{2n+1} + n(n+3)B_{2n})$.

### 2.3. The Case $x = -1$.

We now consider the case $x = -1$ in Theorem 2.1.

The following proposition presents some summing formulas of generalized Fibonacci numbers with positive subscripts.

Taking $x = -1, r = s = t = 1$ in Theorem 2.1, we obtain the following proposition.

Proposition 2.7. *If $r = s = t = 1$ then for $n \geq 0$ we have the following formulas:*

- **(a):** $\sum_{k=0}^{n} k(-1)^k W_k = \frac{1}{2}((-1)^n((n+2)W_{n+3} - (2n+3)W_{n+2} + (n-1)W_{n+1}) + W_2 - W_1 - 2W_0)$.
- **(b):** $\sum_{k=0}^{n} k(-1)^k W_{2k} = \frac{1}{4}((-1)^n((2n+1)W_{2n+2} - 2(n+1)W_{2n+1} + W_{2n}) - W_2 + 2W_1 - W_0)$.
- **(c):** $\sum_{k=0}^{n} k(-1)^k W_{2k+1} = \frac{1}{4}((-1)^n(-W_{2n+2} + 2(n+1)W_{2n+1} + (2n+1)W_{2n}) + W_2 - 2W_1 - W_0)$.

From the above proposition, we have the following corollary which gives sum formulas of Tribonacci numbers (take $W_n = T_n$ with $T_0 = 0, T_1 = 1, T_2 = 1$).

Corollary 2.8. *For $n \geq 0$, Tribonacci numbers have the following properties:*

- **(a):** $\sum_{k=0}^{n} k(-1)^k T_k = \frac{1}{2}(-1)^n((n+2)T_{n+3} - (2n+3)T_{n+2} + (n-1)T_{n+1})$.
- **(b):** $\sum_{k=0}^{n} k(-1)^k T_{2k} = \frac{1}{4}((-1)^n((2n+1)T_{2n+2} - 2(n+1)T_{2n+1} + T_{2n}) + 1)$.
- **(c):** $\sum_{k=0}^{n} k(-1)^k T_{2k+1} = \frac{1}{4}((-1)^n(-T_{2n+2} + 2(n+1)T_{2n+1} + (2n+1)T_{2n}) - 1)$.

Taking $W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3$ in the last proposition, we have the following corollary which presents sum formulas of Tribonacci-Lucas numbers.

Corollary 2.9. *For $n \geq 0$, Tribonacci-Lucas numbers have the following properties:*

- **(a):** $\sum_{k=0}^{n} k(-1)^k K_k = \frac{1}{2}((-1)^n((n+2)K_{n+3} - (2n+3)K_{n+2} + (n-1)K_{n+1}) - 4)$.

**(b):** $\sum_{k=0}^{n} k(-1)^k K_{2k} = \frac{1}{4}((-1)^n ((2n+1) K_{2n+2} - 2(n+1) K_{2n+1} + K_{2n}) - 4)$.

**(c):** $\sum_{k=0}^{n} k(-1)^k K_{2k+1} = \frac{1}{4}((-1)^n (-K_{2n+2} + 2(n+1) K_{2n+1} + (2n+1) K_{2n}) - 2)$.

From the last proposition, we have the following corollary which gives sum formulas of Tribonacci-Perrin numbers (take $W_n = M_n$ with $M_0 = 3, M_1 = 0, M_2 = 2$).

Corollary 2.10. *For $n \geq 0$, Tribonacci-Perrin numbers have the following properties:*

**(a):** $\sum_{k=0}^{n} k(-1)^k M_k = \frac{1}{2}((-1)^n ((n+2) M_{n+3} - (2n+3) M_{n+2} + (n-1) M_{n+1}) - 4)$.

**(b):** $\sum_{k=0}^{n} k(-1)^k M_{2k} = \frac{1}{4}((-1)^n ((2n+1) M_{2n+2} - 2(n+1) M_{2n+1} + M_{2n}) - 5)$.

**(c):** $\sum_{k=0}^{n} k(-1)^k M_{2k+1} = \frac{1}{4}((-1)^n (-M_{2n+2} + 2(n+1) M_{2n+1} + (2n+1) M_{2n}) - 1)$.

Taking $W_n = U_n$ with $U_0 = 1, U_1 = 1, U_2 = 1$ in the last proposition, we have the following corollary which presents sum formulas of modified Tribonacci numbers.

Corollary 2.11. *For $n \geq 0$, modified Tribonacci numbers have the following properties:*

**(a):** $\sum_{k=0}^{n} k(-1)^k U_k = \frac{1}{2}((-1)^n ((n+2) U_{n+3} - (2n+3) U_{n+2} + (n-1) U_{n+1}) - 2)$

**(b):** $\sum_{k=0}^{n} k(-1)^k U_{2k} = \frac{1}{4} (-1)^n ((2n+1) U_{2n+2} - 2(n+1) U_{2n+1} + U_{2n})$.

**(c):** $\sum_{k=0}^{n} k(-1)^k U_{2k+1} = \frac{1}{4}((-1)^n (-U_{2n+2} + 2(n+1) U_{2n+1} + (2n+1) U_{2n}) - 2)$.

From the last proposition, we have the following corollary which gives sum formulas of modified Tribonacci-Lucas numbers (take $W_n = G_n$ with $G_0 = 4, G_1 = 4, G_2 = 10$).

Corollary 2.12. *For $n \geq 0$, modified Tribonacci-Lucas numbers have the following properties:*

**(a):** $\sum_{k=0}^{n} k(-1)^k G_k = \frac{1}{2}((-1)^n ((n+2) G_{n+3} - (2n+3) G_{n+2} + (n-1) G_{n+1}) - 2)$.

**(b):** $\sum_{k=0}^{n} k(-1)^k G_{2k} = \frac{1}{4}((-1)^n ((2n+1) G_{2n+2} - 2(n+1) G_{2n+1} + G_{2n}) - 6)$.

**(c):** $\sum_{k=0}^{n} k(-1)^k G_{2k+1} = \frac{1}{4}((-1)^n (-G_{2n+2} + 2(n+1) G_{2n+1} + (2n+1) G_{2n}) - 2)$.

Taking $W_n = H_n$ with $H_0 = 4, H_1 = 2, H_2 = 0$ in the last proposition, we have the following corollary which presents sum formulas of adjusted Tribonacci-Lucas numbers.

Corollary 2.13. *For $n \geq 0$, adjusted Tribonacci-Lucas numbers have the following properties:*

**(a):** $\sum_{k=0}^{n} k(-1)^k H_k = \frac{1}{2}((-1)^n ((n+2) H_{n+3} - (2n+3) H_{n+2} + (n-1) H_{n+1}) - 10)$.

**(b):** $\sum_{k=0}^{n} k(-1)^k H_{2k} = \frac{1}{4} (-1)^n ((2n+1) H_{2n+2} - 2(n+1) H_{2n+1} + H_{2n})$.

**(c):** $\sum_{k=0}^{n} k(-1)^k H_{2k+1} = \frac{1}{4}((-1)^n (-H_{2n+2} + 2(n+1) H_{2n+1} + (2n+1) H_{2n}) - 8)$.

Taking $r = 2, s = 1, t = 1$ in Theorem 2.1, we obtain the following proposition.

Proposition 2.14. *If $r = 2, s = 1, t = 1$ then for $n \geq 0$ we have the following formulas:*

**(a):** $\sum_{k=0}^{n} k(-1)^k W_k = \frac{1}{3}((-1)^n ((n+2) W_{n+3} - (3n+5) W_{n+2} + 2nW_{n+1}) + W_2 - 2W_1 - 2W_0)$.

**(b):** $\sum_{k=0}^{n} k(-1)^k W_{2k} = \frac{1}{25}((-1)^n ((10n+9) W_{2n+2} - 3(5n+7) W_{2n+1} - (5n+2) W_{2n}) - 9W_2 + 21W_1 + 2W_0)$.

**(c):** $\sum_{k=0}^{n} k(-1)^k W_{2k+1} = \frac{1}{25}((-1)^n ((5n-3) W_{2n+2} + (5n+7) W_{2n+1} + (10n+9) W_{2n}) + 3W_2 - 7W_1 - 9W_0)$.

From the last proposition, we have the following corollary which gives sum formulas of third-order Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 1$).

Corollary 2.15. *For $n \geq 0$, third-order Pell numbers have the following properties:*

**(a):** $\sum_{k=0}^{n} k(-1)^k P_k = \frac{1}{3}(-1)^n ((n+2) P_{n+3} - (3n+5) P_{n+2} + 2nP_{n+1})$.
**(b):** $\sum_{k=0}^{n} k(-1)^k P_{2k} = \frac{1}{25}((-1)^n ((10n+9) P_{2n+2} - 3(5n+7) P_{2n+1} - (5n+2) P_{2n}) + 3)$.
**(c):** $\sum_{k=0}^{n} k(-1)^k P_{2k+1} = \frac{1}{25}((-1)^n ((5n-3) P_{2n+2} + (5n+7) P_{2n+1} + (10n+9) P_{2n}) - 1)$.

Taking $W_n = Q_n$ with $Q_0 = 3, Q_1 = 2, Q_2 = 6$ in the last proposition, we have the following corollary which presents sum formulas of third-order Pell-Lucas numbers.

Corollary 2.16. *For $n \geq 0$, third-order Pell-Lucas numbers have the following properties:*

**(a):** $\sum_{k=0}^{n} k(-1)^k Q_k = \frac{1}{3}((-1)^n ((n+2) Q_{n+3} - (3n+5) Q_{n+2} + 2nQ_{n+1}) - 4)$.
**(b):** $\sum_{k=0}^{n} k(-1)^k Q_{2k} = \frac{1}{25}((-1)^n ((10n+9) Q_{2n+2} - 3(5n+7) Q_{2n+1} - (5n+2) Q_{2n}) - 6)$.
**(c):** $\sum_{k=0}^{n} k(-1)^k Q_{2k+1} = \frac{1}{25}((-1)^n ((5n-3) Q_{2n+2} + (5n+7) Q_{2n+1} + (10n+9) Q_{2n}) - 23)$.

From the last proposition, we have the following corollary which gives sum formulas of third-order modified Pell numbers (take $W_n = E_n$ with $E_0 = 0, E_1 = 1, E_2 = 1$).

Corollary 2.17. *For $n \geq 0$, third-order modified Pell numbers have the following properties:*

**(a):** $\sum_{k=0}^{n} k(-1)^k E_k = \frac{1}{3}((-1)^n ((n+2) E_{n+3} - (3n+5) E_{n+2} + 2nE_{n+1}) - 1)$.
**(b):** $\sum_{k=0}^{n} k(-1)^k E_{2k} = \frac{1}{25}((-1)^n ((10n+9) E_{2n+2} - 3(5n+7) E_{2n+1} - (5n+2) E_{2n}) + 12)$.
**(c):** $\sum_{k=0}^{n} k(-1)^k E_{2k+1} = \frac{1}{25}((-1)^n ((5n-3) E_{2n+2} + (5n+7) E_{2n+1} + (10n+9) E_{2n}) - 4)$.

Taking $W_n = R_n$ with $R_0 = 3, R_1 = 0, R_2 = 2$ in the last proposition, we have the following corollary which presents sum formulas of third-order Pell-Perrin numbers.

Corollary 2.18. *For $n \geq 0$, third-order Pell-Perrin numbers have the following properties:*

**(a):** $\sum_{k=0}^{n} k(-1)^k R_k = \frac{1}{3}((-1)^n ((n+2) R_{n+3} - (3n+5) R_{n+2} + 2nR_{n+1}) - 4)$.
**(b):** $\sum_{k=0}^{n} k(-1)^k R_{2k} = \frac{1}{25}((-1)^n ((10n+9) R_{2n+2} - 3(5n+7) R_{2n+1} - (5n+2) R_{2n}) - 12)$.
**(c):** $\sum_{k=0}^{n} k(-1)^k R_{2k+1} = \frac{1}{25}((-1)^n ((5n-3) R_{2n+2} + (5n+7) R_{2n+1} + (10n+9) R_{2n}) - 21)$.

Taking $r = 0, s = 1, t = 1$ in Theorem 2.1, we obtain the following proposition.

Proposition 2.19. *If $r = 0, s = 1, t = 1$ then for $n \geq 0$ we have the following formulas:*

**(a):** $\sum_{k=0}^{n} k(-1)^k W_k = (-1)^n ((n+2) W_{n+3} - (n+1) W_{n+2} - 2W_{n+1}) + W_2 - 2W_0$.
**(b):** $\sum_{k=0}^{n} k(-1)^k W_{2k} = \frac{1}{25}((-1)^n ((10n+1) W_{2n+2} - (5n+3) W_{2n+1} + (5n+8) W_{2n}) - W_2 + 3W_1 - 8W_0)$.

**(c):** $\sum_{k=0}^{n} k(-1)^k W_{2k+1} = \frac{1}{25}((-1)^n(-(5n+3)W_{2n+2} + (15n+9)W_{2n+1} + (10n+1)W_{2n}) + 3W_2 - 9W_1 - W_0)$.

From the last proposition, we have the following corollary which gives sum formulas of Padovan numbers (take $W_n = P_n$ with $P_0 = 1, P_1 = 1, P_2 = 1$).

Corollary 2.20. *For $n \geq 0$, Padovan numbers have the following properties:*

**(a):** $\sum_{k=0}^{n} k(-1)^k P_k = (-1)^n((n+2)P_{n+3} - (n+1)P_{n+2} - 2P_{n+1}) - 1$.
**(b):** $\sum_{k=0}^{n} k(-1)^k P_{2k} = \frac{1}{25}((-1)^n((10n+1)P_{2n+2} - (5n+3)P_{2n+1} + (5n+8)P_{2n}) - 6)$.
**(c):** $\sum_{k=0}^{n} k(-1)^k P_{2k+1} = \frac{1}{25}((-1)^n(-(5n+3)P_{2n+2} + (15n+9)P_{2n+1} + (10n+1)P_{2n}) - 7)$.

Taking $W_n = E_n$ with $E_0 = 3, E_1 = 0, E_2 = 2$ in the last proposition, we have the following corollary which presents sum formulas of Perrin numbers.

Corollary 2.21. *For $n \geq 0$, Perrin numbers have the following properties:*

**(a):** $\sum_{k=0}^{n} k(-1)^k E_k = (-1)^n((n+2)E_{n+3} - (n+1)E_{n+2} - 2E_{n+1}) - 4$.
**(b):** $\sum_{k=0}^{n} k(-1)^k E_{2k} = \frac{1}{25}((-1)^n((10n+1)E_{2n+2} - (5n+3)E_{2n+1} + (5n+8)E_{2n}) - 26)$.
**(c):** $\sum_{k=0}^{n} k(-1)^k E_{2k+1} = \frac{1}{25}((-1)^n(-(5n+3)E_{2n+2} + (15n+9)E_{2n+1} + (10n+1)E_{2n}) + 3)$.

From the last proposition, we have the following corollary which gives sum formulas of Padovan-Perrin numbers (take $W_n = S_n$ with $S_0 = 0, S_1 = 0, S_2 = 1$).

Corollary 2.22. *For $n \geq 0$, Padovan-Perrin numbers have the following properties:*

**(a):** $\sum_{k=0}^{n} k(-1)^k S_k = (-1)^n((n+2)S_{n+3} - (n+1)S_{n+2} - 2S_{n+1}) + 1$.
**(b):** $\sum_{k=0}^{n} k(-1)^k S_{2k} = \frac{1}{25}((-1)^n((10n+1)S_{2n+2} - (5n+3)S_{2n+1} + (5n+8)S_{2n}) - 1)$.
**(c):** $\sum_{k=0}^{n} k(-1)^k S_{2k+1} = \frac{1}{25}((-1)^n(-(5n+3)S_{2n+2} + (15n+9)S_{2n+1} + (10n+1)S_{2n}) + 3$.

Taking $W_n = A_n$ with $A_0 = 3, A_1 = 1, A_2 = 3$ in the last proposition, we have the following corollary which presents sum formulas of modified Padovan numbers.

Corollary 2.23. *For $n \geq 0$, modified Padovan numbers have the following properties:*

**(a):** $\sum_{k=0}^{n} k(-1)^k A_k = (-1)^n((n+2)A_{n+3} - (n+1)A_{n+2} - 2A_{n+1}) - 3$.
**(b):** $\sum_{k=0}^{n} k(-1)^k A_{2k} = \frac{1}{25}((-1)^n((10n+1)A_{2n+2} - (5n+3)A_{2n+1} + (5n+8)A_{2n}) - 24)$.
**(c):** $\sum_{k=0}^{n} k(-1)^k A_{2k+1} = \frac{1}{25}((-1)^n(-(5n+3)A_{2n+2} + (15n+9)A_{2n+1} + (10n+1)A_{2n}) - 3)$.

Observe that setting $x = 1$, $r = 0, s = 2, t = 1$ (i.e. for the generalized Pell-Padovan case) in Theorem 2.1, (a) makes the right hand side of the sum formulas to be an indeterminate form. Application of L'Hospital rule (using twice) however provides the evaluation of the sum formulas. If $x = -1, r = 0, s = 2, t = 1$ then we have the following theorem (in fact taking $r = 0, s = 2, t = 1$ in Theorem 2.1 (a) and then using L'Hospital rule twice for $x = 1$ we obtain the following theorem).

THEOREM 2.24. *If $r=0,s=2,t=1$ then for $n\geq 0$ we have the following formulas:*

**(a):** $\sum_{k=0}^{n}k(-1)^{k}W_{k}=\frac{1}{2}((-1)^{n}(-(n^{2}+7n+22)W_{n+3}+(n^{2}+5n+16)W_{n+2}+(n^{2}+11n+32)W_{n+1})-16W_{2}+12W_{1}+22W_{0})$.

**(b):** $\sum_{k=0}^{n}k(-1)^{k}W_{2k}=\frac{1}{20}((-1)^{n}((6n+1)W_{2n+2}-(2n+1)W_{2n+1}+(2n+3)W_{2n})-W_{2}+W_{1}-3W_{0})$.

**(c):** $\sum_{k=0}^{n}k(-1)^{k}W_{2k+1}=\frac{1}{20}((-1)^{n}(-(2n+1)W_{2n+2}+(6n+1)W_{2n}+(14n+5)W_{2n+1})+W_{2}-5W_{1}-W_{0})$.

Proof

**(a):** We use Theorem 2.1 (a). If we set $r=0,s=2,t=1$ in Theorem 2.1 (a) then we have

$$\sum_{k=0}^{n}k(-1)^{k}x^{k}W_{k}=\frac{h_{1}(x)}{(x^{3}+2x^{2}-1)^{2}}$$

where

$h_{1}(x)=x^{n+3}(n(x^{3}+2x^{2}-1)+2x^{2}-3)W_{n+3}-x^{n+2}(x^{3}-n(x^{3}+2x^{2}-1)+2)W_{n+2}-x^{n+1}(2x^{3}-4x^{2}+4x^{4}+n(2x^{2}-1)(x^{3}+2x^{2}-1)+1)W_{n+1}+x^{2}(x^{3}+2)W_{2}+x(2x^{3}+2x^{2}+1)W_{1}-x^{3}(2x^{2}-3)W_{0}$

For $x=-1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get

$$\begin{aligned}\sum_{k=0}^{n}k(-1)^{k}W_{k} &= \left.\frac{\frac{d}{dx}(h_{1}(x))}{\frac{d}{dx}((x^{3}+2x^{2}-1)^{2})}\right|_{x=-1}\\ &= \left.\frac{\frac{d^{2}}{dx^{2}}(h_{1}(x))}{\frac{d^{2}}{dx^{2}}((x^{3}+2x^{2}-1)^{2})}\right|_{x=-1}\\ &= \frac{1}{2}((-1)^{n}(-(n^{2}+7n+22)W_{n+3}+(n^{2}+5n+16)W_{n+2}\\ &\quad +(n^{2}+11n+32)W_{n+1})-16W_{2}+12W_{1}+22W_{0}).\end{aligned}$$

**(b):** Taking $x=-1$,$r=0,s=2,t=1$ in Theorem 2.1 (b) we obtain (b).

**(c):** Taking $x=-1$,$r=0,s=2,t=1$ in Theorem 2.1 (c) we obtain (c). □

From the last theorem, we have the following corollary which gives sum formulas of Pell-Padovan numbers (take $W_{n}=R_{n}$ with $R_{0}=1,R_{1}=1,R_{2}=1$).

COROLLARY 2.25. *For $n\geq 0$, Pell-Padovan numbers have the following properties:*

**(a):** $\sum_{k=0}^{n}k(-1)^{k}R_{k}=\frac{1}{2}((-1)^{n}(-(n^{2}+7n+22)R_{n+3}+(n^{2}+5n+16)R_{n+2}+(n^{2}+11n+32)R_{n+1})+18)$.

**(b):** $\sum_{k=0}^{n}k(-1)^{k}R_{2k}=\frac{1}{20}((-1)^{n}((6n+1)R_{2n+2}-(2n+1)R_{2n+1}+(2n+3)R_{2n})-3)$.

**(c):** $\sum_{k=0}^{n}k(-1)^{k}R_{2k+1}=\frac{1}{20}((-1)^{n}(-(2n+1)R_{2n+2}+(6n+1)R_{2n}+(14n+5)R_{2n+1})-5)$.

Taking $W_{n}=C_{n}$ with $C_{0}=3,C_{1}=0,C_{2}=2$ in the last theorem, we have the following corollary which presents sum formulas of Pell-Perrin numbers.

COROLLARY 2.26. *For $n \geq 0$, Pell-Perrin numbers have the following properties:*

**(a):** $\sum_{k=0}^{n} k(-1)^k C_k = \frac{1}{2}((-1)^n (-(n^2+7n+22)C_{n+3} + (n^2+5n+16)C_{n+2} + (n^2+11n+32)C_{n+1}) + 34)$.

**(b):** $\sum_{k=0}^{n} k(-1)^k C_{2k} = \frac{1}{20}((-1)^n ((6n+1)C_{2n+2} - (2n+1)C_{2n+1} + (2n+3)C_{2n}) - 11)$.

**(c):** $\sum_{k=0}^{n} k(-1)^k C_{2k+1} = \frac{1}{20}((-1)^n (-(2n+1)C_{2n+2} + (6n+1)C_{2n} + (14n+5)C_{2n+1}) - 1)$.

From the last theorem, we have the following corollary which gives sum formulas of third order Fibonacci-Pell numbers (take $W_n = G_n$ with $G_0 = 1, G_1 = 0, G_2 = 2$).

COROLLARY 2.27. *For $n \geq 0$, third order Fibonacci-Pell numbers have the following properties:*

**(a):** $\sum_{k=0}^{n} k(-1)^k G_k = \frac{1}{2}((-1)^n (-(n^2+7n+22)G_{n+3} + (n^2+5n+16)G_{n+2} + (n^2+11n+32)G_{n+1}) - 10)$.

**(b):** $\sum_{k=0}^{n} k(-1)^k G_{2k} = \frac{1}{20}((-1)^n ((6n+1)G_{2n+2} - (2n+1)G_{2n+1} + (2n+3)G_{2n}) - 5)$.

**(c):** $\sum_{k=0}^{n} k(-1)^k G_{2k+1} = \frac{1}{20}((-1)^n (-(2n+1)G_{2n+2} + (6n+1)G_{2n} + (14n+5)G_{2n+1}) + 1)$.

Taking $W_n = B_n$ with $B_0 = 3, B_1 = 0, B_2 = 4$ in the last theorem, we have the following corollary which presents sum formulas of third order Lucas-Pell numbers.

COROLLARY 2.28. *For $n \geq 0$, third order Lucas-Pell numbers have the following properties:*

**(a):** $\sum_{k=0}^{n} k(-1)^k B_k = \frac{1}{2}((-1)^n (-(n^2+7n+22)B_{n+3} + (n^2+5n+16)B_{n+2} + (n^2+11n+32)B_{n+1}) + 2)$.

**(b):** $\sum_{k=0}^{n} k(-1)^k B_{2k} = \frac{1}{20}((-1)^n ((6n+1)B_{2n+2} - (2n+1)B_{2n+1} + (2n+3)B_{2n}) - 13)$.

**(c):** $\sum_{k=0}^{n} k(-1)^k B_{2k+1} = \frac{1}{20}((-1)^n (-(2n+1)B_{2n+2} + (6n+1)B_{2n} + (14n+5)B_{2n+1}) + 1)$.

Taking $r = 0, s = 1, t = 2$ in Theorem 2.1, we obtain the following proposition.

PROPOSITION 2.29. *If $r = 0, s = 1, t = 2$ then for $n \geq 1$ we have the following formulas:*

**(a):** $\sum_{k=0}^{n} k(-1)^k W_k = \frac{1}{2}((-1)^n ((n+1)W_{n+3} - nW_{n+2} - 2W_{n+1}) + W_1 - 2W_0)$.

**(b):** $\sum_{k=0}^{n} k(-1)^k W_{2k} = \frac{1}{8}((-1)^n ((2n-1)W_{2n+2} - 2nW_{2n+1} + 4(n+1)W_{2n}) + W_2 - 4W_0)$.

**(c):** $\sum_{k=0}^{n} k(-1)^k W_{2k+1} = \frac{1}{8}((-1)^n (-2nW_{2n+2} + 3(2n+1)W_{2n+1} + 2(2n-1)W_{2n}) - 3W_1 + 2W_0)$.

From the last proposition, we have the following corollary which gives sum formulas of Jacobsthal-Padovan numbers (take $W_n = Q_n$ with $Q_0 = 1, Q_1 = 1, Q_2 = 1$).

COROLLARY 2.30. *For $n \geq 0$, Jacobsthal-Padovan numbers have the following properties:*

**(a):** $\sum_{k=0}^{n} k(-1)^k Q_k = \frac{1}{2}((-1)^n ((n+1)Q_{n+3} - nQ_{n+2} - 2Q_{n+1}) - 1)$.

**(b):** $\sum_{k=0}^{n} k(-1)^k Q_{2k} = \frac{1}{8}((-1)^n ((2n-1)Q_{2n+2} - 2nQ_{2n+1} + 4(n+1)Q_{2n}) - 3)$.

**(c):** $\sum_{k=0}^{n} k(-1)^k Q_{2k+1} = \frac{1}{8}((-1)^n (-2nQ_{2n+2} + 3(2n+1)Q_{2n+1} + 2(2n-1)Q_{2n}) - 1)$.

Taking $W_n = L_n$ with $L_0 = 3, L_1 = 0, L_2 = 2$ in the last proposition, we have the following corollary which presents sum formulas of Jacobsthal-Perrin numbers.

COROLLARY 2.31. *For $n \geq 0$, Jacobsthal-Perrin numbers have the following properties:*

**(a):** $\sum_{k=0}^{n} k(-1)^k L_k = \frac{1}{2}((-1)^n ((n+1) L_{n+3} - nL_{n+2} - 2L_{n+1}) - 6)$.
**(b):** $\sum_{k=0}^{n} k(-1)^k L_{2k} = \frac{1}{8}((-1)^n ((2n-1) L_{2n+2} - 2nL_{2n+1} + 4(n+1) L_{2n}) - 10)$.
**(c):** $\sum_{k=0}^{n} k(-1)^k L_{2k+1} = \frac{1}{8}((-1)^n (-2nL_{2n+2} + 3(2n+1) L_{2n+1} + 2(2n-1) L_{2n}) + 6)$.

From the last proposition, we have the following corollary which gives sum formulas of adjusted Jacobsthal-Padovan numbers (take $W_n = K_n$ with $K_0 = 0, K_1 = 1, K_2 = 0$).

COROLLARY 2.32. *For $n \geq 0$, adjusted Jacobsthal-Padovan numbers have the following properties:*

**(a):** $\sum_{k=0}^{n} k(-1)^k K_k = \frac{1}{2}((-1)^n ((n+1) K_{n+3} - nK_{n+2} - 2K_{n+1}) + 1)$.
**(b):** $\sum_{k=0}^{n} k(-1)^k K_{2k} = \frac{1}{8}(-1)^n ((2n-1) K_{2n+2} - 2nK_{2n+1} + 4(n+1) K_{2n})$.
**(c):** $\sum_{k=0}^{n} k(-1)^k K_{2k+1} = \frac{1}{8}((-1)^n (-2nK_{2n+2} + 3(2n+1) K_{2n+1} + 2(2n-1) K_{2n}) - 3)$.

Taking $W_n = M_n$ with $M_0 = 3, M_1 = 1, M_2 = 3$ in the last proposition, we have the following corollary which presents sum formulas of modified Jacobsthal-Padovan numbers.

COROLLARY 2.33. *For $n \geq 0$, modified Jacobsthal-Padovan numbers have the following properties:*

**(a):** $\sum_{k=0}^{n} k(-1)^k M_k = \frac{1}{2}((-1)^n ((n+1) M_{n+3} - nM_{n+2} - 2M_{n+1}) - 5)$.
**(b):** $\sum_{k=0}^{n} k(-1)^k M_{2k} = \frac{1}{8}((-1)^n ((2n-1) M_{2n+2} - 2nM_{2n+1} + 4(n+1) M_{2n}) - 9)$.
**(c):** $\sum_{k=0}^{n} k(-1)^k M_{2k+1} = \frac{1}{8}((-1)^n (-2nM_{2n+2} + 3(2n+1) M_{2n+1} + 2(2n-1) M_{2n}) + 3)$.

Taking $r = 1, s = 0, t = 1$ in Theorem 2.1, we obtain the following proposition.

PROPOSITION 2.34. *If $r = 1, s = 0, t = 1$ then for $n \geq 0$ we have the following formulas:*

**(a):** $\sum_{k=0}^{n} k(-1)^k W_k = \frac{1}{9}((-1)^n ((6n+1) W_{n+1} + (3n+5) W_{n+3} - (6n+7) W_{n+2}) + 2W_2 - W_1 - 5W_0)$.
**(b):** $\sum_{k=0}^{n} k(-1)^k W_{2k} = (-1)^n ((n+1) W_{2n+2} - (n+2) W_{2n+1} + W_{2n}) - W_2 + 2W_1 - W_0$.
**(c):** $\sum_{k=0}^{n} k(-1)^k W_{2k+1} = (-1)^n (-W_{2n+2} + W_{2n+1} + (n+1) W_{2n}) + W_2 - W_1 - W_0$.

From the last proposition, we have the following corollary which gives sum formulas of Narayana numbers (take $W_n = N_n$ with $N_0 = 0, N_1 = 1, N_2 = 1$).

COROLLARY 2.35. *For $n \geq 0$, Narayana numbers have the following properties:*

**(a):** $\sum_{k=0}^{n} k(-1)^k N_k = \frac{1}{9}((-1)^n ((6n+1) N_{n+1} + (3n+5) N_{n+3} - (6n+7) N_{n+2}) + 1)$.
**(b):** $\sum_{k=0}^{n} k(-1)^k N_{2k} = (-1)^n ((n+1) N_{2n+2} - (n+2) N_{2n+1} + N_{2n}) + 1$.
**(c):** $\sum_{k=0}^{n} k(-1)^k N_{2k+1} = (-1)^n (-N_{2n+2} + N_{2n+1} + (n+1) N_{2n})$.

Taking $W_n = U_n$ with $U_0 = 3, U_1 = 1, U_2 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Narayana-Lucas numbers.

COROLLARY 2.36. *For $n \geq 0$, Narayana-Lucas numbers have the following properties:*

**(a):** $\sum_{k=0}^{n} k(-1)^k U_k = \frac{1}{9}((-1)^n ((6n+1) U_{n+1} + (3n+5) U_{n+3} - (6n+7) U_{n+2}) - 14)$.

**(b):** $\sum_{k=0}^{n} k(-1)^k U_{2k} = (-1)^n ((n+1) U_{2n+2} - (n+2) U_{2n+1} + U_{2n}) - 2$.

**(c):** $\sum_{k=0}^{n} k(-1)^k U_{2k+1} = (-1)^n (-U_{2n+2} + U_{2n+1} + (n+1) U_{2n}) - 3$.

From the last proposition, we have the following corollary which gives sum formulas of Narayana-Perrin numbers (take $W_n = H_n$ with $H_0 = 3, H_1 = 0, H_2 = 2$).

Corollary 2.37. *For $n \geq 0$, Narayana-Perrin numbers have the following properties:*

**(a):** $\sum_{k=0}^{n} k(-1)^k H_k = \frac{1}{9}((-1)^n ((6n+1) H_{n+1} + (3n+5) H_{n+3} - (6n+7) H_{n+2}) - 11)$.

**(b):** $\sum_{k=0}^{n} k(-1)^k H_{2k} = (-1)^n ((n+1) H_{2n+2} - (n+2) H_{2n+1} + H_{2n}) - 5$.

**(c):** $\sum_{k=0}^{n} k(-1)^k H_{2k+1} = (-1)^n (-H_{2n+2} + H_{2n+1} + (n+1) H_{2n}) - 1$.

Taking $r = 1, s = 1, t = 2$ in Theorem 2.1, we obtain the following proposition.

Proposition 2.38. *If $r = 1, s = 1, t = 2$ then for $n \geq 0$ we have the following formulas:*

**(a):** $\sum_{k=0}^{n} k(-1)^k W_k = \frac{1}{9}((-1)^n ((3n+4) W_{n+3} - (6n+5) W_{n+2} + (3n-5) W_{n+1}) + W_2 + W_1 - 8W_0)$.

**(b):** $\sum_{k=0}^{n} k(-1)^k W_{2k} = \frac{1}{25}((-1)^n ((10n-3) W_{2n+2} - 3(5n+1) W_{2n+1} + 2(5n+11) W_{2n}) + 3W_2 + 3W_1 - 22W_0)$.

**(c):** $\sum_{k=0}^{n} k(-1)^k W_{2k+1} = \frac{1}{25}((-1)^n (-(5n+6) W_{2n+2} + (20n+19) W_{2n+1} + 2(10n-3) W_{2n}) + 6W_2 - 19W_1 + 6W_0)$.

From the last proposition, we have the following corollary which gives sum formulas of third order Jacobsthal numbers (take $W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1$).

Corollary 2.39. *For $n \geq 0$, third order Jacobsthal numbers have the following properties:*

**(a):** $\sum_{k=0}^{n} k(-1)^k J_k = \frac{1}{9}((-1)^n ((3n+4) J_{n+3} - (6n+5) J_{n+2} + (3n-5) J_{n+1}) + 2)$.

**(b):** $\sum_{k=0}^{n} k(-1)^k J_{2k} = \frac{1}{25}((-1)^n ((10n-3) J_{2n+2} - 3(5n+1) J_{2n+1} + 2(5n+11) J_{2n}) + 6)$.

**(c):** $\sum_{k=0}^{n} k(-1)^k J_{2k+1} = \frac{1}{25}((-1)^n (-(5n+6) J_{2n+2} + (20n+19) J_{2n+1} + 2(10n-3) J_{2n}) - 13)$.

Taking $W_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5$ in the last proposition, we have the following corollary which presents sum formulas of third order Jacobsthal-Lucas numbers.

Corollary 2.40. *For $n \geq 0$, third order Jacobsthal-Lucas numbers have the following properties:*

**(a):** $\sum_{k=0}^{n} k(-1)^k j_k = \frac{1}{9}((-1)^n ((3n+4) j_{n+3} - (6n+5) j_{n+2} + (3n-5) j_{n+1}) - 10)$.

**(b):** $\sum_{k=0}^{n} k(-1)^k j_{2k} = \frac{1}{25}((-1)^n ((10n-3) j_{2n+2} - 3(5n+1) j_{2n+1} + 2(5n+11) j_{2n}) - 26)$.

**(c):** $\sum_{k=0}^{n} k(-1)^k j_{2k+1} = \frac{1}{25}((-1)^n (-(5n+6) j_{2n+2} + (20n+19) j_{2n+1} + 2(10n-3) j_{2n}) + 23)$.

From the last proposition, we have the following corollary which gives sum formulas of modified third order Jacobsthal-Lucas numbers (take $W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3$).

Corollary 2.41. *For $n \geq 0$, modified third order Jacobsthal-Lucas numbers have the following properties:*

**(a):** $\sum_{k=0}^{n} k(-1)^k K_k = \frac{1}{9}((-1)^n((3n+4)K_{n+3} - (6n+5)K_{n+2} + (3n-5)K_{n+1}) - 20).$

**(b):** $\sum_{k=0}^{n} k(-1)^k K_{2k} = \frac{1}{25}((-1)^n((10n-3)K_{2n+2} - 3(5n+1)K_{2n+1} + 2(5n+11)K_{2n}) - 54).$

**(c):** $\sum_{k=0}^{n} k(-1)^k K_{2k+1} = \frac{1}{25}((-1)^n(-(5n+6)K_{2n+2} + (20n+19)K_{2n+1} + 2(10n-3)K_{2n}) + 17).$

Taking $W_n = Q_n$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2$ in the last proposition, we have the following corollary which presents sum formulas of third order Jacobsthal-Perrin numbers.

Corollary 2.42. *For $n \geq 0$, third order Jacobsthal-Perrin numbers have the following properties:*

**(a):** $\sum_{k=0}^{n} k(-1)^k Q_k = \frac{1}{9}((-1)^n((3n+4)Q_{n+3} - (6n+5)Q_{n+2} + (3n-5)Q_{n+1}) - 22).$

**(b):** $\sum_{k=0}^{n} k(-1)^k Q_{2k} = \frac{1}{25}((-1)^n((10n-3)Q_{2n+2} - 3(5n+1)Q_{2n+1} + 2(5n+11)Q_{2n}) - 60).$

**(c):** $\sum_{k=0}^{n} k(-1)^k Q_{2k+1} = \frac{1}{25}((-1)^n(-(5n+6)Q_{2n+2} + (20n+19)Q_{2n+1} + 2(10n-3)Q_{2n}) + 30).$

Taking $r = 2, s = 3, t = 5$ in Theorem 2.1, we obtain the following proposition.

Proposition 2.43. *If $r = 2, s = 3, t = 5$ then for $n \geq 0$ we have the following formulas:*

**(a):** $\sum_{k=0}^{n} k(-1)^k W_k = \frac{1}{25}((-1)^n((5n+4)W_{n+3} - (15n+7)W_{n+2} - 20W_{n+1}) - W_2 + 8W_1 - 20W_0).$

**(b):** $\sum_{k=0}^{n} k(-1)^k W_{2k} = \frac{1}{625}((-1)^n((100n-77)W_{2n+2} - 11(25n-13)W_{2n+1} + 5(75n+86)W_{2n}) + 77W_2 - 143W_1 - 430W_0).$

**(c):** $\sum_{k=0}^{n} k(-1)^k W_{2k+1} = \frac{1}{625}((-1)^n(-(75n+11)W_{2n+2} + (675n+199)W_{2n+1} + 5(100n-77)W_{2n}) + 11W_2 - 199W_1 + 385W_0).$

From the last proposition, we have the following corollary which gives sum formulas of 3-primes numbers (take $W_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = 2$).

Corollary 2.44. *For $n \geq 0$, 3-primes numbers have the following properties:*

**(a):** $\sum_{k=0}^{n} k(-1)^k G_k = \frac{1}{25}((-1)^n((5n+4)G_{n+3} - (15n+7)G_{n+2} - 20G_{n+1}) + 6).$

**(b):** $\sum_{k=0}^{n} k(-1)^k G_{2k} = \frac{1}{625}((-1)^n((100n-77)G_{2n+2} - 11(25n-13)G_{2n+1} + 5(75n+86)G_{2n}) + 11).$

**(c):** $\sum_{k=0}^{n} k(-1)^k G_{2k+1} = \frac{1}{625}((-1)^n(-(75n+11)G_{2n+2} + (675n+199)G_{2n+1} + 5(100n-77)G_{2n}) - 177).$

Taking $W_n = H_n$ with $H_0 = 3, H_1 = 2, H_2 = 10$ in the last proposition, we have the following corollary which presents sum formulas of Lucas 3-primes numbers.

Corollary 2.45. *For $n \geq 0$, Lucas 3-primes numbers have the following properties:*

**(a):** $\sum_{k=0}^{n} k(-1)^k H_k = \frac{1}{25}((-1)^n((5n+4)H_{n+3} - (15n+7)H_{n+2} - 20H_{n+1}) - 54).$

**(b):** $\sum_{k=0}^{n} k(-1)^k H_{2k} = \frac{1}{625}((-1)^n((100n-77)H_{2n+2} - 11(25n-13)H_{2n+1} + 5(75n+86)H_{2n}) - 806).$

**(c):** $\sum_{k=0}^{n} k(-1)^k H_{2k+1} = \frac{1}{625}((-1)^n (-(75n+11) H_{2n+2} + (675n+199) H_{2n+1} + 5(100n-77) H_{2n}) + 867)$.

From the last proposition, we have the following corollary which gives sum formulas of modified 3-primes numbers (take $W_n = E_n$ with $E_0 = 0, E_1 = 1, E_2 = 1$).

Corollary 2.46. *For $n \geq 0$, modified 3-primes numbers have the following properties:*

**(a):** $\sum_{k=0}^{n} k(-1)^k E_k = \frac{1}{25}((-1)^n ((5n+4) E_{n+3} - (15n+7) E_{n+2} - 20E_{n+1}) + 7)$.

**(b):** $\sum_{k=0}^{n} k(-1)^k E_{2k} = \frac{1}{625}((-1)^n ((100n-77) E_{2n+2} - 11(25n-13) E_{2n+1} + 5(75n+86) E_{2n}) - 66)$.

**(c):** $\sum_{k=0}^{n} k(-1)^k E_{2k+1} = \frac{1}{625}((-1)^n (-(75n+11) E_{2n+2} + (675n+199) E_{2n+1} + 5(100n-77) E_{2n}) - 188)$.

Taking $r = 5, s = 3, t = 2$ in Theorem 2.1, we obtain the following proposition.

Proposition 2.47. *If $r = 5, s = 3, t = 2$ then for $n \geq 0$, we have the following formulas:*

**(a):** $\sum_{k=0}^{n} k(-1)^k W_k = \frac{1}{5}((-1)^n ((n+2) W_{n+3} - (6n+11) W_{n+2} + (3n-1) W_{n+1}) + W_2 - 5W_1 - 4W_0)$.

**(b):** $\sum_{k=0}^{n} k(-1)^k W_{2k} = \frac{1}{625}((-1)^n ((100n+91) W_{2n+2} - 17(25n+29) W_{2n+1} - 2(75n+37) W_{2n}) - 91W_2 + 493W_1 + 74W_0)$.

**(c):** $\sum_{k=0}^{n} k(-1)^k W_{2k+1} = \frac{1}{625}((-1)^n ((75n-38) W_{2n+2} + (150n+199) W_{2n+1} + 2(100n+91) W_{2n}) + 38W_2 - 199W_1 - 182W_0)$.

From the last proposition, we have the following corollary which gives sum formulas of reverse 3-primes numbers (take $W_n = N_n$ with $N_0 = 0, N_1 = 1, N_2 = 5$).

Corollary 2.48. *For $n \geq 0$, reverse 3-primes numbers have the following properties:*

**(a):** $\sum_{k=0}^{n} k(-1)^k N_k = \frac{1}{5}(-1)^n ((n+2) N_{n+3} - (6n+11) N_{n+2} + (3n-1) N_{n+1})$.

**(b):** $\sum_{k=0}^{n} k(-1)^k N_{2k} = \frac{1}{625}((-1)^n ((100n+91) N_{2n+2} - 17(25n+29) N_{2n+1} - 2(75n+37) N_{2n}) + 38)$.

**(c):** $\sum_{k=0}^{n} k(-1)^k N_{2k+1} = \frac{1}{625}((-1)^n ((75n-38) N_{2n+2} + (150n+199) N_{2n+1} + 2(100n+91) N_{2n}) - 9)$.

Taking $W_n = S_n$ with $S_0 = 3, S_1 = 5, S_2 = 31$ in the last proposition, we have the following corollary which presents sum formulas of reverse Lucas 3-primes numbers.

Corollary 2.49. *For $n \geq 0$, reverse Lucas 3-primes numbers have the following properties:*

**(a):** $\sum_{k=0}^{n} k(-1)^k S_k = \frac{1}{5}((-1)^n ((n+2) S_{n+3} - (6n+11) S_{n+2} + (3n-1) S_{n+1}) - 6)$.

**(b):** $\sum_{k=0}^{n} k(-1)^k S_{2k} = \frac{1}{625}((-1)^n ((100n+91) S_{2n+2} - 17(25n+29) S_{2n+1} - 2(75n+37) S_{2n}) - 134)$.

**(c):** $\sum_{k=0}^{n} k(-1)^k S_{2k+1} = \frac{1}{625}((-1)^n((75n-38)S_{2n+2}+(150n+199)S_{2n+1}+2(100n+91)S_{2n}) - 363)$.

From the last proposition, we have the following corollary which gives sum formulas of reverse modified 3-primes numbers (take $W_n = U_n$ with $U_0 = 0, U_1 = 1, U_2 = 4$).

Corollary 2.50. *For $n \geq 0$, reverse modified 3-primes numbers have the following properties:*

**(a):** $\sum_{k=0}^{n} k(-1)^k U_k = \frac{1}{5}((-1)^n((n+2)U_{n+3}-(6n+11)U_{n+2}+(3n-1)U_{n+1}) - 1)$.

**(b):** $\sum_{k=0}^{n} k(-1)^k U_{2k} = \frac{1}{625}((-1)^n((100n+91)U_{2n+2}-17(25n+29)U_{2n+1}-2(75n+37)U_{2n}) + 129)$.

**(c):** $\sum_{k=0}^{n} k(-1)^k U_{2k+1} = \frac{1}{625}((-1)^n((75n-38)U_{2n+2}+(150n+199)U_{2n+1}+2(100n+91)U_{2n}) - 47)$.

**2.4. The Case $x = i$.** We now consider the complex case $x = i$ in Theorem 2.1. Taking $x = i, r = s = t = 1$ in Theorem 2.1, we obtain the following proposition.

Proposition 2.51. *If $r = s = t = 1$ then for $n \geq 0$ we have the following formulas:*

**(a):** $\sum_{k=0}^{n} ki^k W_k = \frac{1}{4}(i^n(i(2n+4-2i)W_{n+3}+((2-2i)n-6i)W_{n+2}-i((4-2i)n+2-6i)W_{n+1}) - (2-2i)W_2+(4-2i)W_1-(2+4i)W_0)$.

**(b):** $\sum_{k=0}^{n} ki^k W_{2k} = \frac{i}{8}(i^n(i(4in+4+4i)W_{2n+2}+2((2-2i)n+2-4i)W_{2n+1}+(4n+8)W_{2n}) + (4-4i)W_2-(4-8i)W_1-8W_0)$.

**(c):** $\sum_{k=0}^{n} ki^k W_{2k+1} = \frac{i}{8}(i^n((4+4i)W_{2n+1}-i(4n+4)W_{2n+2}+(4i-4-4n)W_{2n})+4iW_2-(4+4i)W_1 + (4-4i)W_0)$.

From the above proposition, we have the following corollary which gives sum formulas of Tribonacci numbers (take $W_n = T_n$ with $T_0 = 0, T_1 = 1, T_2 = 1$).

Corollary 2.52. *For $n \geq 0$, Tribonacci numbers have the following properties:*

**(a):** $\sum_{k=0}^{n} ki^k T_k = \frac{1}{4}(i^n(i(2n+4-2i)T_{n+3}+((2-2i)n-6i)T_{n+2}-i((4-2i)n+2-6i)T_{n+1}) + 2)$.

**(b):** $\sum_{k=0}^{n} ki^k T_{2k} = \frac{i}{8}(i^n(i(4in+4+4i)T_{2n+2}+2((2-2i)n+2-4i)T_{2n+1}+(4n+8)T_{2n}) + 4i)$.

**(c):** $\sum_{k=0}^{n} ki^k T_{2k+1} = \frac{i}{8}(i^n((4+4i)T_{2n+1}-i(4n+4)T_{2n+2}+(4i-4-4n)T_{2n}) - 4)$.

Taking $W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3$ in the last proposition, we have the following corollary which presents sum formulas of Tribonacci-Lucas numbers.

Corollary 2.53. *For $n \geq 0$, Tribonacci-Lucas numbers have the following properties:*

**(a):** $\sum_{k=0}^{n} ki^k K_k = \frac{1}{4}(i^n(i(2n+4-2i)K_{n+3}+((2-2i)n-6i)K_{n+2}-i((4-2i)n+2-6i)K_{n+1}) - 8-8i)$.

**(b):** $\sum_{k=0}^{n} ki^k K_{2k} = \frac{i}{8}(i^n(i\,(4in+4+4i)\,K_{2n+2}+2\,((2-2i)\,n+2-4i)\,K_{2n+1}+(4n+8)\,K_{2n})-16-4i)$.

**(c):** $\sum_{k=0}^{n} ki^k K_{2k+1} = \frac{i}{8}(i^n((4+4i)\,K_{2n+1}-i\,(4n+4)\,K_{2n+2}+(4i-4-4n)\,K_{2n})+8-4i)$.

From the last proposition, we have the following corollary which gives sum formulas of Tribonacci-Perrin numbers (take $W_n = M_n$ with $M_0 = 3, M_1 = 0, M_2 = 2$).

Corollary 2.54. *For $n \geq 0$, Tribonacci-Perrin numbers have the following properties:*

**(a):** $\sum_{k=0}^{n} ki^k M_k = \frac{1}{4}(i^n(i\,(2n+4-2i)\,M_{n+3}+((2-2i)\,n-6i)\,M_{n+2}-i\,((4-2i)\,n+2-6i)\,M_{n+1})-10-8i)$.

**(b):** $\sum_{k=0}^{n} ki^k M_{2k} = \frac{i}{8}(i^n(i\,(4in+4+4i)\,M_{2n+2}+2\,((2-2i)\,n+2-4i)\,M_{2n+1}+(4n+8)\,M_{2n})-16-8i)$.

**(c):** $\sum_{k=0}^{n} ki^k M_{2k+1} = \frac{i}{8}(i^n((4+4i)\,M_{2n+1}-i\,(4n+4)\,M_{2n+2}+(4i-4-4n)\,M_{2n})+12-4i)$.

Taking $W_n = U_n$ with $U_0 = 1, U_1 = 1, U_2 = 1$ in the last proposition, we have the following corollary which presents sum formulas of modified Tribonacci numbers.

Corollary 2.55. *For $n \geq 0$, modified Tribonacci numbers have the following properties:*

**(a):** $\sum_{k=0}^{n} ki^k U_k = \frac{1}{4}(i^n(i\,(2n+4-2i)\,U_{n+3}+((2-2i)\,n-6i)\,U_{n+2}-i\,((4-2i)\,n+2-6i)\,U_{n+1})-4i)$.

**(b):** $\sum_{k=0}^{n} ki^k U_{2k} = \frac{i}{8}(i^n(i\,(4in+4+4i)\,U_{2n+2}+2\,((2-2i)\,n+2-4i)\,U_{2n+1}+(4n+8)\,U_{2n})-8+4i)$.

**(c):** $\sum_{k=0}^{n} ki^k U_{2k+1} = \frac{i}{8}(i^n((4+4i)\,U_{2n+1}-i\,(4n+4)\,U_{2n+2}+(4i-4-4n)\,U_{2n})-4i)$.

From the last proposition, we have the following corollary which gives sum formulas of modified Tribonacci-Lucas numbers (take $W_n = G_n$ with $G_0 = 4, G_1 = 4, G_2 = 10$).

Corollary 2.56. *For $n \geq 0$, modified Tribonacci-Lucas numbers have the following properties:*

**(a):** $\sum_{k=0}^{n} ki^k G_k = \frac{1}{4}(i^n(i\,(2n+4-2i)\,G_{n+3}+((2-2i)\,n-6i)\,G_{n+2}-i\,((4-2i)\,n+2-6i)\,G_{n+1})-12-4i)$.

**(b):** $\sum_{k=0}^{n} ki^k G_{2k} = \frac{i}{8}(i^n(i\,(4in+4+4i)\,G_{2n+2}+2\,((2-2i)\,n+2-4i)\,G_{2n+1}+(4n+8)\,G_{2n})-8-8i)$.

**(c):** $\sum_{k=0}^{n} ki^k G_{2k+1} = \frac{i}{8}(i^n((4+4i)\,G_{2n+1}-i\,(4n+4)\,G_{2n+2}+(4i-4-4n)\,G_{2n})+8i)$.

Taking $W_n = H_n$ with $H_0 = 4, H_1 = 2, H_2 = 0$ in the last proposition, we have the following corollary which presents sum formulas of adjusted Tribonacci-Lucas numbers.

Corollary 2.57. *For $n \geq 0$, adjusted Tribonacci-Lucas numbers have the following properties:*

**(a):** $\sum_{k=0}^{n} ki^k H_k = \frac{1}{4}(i^n(i\,(2n+4-2i)\,H_{n+3}+((2-2i)\,n-6i)\,H_{n+2}-i\,((4-2i)\,n+2-6i)\,H_{n+1})-20i)$.

**(b):** $\sum_{k=0}^{n} ki^k H_{2k} = \frac{i}{8}(i^n (i\,(4in+4+4i)\,H_{2n+2} + 2\,((2-2i)\,n+2-4i)\,H_{2n+1} + (4n+8)\,H_{2n}) - 40 + 16i)$.

**(c):** $\sum_{k=0}^{n} ki^k H_{2k+1} = \frac{i}{8}(i^n ((4+4i)\,H_{2n+1} - i\,(4n+4)\,H_{2n+2} + (4i-4-4n)\,H_{2n}) + 8 - 24i)$.

Corresponding sums of the other third order linear sequences can be calculated similarly when $x = i$.

## 3. Sum Formulas of Generalized Tribonacci Numbers with Negative Subscripts

The following Theorem presents some sum formulas (identities) of generalized Tribonacci numbers with negative subscripts.

Theorem 3.1. *Let $x$ be a complex number. For $n \geq 1$, we have the following formulas:*

**(a):** *If $(t+rx^2+sx-x^3) \neq 0$ then*

$$\sum_{k=1}^{n} kx^k W_{-k} = \frac{\Lambda_1}{(t+rx^2+sx-x^3)^2}$$

**(b):** *If $(2sx^2 - s^2x + r^2x^2 + t^2 - x^3 + 2rtx) \neq 0$ then*

$$\sum_{k=1}^{n} kx^k W_{-2k} = \frac{\Lambda_2}{(2sx^2 - s^2x + r^2x^2 + t^2 - x^3 + 2rtx)^2}$$

*and*

**(c):** *If $(2sx^2 - s^2x + r^2x^2 + t^2 - x^3 + 2rtx) \neq 0$ then*

$$\sum_{k=1}^{n} kx^k W_{-2k+1} = \frac{\Lambda_3}{(2sx^2 - s^2x + r^2x^2 + t^2 - x^3 + 2rtx)^2}$$

*where*

$$\Lambda_1 = \sum_{k=1}^{6} \lambda_k,\ \ \Lambda_2 = \sum_{k=1}^{6} \theta_k,\ \ \Lambda_3 = \sum_{k=1}^{6} \mu_k,$$

*with*

$\lambda_1 = -x^{n+1}(n(t+rx^2+sx-x^3)(t+rx^2+sx)+sx^4+2tx^3+r^2x^4+s^2x^2+2rsx^3+2rtx^2+2stx+t^2)$ $W_{-n-1}$,

$\lambda_2 = -x^{n+2}(n(t+sx)(t+rx^2+sx-x^3)+rsx^3+2s^2x^2+tx^3+4stx+2t^2)W_{-n-2}$,

$\lambda_3 = -tx^{n+3}(n(t+rx^2+sx-x^3)+rx^2+2sx+3t)W_{-n-3}$,

$\lambda_4 = +x(t-rx^2+2x^3)W_2$,

$\lambda_5 = +x(-2rx^3+sx^2+r^2x^2-rt+2tx+x^4)W_1$,

$\lambda_6 = +tx(-s-2rx+3x^2)W_0$,

$\theta_1 = -x^{n+1}(n(t+rx)(2sx^2-s^2x+r^2x^2+t^2-x^3+2rtx)+rx^4+2tx^3+2rt^2x-rs^2x^2+r^2tx^2-2stx^2+t^3)W_{-2n+1}$,

$\theta_2 = +x^{n+1}(n(r^2x+rt+sx-x^2)(2sx^2-s^2x+r^2x^2+t^2-x^3+2rtx)-sx^4+2s^2x^3-s^3x^2-2rtx^3+2st^2x+2r^2t^2x+r^3tx^2-3t^2x^2-r^2s^2x^2+rt^3)W_{-2n}$,

$\theta_3 = +tx^{n+1}(n(s-x)(2sx^2-s^2x+r^2x^2+t^2-x^3+2rtx)+st^2+2sx^3-2t^2x-2rtx^2-r^2sx^2-s^2x^2-x^4)W_{-2n-1}$,

$\theta_4 = +x(-st^2 - 2sx^3 + 2t^2x + s^2x^2 + x^4 + 2rtx^2 + r^2sx^2)W_2,$
$\theta_5 = +x(t + rs)(-2sx^2 - r^2x^2 + t^2 + 2x^3)W_1,$
$\theta_6 = -tx(rt^2 - s^2t - 2rx^3 - 3tx^2 + r^3x^2 + 2rsx^2 + 2r^2tx + 4stx)W_0,$
$\mu_1 = x^{n+2}(n(s - x)(2sx^2 - s^2x + r^2x^2 + t^2 - x^3 + 2rtx) + 2st^2 + 2rstx - sx^3 - s^3x - 3t^2x - 4rtx^2 - r^2x^3 + 2s^2x^2)W_{-2n+1},$
$\mu_2 = -x^{n+2}(t + rs)(n(2sx^2 - s^2x + r^2x^2 + t^2 - x^3 + 2rtx) + 2rtx - s^2x + 2t^2 + x^3)W_{-2n},$
$\mu_3 = -tx^{n+1}(n(t + rx)(2sx^2 - s^2x + r^2x^2 + t^2 - x^3 + 2rtx) + rx^4 + 2tx^3 + t^3 + 2rt^2x - rs^2x^2 + r^2tx^2 - 2stx^2)W_{-2n-1},$
$\mu_4 = +x(rx^4 + 2tx^3 + t^3 + 2rt^2x - 2stx^2 - rs^2x^2 + r^2tx^2)W_2,$
$\mu_5 = -x(rt^3 - sx^4 + 2s^2x^3 - s^3x^2 - 3t^2x^2 - r^2s^2x^2 - 2rtx^3 + 2st^2x + 2r^2t^2x + r^3tx^2)W_1,$
$\mu_6 = +tx(-st^2 - 2sx^3 + 2t^2x + s^2x^2 + x^4 + 2rtx^2 + r^2sx^2)W_0.$

*Proof.*

**(a):** Using the recurrence relation

$$W_{-n+3} = r \times W_{-n+2} + s \times W_{-n+1} + t \times W_{-n} \Rightarrow W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

i.e.

$$tW_{-n} = W_{-n+3} - rW_{-n+2} - sW_{-n+1}$$

or

$$W_{-n} = \frac{1}{t}W_{-n+3} - \frac{r}{t}W_{-n+2} - \frac{s}{t}W_{-n+1}$$

we obtain

$$\begin{array}{rcl}
tnx^nW_{-n} & = & nx^nW_{-n+3} - rnx^nW_{-n+2} - snx^nW_{-n+1} \\
t(n-1)x^{n-1}W_{-n+1} & = & (n-1)x^{n-1}W_{-n+4} - r(n-1)x^{n-1}W_{-n+3} - s(n-1)x^{n-1}W_{-n+2} \\
t(n-2)x^{n-2}W_{-n+2} & = & (n-2)x^{n-2}W_{-n+5} - r(n-2)x^{n-2}W_{-n+4} - s(n-2)x^{n-2}W_{-n+3} \\
 & & \vdots \\
t \times 3 \times x^3W_{-3} & = & 3 \times x^3W_0 - r \times 3 \times x^3W_{-1} - s \times 3 \times x^3W_{-2} \\
t \times 2 \times x^2W_{-2} & = & 2 \times x^2W_1 - r \times 2 \times x^2W_0 - s \times 2 \times x^2W_{-1} \\
t \times 1 \times x^1W_{-1} & = & 1 \times x^1W_2 - r \times 1 \times x^1W_1 - s \times 1 \times x^1W_0
\end{array}$$

If we add the equations side by side, we get

$$
\begin{aligned}
& t((n+1)x^{n+1}W_{-n-1}+(n+2)x^{n+2}W_{-n-2}+(n+3)x^{n+3}W_{-n-3}+\sum_{k=1}^{n}kx^{k}W_{-k}) \qquad (3.1)\\
= & (1\times x^{1}W_{2}+2\times x^{2}W_{1}+3\times x^{3}W_{0}+\sum_{k=1}^{n}(k+3)x^{k+3}W_{-k})-r((n+3)x^{n+3}W_{-n-1}\\
& +1\times x^{1}W_{1}+2\times x^{2}W_{0}+\sum_{k=1}^{n}(k+2)x^{k+2}W_{-k})\\
& -s((n+2)x^{n+2}W_{-n-1}+(n+3)x^{n+3}W_{-n-2}+1\times x^{1}W_{0}+\sum_{k=1}^{n}(k+1)x^{k+1}W_{-k}).
\end{aligned}
$$

Then, using Theorem 1.2 (a) and solving (3.1), the required result of (a) follows.

**(b) and (c):** Using the recurrence relation

$$W_{-n+3}=rW_{-n+2}+sW_{-n+1}+tW_{-n}$$

i.e.

$$sW_{-n+1}=W_{-n+3}-rW_{-n+2}-tW_{-n}$$

we obtain

$$
\begin{aligned}
snx^{n}W_{-2n+1} &= nx^{n}W_{-2n+3}-rnx^{n}W_{-2n+2}-tnx^{n}W_{-2n}\\
s(n-1)x^{n-1}W_{-2n+3} &= (n-1)x^{n-1}W_{-2n+5}-r(n-1)x^{n-1}W_{-2n+4}-t(n-1)x^{n-1}W_{-2n+2}\\
s(n-2)x^{n-2}W_{-2n+5} &= (n-2)x^{n-2}W_{-2n+7}-r(n-2)x^{n-2}W_{-2n+6}-t(n-2)x^{n-2}W_{-2n+4}\\
s(n-3)x^{n-3}W_{-2n+7} &= (n-3)x^{n-3}W_{-2n+9}-r(n-3)x^{n-3}W_{-2n+8}-t(n-3)x^{n-3}W_{-2n+6}\\
&\vdots\\
s\times 3\times x^{3}W_{-5} &= 3\times x^{3}W_{-3}-r\times 3\times x^{3}W_{-4}-t\times 3\times x^{3}W_{-6}\\
s\times 2\times x^{2}W_{-3} &= 2\times x^{2}W_{-1}-r\times 2\times x^{2}W_{-2}-t\times 2\times x^{2}W_{-4}\\
s\times 1\times x^{1}W_{-1} &= 1\times x^{1}W_{1}-r\times 1\times x^{1}W_{0}-t\times 1\times x^{1}W_{-2}
\end{aligned}
$$

If we add the equations side by side, we get

$$
\begin{aligned}
s\sum_{k=1}^{n}kx^{k}W_{-2k+1} &= (-(n+1)x^{n+1}W_{-2n+1}+1\times x^{1}W_{1}+\sum_{k=1}^{n}(k+1)x^{k+1}W_{-2k+1}) \qquad (3.2)\\
&\quad -r(-(n+1)x^{n+1}W_{-2n}+1\times x^{1}W_{0}+\sum_{k=1}^{n}(k+1)x^{k+1}W_{-2k})-t\sum_{k=1}^{n}kx^{k}W_{-2k}.
\end{aligned}
$$

Similarly, using the recurrence relation

$$W_{-n+3}=rW_{-n+2}+sW_{-n+1}+tW_{-n}$$

i.e.

$$sW_{-n+1}=W_{-n+3}-rW_{-n+2}-tW_{-n}$$

we obtain

$$\begin{aligned}
snx^nW_{-2n} &= nx^nW_{-2n+2} - rnx^nW_{-2n+1} - tnx^nW_{-2n-1} \\
s(n-1)x^{n-1}W_{-2n+2} &= (n-1)x^{n-1}W_{-2n+4} - r(n-1)x^{n-1}W_{-2n+3} - t(n-1)x^{n-1}W_{-2n+1} \\
s(n-2)x^{n-2}W_{-2n+4} &= (n-2)x^{n-2}W_{-2n+6} - r(n-2)x^{n-2}W_{-2n+5} - t(n-2)x^{n-2}W_{-2n+3} \\
&\vdots \\
s\times 4\times x^4W_{-8} &= 4n\times x^4W_{-6} - r\times 4\times x^4W_{-7} - t\times 4\times x^4W_{-9} \\
s\times 3\times x^3W_{-6} &= 3\times x^3W_{-4} - r\times 3\times x^3W_{-5} - t\times 3\times x^3W_{-7} \\
s\times 2\times x^2W_{-4} &= 2\times x^2W_{-2} - r\times 2\times x^2W_{-3} - t\times 2\times x^2W_{-5} \\
s\times 1\times x^1W_{-2} &= 1\times x^1W_0 - r\times 1\times x^1W_{-1} - t\times 1\times x^1W_{-3}
\end{aligned}$$

If we add the equations side by side, we get

$$\begin{aligned}
s\sum_{k=1}^{n}kx^kW_{-2k} &= (-(n+1)x^{n+1}W_{-2n} + 1\times x^1W_0 + \sum_{k=1}^{n}(k+1)x^{k+1}W_{-2k}) \qquad (3.3) \\
&\quad -(r\sum_{k=1}^{n}kx^kW_{-2k+1}) - t(nx^nW_{-2n-1} - 0\times x^0W_{-1} + \sum_{k=1}^{n}(k-1)x^{k-1}W_{-2k+1}).
\end{aligned}$$

Then, using Theorem 1.2 (b) and (c) and solving system (3.2)-(3.3) the required result of (b) and (c) follow. □

□

**3.1. Special Cases.** In this section, we present the closed form solutions (identities) of the sums $\sum_{k=1}^{n}kx^kW_{-k}$, $\sum_{k=1}^{n}kx^kW_{-2k}$ and $\sum_{k=1}^{n}kx^kW_{-2k+1}$ for the specific case of sequence $\{W_n\}$.

**3.2. The Case $x=1$.** We now consider the case $x=1$ in Theorem 3.1. In this subsection, we only consider the case $x=1$, $r=0, s=2, t=1$ (this special case was not given in [36] because we can not use Theorem 3.1 directly). Observe that setting $x=1$, $r=0, s=2, t=1$ (i.e. for the generalized Pell-Padovan case) in Theorem 3.1, (b) and (c) make the right hand side of the sum formulas to be an indeterminate form. Application of L'Hospital rule (using twice) however provides the evaluation of the sum formulas. If $x=1$, $r=0, s=2, t=1$ then we have the following theorem (in fact taking $r=0, s=2, t=1$ in Theorem 3.1 and then using L'Hospital rule twice for $x=1$ we obtain the following theorem).

THEOREM 3.2. *If $r=0, s=2, t=1$ then for $n\geq 1$, we have the following formulas:*

- **(a):** $\sum_{k=1}^{n}kW_{-k} = \frac{1}{4}(-(6n+13)W_{-n-1} - (6n+19)W_{-n-2} - (2n+7)W_{-n-3} + 3W_2 + 5W_1 + W_0)$.
- **(b):** $\sum_{k=1}^{n}kW_{-2k} = \frac{1}{2}(-n(n-1)W_{-2n+1} + (n+1)(n-2)W_{-2n} + n(n-3)W_{-2n-1} + 2W_0)$.
- **(c):** $\sum_{k=1}^{n}kW_{-2k+1} = \frac{1}{2}((n+1)(n-2)W_{-2n+1} - n(n+1)W_{-2n} - n(n-1)W_{-2n-1} + 2W_1)$.

Proof

**(a):** We use Theorem 3.1 (a). If we set $x=1, r=0, s=2, t=1$ in Theorem 3.1 (a) then we have (a).

**(b):** We use Theorem 3.1 (b). If we set $r=0, s=2, t=1$ in Theorem 3.1 (b) then we have

$$\sum_{k=1}^{n} kx^k W_{-2k} = \frac{g_3(x)}{(-x^3+4x^2-4x+1)^2}$$

where

$g_3(x) = -x^{n+1}(n(-x^3+4x^2-4x+1)-4x^2+2x^3+1)W_{-2n+1}+x^{n+1}(4x-11x^2+8x^3-2x^4+n(2x-x^2)(-x^3+4x^2-4x+1))W_{-2n}-x^{n+1}(2x+4x^2-4x^3+x^4+n\,(x-2)\,(-x^3+4x^2-4x+1)-2)W_{-2n-1}+x(x^4-4x^3+4x^2+2x-2)W_2+x(2x^3-4x^2+1)W_1+x(3x^2-8x+4)W_0.$

For $x=1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule. Then we get

$$\begin{aligned}
\sum_{k=1}^{n} kW_{-2k} &= \left.\frac{\frac{d}{dx}(g_3(x))}{\frac{d}{dx}((-x^3+4x^2-4x+1)^2)}\right|_{x=1} \\
&= \left.\frac{\frac{d^2}{dx^2}(g_3(x))}{\frac{d^2}{dx^2}((-x^3+4x^2-4x+1)^2)}\right|_{x=1} \\
&= \frac{1}{2}(-n\,(n-1)\,W_{-2n+1}+(n+1)\,(n-2)\,W_{-2n}+n\,(n-3)\,W_{-2n-1}+2W_0).
\end{aligned}$$

**(c):** We use Theorem 3.1 (c). If we set $r=0, s=2, t=1$ in Theorem 3.1 (c) then we have

$$\sum_{k=1}^{n} kx^k W_{-2k+1} = \frac{g_4(x)}{(-x^3+4x^2-4x+1)^2}$$

where

$g_4(x) = -x^{n+2}(n\,(x-2)\,(-x^3+4x^2-4x+1)+11x-8x^2+2x^3-4)W_{-2n+1}-x^{n+2}(n(-x^3+4x^2-4x+1)-4x+x^3+2)W_{-2n}-x^{n+1}(n(-x^3+4x^2-4x+1)-4x^2+2x^3+1)W_{-2n-1}+x(2x^3-4x^2+1)W_2+x(2x^4-8x^3+11x^2-4x)W_1+x(x^4-4x^3+4x^2+2x-2)W_0.$

For $x=1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule. Then we get

$$\begin{aligned}
\sum_{k=1}^{n} kW_{-2k+1} &= \left.\frac{\frac{d}{dx}(g_4(x))}{\frac{d}{dx}((-x^3+4x^2-4x+1)^2)}\right|_{x=1} \\
&= \left.\frac{\frac{d^2}{dx^2}(g_4(x))}{\frac{d^2}{dx^2}((-x^3+4x^2-4x+1)^2)}\right|_{x=1} \\
&= \frac{1}{2}((n+1)\,(n-2)\,W_{-2n+1}-n\,(n+1)\,W_{-2n}-n\,(n-1)\,W_{-2n-1}+2W_1).
\end{aligned}$$

□

From the last theorem, we have the following corollary which gives sum formulas of Pell-Padovan numbers (take $W_n = R_n$ with $R_0=1, R_1=1, R_2=1$).

Corollary 3.3. *For $n \geq 1$, Pell-Padovan numbers have the following properties:*

**(a):** $\sum_{k=1}^{n} kR_{-k} = \frac{1}{4}(-(6n+13)\,R_{-n-1}-(6n+19)\,R_{-n-2}-(2n+7)\,R_{-n-3}+9).$

**(b):** $\sum_{k=1}^{n} kR_{-2k} = \frac{1}{2}(-n\,(n-1)\,R_{-2n+1}+(n+1)\,(n-2)\,R_{-2n}+n\,(n-3)\,R_{-2n-1}+2).$

**(c):** $\sum_{k=1}^{n} kR_{-2k+1} = \frac{1}{2}((n+1)(n-2)R_{-2n+1} - n(n+1)R_{-2n} - n(n-1)R_{-2n-1} + 2)$.

Taking $W_n = C_n$ with $C_0 = 3, C_1 = 0, C_2 = 2$ in the last theorem, we have the following corollary which presents sum formulas of Pell-Perrin numbers.

COROLLARY 3.4. *For $n \geq 1$, Pell-Perrin numbers have the following properties:*

**(a):** $\sum_{k=1}^{n} kC_{-k} = \frac{1}{4}(-(6n+13)C_{-n-1} - (6n+19)C_{-n-2} - (2n+7)C_{-n-3} + 9)$.

**(b):** $\sum_{k=1}^{n} kC_{-2k} = \frac{1}{2}(-n(n-1)C_{-2n+1} + (n+1)(n-2)C_{-2n} + n(n-3)C_{-2n-1} + 6)$.

**(c):** $\sum_{k=1}^{n} kC_{-2k+1} = \frac{1}{2}((n+1)(n-2)C_{-2n+1} - n(n+1)C_{-2n} - n(n-1)C_{-2n-1})$.

From the last theorem, we have the following corollary which gives sum formulas of third order Fibonacci-Pell numbers (take $W_n = G_n$ with $G_0 = 1, G_1 = 0, G_2 = 2$).

COROLLARY 3.5. *For $n \geq 1$, third order Fibonacci-Pell numbers have the following properties:*

**(a):** $\sum_{k=1}^{n} kG_{-k} = \frac{1}{4}(-(6n+13)G_{-n-1} - (6n+19)G_{-n-2} - (2n+7)G_{-n-3} + 7)$.

**(b):** $\sum_{k=1}^{n} kG_{-2k} = \frac{1}{2}(-n(n-1)G_{-2n+1} + (n+1)(n-2)G_{-2n} + n(n-3)G_{-2n-1} + 2)$.

**(c):** $\sum_{k=1}^{n} kG_{-2k+1} = \frac{1}{2}((n+1)(n-2)G_{-2n+1} - n(n+1)G_{-2n} - n(n-1)G_{-2n-1})$.

Taking $W_n = B_n$ with $B_0 = 3, B_1 = 0, B_2 = 4$ in the last theorem, we have the following corollary which presents sum formulas of third order Lucas-Pell numbers.

COROLLARY 3.6. *For $n \geq 1$, third order Lucas-Pell numbers have the following properties:*

**(a):** $\sum_{k=1}^{n} kB_{-k} = \frac{1}{4}(-(6n+13)B_{-n-1} - (6n+19)B_{-n-2} - (2n+7)B_{-n-3} + 15)$.

**(b):** $\sum_{k=1}^{n} kB_{-2k} = \frac{1}{2}(-n(n-1)B_{-2n+1} + (n+1)(n-2)B_{-2n} + n(n-3)B_{-2n-1} + 6)$.

**(c):** $\sum_{k=1}^{n} kB_{-2k+1} = \frac{1}{2}((n+1)(n-2)B_{-2n+1} - n(n+1)B_{-2n} - n(n-1)B_{-2n-1})$.

**3.3. The Case $x = -1$.** We now consider the case $x = -1$ in Theorem 3.1. Taking $r = s = t = 1$ in Theorem 3.1, we obtain the following proposition.

PROPOSITION 3.7. *If $r = s = t = 1$ then for $n \geq 1$ we have the following formulas:*

**(a):** $\sum_{k=1}^{n} k(-1)^k W_{-k} = \frac{1}{2}((-1)^n (nW_{-n-1} + W_{-n-2} + (n+1)W_{-n-3}) + W_2 - W_1 - 2W_0)$.

**(b):** $\sum_{k=1}^{n} k(-1)^k W_{-2k} = \frac{1}{4}((-1)^n (-W_{-2n+1} + 2(n+1)W_{-2n} - (2n-1)W_{-2n-1}) - W_2 + 2W_1 - W_0)$.

**(c):** $\sum_{k=1}^{n} k(-1)^k W_{-2k+1} = \frac{1}{4}((-1)^n ((2n+1)W_{-2n+1} - 2nW_{-2n} - W_{-2n-1}) + W_2 - 2W_1 - W_0)$.

From the above proposition, we have the following corollary which gives sum formulas of Tribonacci numbers (take $W_n = T_n$ with $T_0 = 0, T_1 = 1, T_2 = 1$).

COROLLARY 3.8. *For $n \geq 1$, Tribonacci numbers have the following properties:*

**(a):** $\sum_{k=1}^{n} k(-1)^k T_{-k} = \frac{1}{2}(-1)^n (nT_{-n-1} + T_{-n-2} + (n+1)T_{-n-3})$.

**(b):** $\sum_{k=1}^{n} k(-1)^k T_{-2k} = \frac{1}{4}((-1)^n (-T_{-2n+1} + 2(n+1)T_{-2n} - (2n-1)T_{-2n-1}) + 1)$.

**(c):** $\sum_{k=1}^{n} k(-1)^k T_{-2k+1} = \frac{1}{4}((-1)^n ((2n+1)T_{-2n+1} - 2nT_{-2n} - T_{-2n-1}) - 1)$.

Taking $W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3$ in the last proposition, we have the following corollary which presents sum formulas of Tribonacci-Lucas numbers.

Corollary 3.9. *For $n \geq 1$, Tribonacci-Lucas numbers have the following properties:*

- **(a):** $\sum_{k=1}^{n} k(-1)^k K_{-k} = \frac{1}{2}((-1)^n \left(nK_{-n-1} + K_{-n-2} + (n+1) K_{-n-3}\right) - 4)$.
- **(b):** $\sum_{k=1}^{n} k(-1)^k K_{-2k} = \frac{1}{4}((-1)^n \left(-K_{-2n+1} + 2(n+1) K_{-2n} - (2n-1) K_{-2n-1}\right) - 4)$.
- **(c):** $\sum_{k=1}^{n} k(-1)^k K_{-2k+1} = \frac{1}{4}((-1)^n \left((2n+1) K_{-2n+1} - 2nK_{-2n} - K_{-2n-1}\right) - 2)$.

From the last proposition, we have the following corollary which gives sum formulas of Tribonacci-Perrin numbers (take $W_n = M_n$ with $M_0 = 3, M_1 = 0, M_2 = 2$).

Corollary 3.10. *For $n \geq 1$, Tribonacci-Perrin numbers have the following properties:*

- **(a):** $\sum_{k=1}^{n} k(-1)^k M_{-k} = \frac{1}{2}((-1)^n \left(nM_{-n-1} + M_{-n-2} + (n+1) M_{-n-3}\right) - 4)$.
- **(b):** $\sum_{k=1}^{n} k(-1)^k M_{-2k} = \frac{1}{4}((-1)^n \left(-M_{-2n+1} + 2(n+1) M_{-2n} - (2n-1) M_{-2n-1}\right) - 5)$.
- **(c):** $\sum_{k=1}^{n} k(-1)^k M_{-2k+1} = \frac{1}{4}((-1)^n \left((2n+1) M_{-2n+1} - 2nM_{-2n} - M_{-2n-1}\right) - 1)$.

Taking $W_n = U_n$ with $U_0 = 1, U_1 = 1, U_2 = 1$ in the last proposition, we have the following corollary which presents sum formulas of modified Tribonacci numbers.

Corollary 3.11. *For $n \geq 1$, modified Tribonacci numbers have the following properties:*

- **(a):** $\sum_{k=1}^{n} k(-1)^k U_{-k} = \frac{1}{2}((-1)^n \left(nU_{-n-1} + U_{-n-2} + (n+1) U_{-n-3}\right) - 2)$.
- **(b):** $\sum_{k=1}^{n} k(-1)^k U_{-2k} = \frac{1}{4} (-1)^n \left(-U_{-2n+1} + 2(n+1) U_{-2n} - (2n-1) U_{-2n-1}\right)$.
- **(c):** $\sum_{k=1}^{n} k(-1)^k U_{-2k+1} = \frac{1}{4}((-1)^n \left((2n+1) U_{-2n+1} - 2nU_{-2n} - U_{-2n-1}\right) - 2)$.

From the last proposition, we have the following corollary which gives sum formulas of modified Tribonacci-Lucas numbers (take $W_n = G_n$ with $G_0 = 4, G_1 = 4, G_2 = 10$).

Corollary 3.12. *For $n \geq 1$, modified Tribonacci-Lucas numbers have the following properties:*

- **(a):** $\sum_{k=1}^{n} k(-1)^k G_{-k} = \frac{1}{2}((-1)^n \left(nG_{-n-1} + G_{-n-2} + (n+1) G_{-n-3}\right) - 2)$.
- **(b):** $\sum_{k=1}^{n} k(-1)^k G_{-2k} = \frac{1}{4}((-1)^n \left(-G_{-2n+1} + 2(n+1) G_{-2n} - (2n-1) G_{-2n-1}\right) - 6)$.
- **(c):** $\sum_{k=1}^{n} k(-1)^k G_{-2k+1} = \frac{1}{4}((-1)^n \left((2n+1) G_{-2n+1} - 2nG_{-2n} - G_{-2n-1}\right) - 2)$.

Taking $W_n = H_n$ with $H_0 = 4, H_1 = 2, H_2 = 0$ in the last proposition, we have the following corollary which presents sum formulas of adjusted Tribonacci-Lucas numbers.

Corollary 3.13. *For $n \geq 1$, adjusted Tribonacci-Lucas numbers have the following properties:*

- **(a):** $\sum_{k=1}^{n} k(-1)^k H_{-k} = \frac{1}{2}((-1)^n \left(nH_{-n-1} + H_{-n-2} + (n+1) H_{-n-3}\right) - 10)$.
- **(b):** $\sum_{k=1}^{n} k(-1)^k H_{-2k} = \frac{1}{4} (-1)^n \left(-H_{-2n+1} + 2(n+1) H_{-2n} - (2n-1) H_{-2n-1}\right)$.
- **(c):** $\sum_{k=1}^{n} k(-1)^k H_{-2k+1} = \frac{1}{4}((-1)^n \left((2n+1) H_{-2n+1} - 2nH_{-2n} - H_{-2n-1}\right) - 8)$.

Taking $r = 2, s = 1, t = 1$ in Theorem 3.1, we obtain the following proposition.

PROPOSITION 3.14. *If $r = 2, s = 1, t = 1$ then for $n \geq 1$ we have the following formulas:*

**(a):** $\sum_{k=1}^{n} k(-1)^k W_{-k} = \frac{1}{3}((-1)^n ((2n+1) W_{-n-1} + W_{-n-2} + (n+1) W_{-n-3}) + W_2 - 2W_1 - 2W_0)$.

**(b):** $\sum_{k=1}^{n} k(-1)^k W_{-2k} = \frac{1}{25}((-1)^n (-(5n+3) W_{-2n+1} + (20n+7) W_{-2n} - (10n-9) W_{-2n-1}) - 9W_2 + 21W_1 + 2W_0)$.

**(c):** $\sum_{k=1}^{n} k(-1)^k W_{-2k+1} = \frac{1}{25}((-1)^n ((10n+1) W_{-2n+1} - 3(5n-2) W_{-2n} - (5n+3) W_{-2n-1}) + 3W_2 - 7W_1 - 9W_0)$.

From the last proposition, we have the following corollary which gives sum formulas of Third-order Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 1$).

COROLLARY 3.15. *For $n \geq 1$, third-order Pell numbers have the following properties:*

**(a):** $\sum_{k=1}^{n} k(-1)^k P_{-k} = \frac{1}{3}(-1)^n ((2n+1) P_{-n-1} + P_{-n-2} + (n+1) P_{-n-3})$.

**(b):** $\sum_{k=1}^{n} k(-1)^k P_{-2k} = \frac{1}{25}((-1)^n (-(5n+3) P_{-2n+1} + (20n+7) P_{-2n} - (10n-9) P_{-2n-1}) + 3)$.

**(c):** $\sum_{k=1}^{n} k(-1)^k P_{-2k+1} = \frac{1}{25}((-1)^n ((10n+1) P_{-2n+1} - 3(5n-2) P_{-2n} - (5n+3) P_{-2n-1}) - 1)$.

Taking $W_n = Q_n$ with $Q_0 = 3, Q_1 = 2, Q_2 = 6$ in the last proposition, we have the following corollary which presents sum formulas of third-order Pell-Lucas numbers.

COROLLARY 3.16. *For $n \geq 1$, third-order Pell-Lucas numbers have the following properties:*

**(a):** $\sum_{k=1}^{n} k(-1)^k Q_{-k} = \frac{1}{3}((-1)^n ((2n+1) Q_{-n-1} + Q_{-n-2} + (n+1) Q_{-n-3}) - 4)$.

**(b):** $\sum_{k=1}^{n} k(-1)^k Q_{-2k} = \frac{1}{25}((-1)^n (-(5n+3) Q_{-2n+1} + (20n+7) Q_{-2n} - (10n-9) Q_{-2n-1}) - 6)$.

**(c):** $\sum_{k=1}^{n} k(-1)^k Q_{-2k+1} = \frac{1}{25}((-1)^n ((10n+1) Q_{-2n+1} - 3(5n-2) Q_{-2n} - (5n+3) Q_{-2n-1}) - 23)$.

From the last proposition, we have the following corollary which gives sum formulas of third-order modified Pell numbers (take $W_n = E_n$ with $E_0 = 0, E_1 = 1, E_2 = 1$).

COROLLARY 3.17. *For $n \geq 1$, third-order modified Pell numbers have the following properties:*

**(a):** $\sum_{k=1}^{n} k(-1)^k E_{-k} = \frac{1}{3}((-1)^n ((2n+1) E_{-n-1} + E_{-n-2} + (n+1) E_{-n-3}) - 1)$.

**(b):** $\sum_{k=1}^{n} k(-1)^k E_{-2k} = \frac{1}{25}((-1)^n (-(5n+3) E_{-2n+1} + (20n+7) E_{-2n} - (10n-9) E_{-2n-1}) + 12)$.

**(c):** $\sum_{k=1}^{n} k(-1)^k E_{-2k+1} = \frac{1}{25}((-1)^n ((10n+1) E_{-2n+1} - 3(5n-2) E_{-2n} - (5n+3) E_{-2n-1}) - 4)$.

Taking $W_n = R_n$ with $R_0 = 3, R_1 = 0, R_2 = 2$ in the last proposition, we have the following corollary which presents sum formulas of third-order Pell-Perrin numbers.

COROLLARY 3.18. *For $n \geq 1$, third-order Pell-Perrin numbers have the following properties:*

**(a):** $\sum_{k=1}^{n} k(-1)^k R_{-k} = \frac{1}{3}((-1)^n ((2n+1) R_{-n-1} + R_{-n-2} + (n+1) R_{-n-3}) - 4)$.

**(b):** $\sum_{k=1}^{n} k(-1)^k R_{-2k} = \frac{1}{25}((-1)^n (-(5n+3) R_{-2n+1} + (20n+7) R_{-2n} - (10n-9) R_{-2n-1}) - 12)$.

**(c):** $\sum_{k=1}^{n} k(-1)^k R_{-2k+1} = \frac{1}{25}((-1)^n ((10n+1) R_{-2n+1} - 3(5n-2) R_{-2n} - (5n+3) R_{-2n-1}) - 21)$.

Taking $r = 0, s = 1, t = 1$ in Theorem 3.1, we obtain the following proposition.

Proposition 3.19. *If $r=0,s=1,t=1$ then for $n\geq 1$ we have the following formulas:*

**(a):** $\sum_{k=1}^{n}k(-1)^{k}W_{-k}=(-1)^{n}\left(-W_{-n-1}+W_{-n-2}+(n+1)W_{-n-3}\right)+W_{2}-2W_{0}$.

**(b):** $\sum_{k=1}^{n}k(-1)^{k}W_{-2k}=\frac{1}{25}((-1)^{n}\left((5n-3)W_{-2n+1}+(10n+9)W_{-2n}-(10n-1)W_{-2n-1}\right)-W_{2}+3W_{1}-8W_{0})$.

**(c):** $\sum_{k=1}^{n}k(-1)^{k}W_{-2k+1}=\frac{1}{25}((-1)^{n}\left((10n+9)W_{-2n+1}-(5n+2)W_{-2n}+(5n-3)W_{-2n-1}\right)+3W_{2}-9W_{1}-W_{0})$.

From the last proposition, we have the following corollary which gives sum formulas of Padovan numbers (take $W_n=P_n$ with $P_0=1,P_1=1,P_2=1$).

Corollary 3.20. *For $n\geq 1$, Padovan numbers have the following properties:*

**(a):** $\sum_{k=1}^{n}k(-1)^{k}P_{-k}=(-1)^{n}\left(-P_{-n-1}+P_{-n-2}+(n+1)P_{-n-3}\right)-1$.

**(b):** $\sum_{k=1}^{n}k(-1)^{k}P_{-2k}=\frac{1}{25}((-1)^{n}\left((5n-3)P_{-2n+1}+(10n+9)P_{-2n}-(10n-1)P_{-2n-1}\right)-6)$.

**(c):** $\sum_{k=1}^{n}k(-1)^{k}P_{-2k+1}=\frac{1}{25}((-1)^{n}\left((10n+9)P_{-2n+1}-(5n+2)P_{-2n}+(5n-3)P_{-2n-1}\right)-7$.

Taking $W_n=E_n$ with $E_0=3,E_1=0,E_2=2$ in the last proposition, we have the following corollary which presents sum formulas of Perrin numbers. :

Corollary 3.21. *For $n\geq 1$, Perrin numbers have the following properties:*

**(a):** $\sum_{k=1}^{n}k(-1)^{k}E_{-k}=(-1)^{n}\left(-E_{-n-1}+E_{-n-2}+(n+1)E_{-n-3}\right)-4$.

**(b):** $\sum_{k=1}^{n}k(-1)^{k}E_{-2k}=\frac{1}{25}((-1)^{n}\left((5n-3)E_{-2n+1}+(10n+9)E_{-2n}-(10n-1)E_{-2n-1}\right)-26)$.

**(c):** $\sum_{k=1}^{n}k(-1)^{k}E_{-2k+1}=\frac{1}{25}((-1)^{n}\left((10n+9)E_{-2n+1}-(5n+2)E_{-2n}+(5n-3)E_{-2n-1}\right)+3)$.

From the last proposition, we have the following corollary which gives sum formulas of Padovan-Perrin numbers (take $W_n=S_n$ with $S_0=0,S_1=0,S_2=1$).

Corollary 3.22. *For $n\geq 1$, Padovan-Perrin numbers have the following properties:*

**(a):** $\sum_{k=1}^{n}k(-1)^{k}S_{-k}=(-1)^{n}\left(-S_{-n-1}+S_{-n-2}+(n+1)S_{-n-3}\right)+1$.

**(b):** $\sum_{k=1}^{n}k(-1)^{k}S_{-2k}=\frac{1}{25}((-1)^{n}\left((5n-3)S_{-2n+1}+(10n+9)S_{-2n}-(10n-1)S_{-2n-1}\right)-1)$.

**(c):** $\sum_{k=1}^{n}k(-1)^{k}S_{-2k+1}=\frac{1}{25}((-1)^{n}\left((10n+9)S_{-2n+1}-(5n+2)S_{-2n}+(5n-3)S_{-2n-1}\right)+3)$.

Taking $W_n=A_n$ with $A_0=3,A_1=1,A_2=3$ in the last proposition, we have the following corollary which presents sum formulas of modified Padovan numbers.

Corollary 3.23. *For $n\geq 1$, modified Padovan numbers have the following properties:*

**(a):** $\sum_{k=1}^{n}k(-1)^{k}A_{-k}=(-1)^{n}\left(-A_{-n-1}+A_{-n-2}+(n+1)A_{-n-3}\right)-3$.

**(b):** $\sum_{k=1}^{n}k(-1)^{k}A_{-2k}=\frac{1}{25}((-1)^{n}\left((5n-3)A_{-2n+1}+(10n+9)A_{-2n}-(10n-1)A_{-2n-1}\right)-24)$.

**(c):** $\sum_{k=1}^{n}k(-1)^{k}A_{-2k+1}=\frac{1}{25}((-1)^{n}\left((10n+9)A_{-2n+1}-(5n+2)A_{-2n}+(5n-3)A_{-2n-1}\right)-3)$.

Observe that setting $x=-1$, $r=0,s=2,t=1$ (i.e. for the generalized Pell-Padovan case) in Theorem 3.1, (a) makes the right hand side of the sum formulas to be an indeterminate form. Application of L'Hospital rule (using twice) however provides the evaluation of the sum formulas. If $x=-1$, $r=0,s=2,t=1$ then we have the following theorem (in fact taking $r=0,s=2,t=1$ in Theorem 3.1 (a) and then using L'Hospital rule twice for $x=-1$ we obtain the following theorem).

THEOREM 3.24. *If $r=0,s=2,t=1$ then for $n\geq 1$ we have the following formulas:*

**(a):** $\sum_{k=1}^{n} k(-1)^k W_{-k} = \frac{1}{2}((-1)^n\left(-(n^2-n-32)W_{-n-1}+(n^2+n-32)W_{-n-2}+(n+5)\left(n-6\right)W_{-n-3}\right)+24W_2-28W_1-18W_0)$

**(b):** $\sum_{k=1}^{n} k(-1)^k W_{-2k} = \frac{1}{20}((-1)^n\left(\left(2n-1\right)W_{-2n+1}+\left(6n+5\right)W_{-2n}-\left(6n-1\right)W_{-2n-1}\right)-W_2+W_1-3W_0)$

**(c):** $\sum_{k=1}^{n} k(-1)^k W_{-2k+1} = \frac{1}{20}((-1)^n\left(\left(6n+5\right)W_{-2n+1}-\left(2n+1\right)W_{-2n}+\left(2n-1\right)W_{-2n-1}\right)+W_2-5W_1-W_0)$

Proof

**(a):** We use Theorem 3.1 (a). If we set $r=0,s=2,t=1$ in Theorem 3.1 (a) then we have

$$\sum_{k=1}^{n} k(-1)^k x^k W_{-k} = \frac{h_2(x)}{(-x^3+2x+1)^2}$$

where

$h_2(x) = -x^{n+1}(n\left(2x+1\right)\left(-x^3+2x+1\right)+4x+4x^2+2x^3+2x^4+1)W_{-n-1} - x^{n+2}(n\left(2x+1\right)\left(-x^3+2x+1\right)+8x+8x^2+x^3+2)W_{-n-2} + x(3x^2-2)W_0 - x^{n+3}(n(-x^3+2x+1)+4x+3)W_{-n-3} + x(2x^3+1)W_2 + x(x^4+2x^2+2x)W_1.$

For $x=-1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get

$$\begin{aligned}
\sum_{k=1}^{n} k(-1)^k W_{-k} &= \left.\frac{\frac{d}{dx}(h_2(x))}{\frac{d}{dx}((-x^3+2x+1)^2)}\right|_{x=-1} \\
&= \left.\frac{\frac{d^2}{dx^2}(h_2(x))}{\frac{d^2}{dx^2}((-x^3+2x+1)^2)}\right|_{x=-1} \\
&= \frac{1}{2}((-1)^n\left(-(n^2-n-32)W_{-n-1}+(n^2+n-32)W_{-n-2}\right. \\
&\quad \left. +\left(n+5\right)\left(n-6\right)W_{-n-3}\right)+24W_2-28W_1-18W_0).
\end{aligned}$$

**(b):** Taking $x=-1$,$r=0,s=2,t=1$ in Theorem 3.1 (b) we obtain (b).

**(c):** Taking $x=-1$,$r=0,s=2,t=1$ in Theorem 3.1 (c) we obtain (c).□

From the last theorem, we have the following corollary which gives sum formulas of Pell-Padovan numbers (take $W_n=R_n$ with $R_0=1,R_1=1,R_2=1$).

COROLLARY 3.25. *For $n\geq 1$, Pell-Padovan numbers have the following properties:*

**(a):** $\sum_{k=1}^{n} k(-1)^k R_{-k} = \frac{1}{2}((-1)^n (-(n^2-n-32)R_{-n-1}+(n^2+n-32)R_{-n-2}+(n+5)(n-6)R_{-n-3})-22)$.

**(b):** $\sum_{k=1}^{n} k(-1)^k R_{-2k} = \frac{1}{20}((-1)^n ((2n-1)R_{-2n+1}+(6n+5)R_{-2n}-(6n-1)R_{-2n-1})-3)$.

**(c):** $\sum_{k=1}^{n} k(-1)^k R_{-2k+1} = \frac{1}{20}((-1)^n ((6n+5)R_{-2n+1}-(2n+1)R_{-2n}+(2n-1)R_{-2n-1})-5)$.

Taking $R_n = C_n$ with $C_0 = 3, C_1 = 0, C_2 = 2$ in the last theorem, we have the following corollary which presents sum formulas of Pell-Perrin numbers.

Corollary 3.26. *For $n \geq 1$, Pell-Perrin numbers have the following properties:*

**(a):** $\sum_{k=1}^{n} k(-1)^k C_{-k} = \frac{1}{2}((-1)^n (-(n^2-n-32)C_{-n-1}+(n^2+n-32)C_{-n-2}+(n+5)(n-6)C_{-n-3})-6)$.

**(b):** $\sum_{k=1}^{n} k(-1)^k C_{-2k} = \frac{1}{20}((-1)^n ((2n-1)C_{-2n+1}+(6n+5)C_{-2n}-(6n-1)C_{-2n-1})-11)$.

**(c):** $\sum_{k=1}^{n} k(-1)^k C_{-2k+1} = \frac{1}{20}((-1)^n ((6n+5)C_{-2n+1}-(2n+1)C_{-2n}+(2n-1)C_{-2n-1})-1)$.

From the last theorem, we have the following corollary which gives sum formulas of third order Fibonacci-Pell numbers (take $C_n = G_n$ with $G_0 = 1, G_1 = 0, G_2 = 2$).

Corollary 3.27. *For $n \geq 1$, third order Fibonacci-Pell numbers have the following properties:*

**(a):** $\sum_{k=1}^{n} k(-1)^k G_{-k} = \frac{1}{2}((-1)^n (-(n^2-n-32)G_{-n-1}+(n^2+n-32)G_{-n-2}+(n+5)(n-6)G_{-n-3})+30)$.

**(b):** $\sum_{k=1}^{n} k(-1)^k G_{-2k} = \frac{1}{20}((-1)^n ((2n-1)G_{-2n+1}+(6n+5)G_{-2n}-(6n-1)G_{-2n-1})-5)$.

**(c):** $\sum_{k=1}^{n} k(-1)^k G_{-2k+1} = \frac{1}{20}((-1)^n ((6n+5)G_{-2n+1}-(2n+1)G_{-2n}+(2n-1)G_{-2n-1})+1)$.

Taking $G_n = B_n$ with $B_0 = 3, B_1 = 0, B_2 = 4$ in the last theorem, we have the following corollary which presents sum formulas of third order Lucas-Pell numbers.

Corollary 3.28. *For $n \geq 1$, third order Lucas-Pell numbers have the following properties:*

**(a):** $\sum_{k=1}^{n} k(-1)^k B_{-k} = \frac{1}{2}((-1)^n (-(n^2-n-32)B_{-n-1}+(n^2+n-32)B_{-n-2}+(n+5)(n-6)B_{-n-3})+42)$.

**(b):** $\sum_{k=1}^{n} k(-1)^k B_{-2k} = \frac{1}{20}((-1)^n ((2n-1)B_{-2n+1}+(6n+5)B_{-2n}-(6n-1)B_{-2n-1})-13)$.

**(c):** $\sum_{k=1}^{n} k(-1)^k B_{-2k+1} = \frac{1}{20}((-1)^n ((6n+5)B_{-2n+1}-(2n+1)B_{-2n}+(2n-1)B_{-2n-1})+1)$.

Taking $r = 0, s = 1, t = 2$ in Theorem 3.1, we obtain the following proposition.

Proposition 3.29. *If $r = 0, s = 1, t = 2$ then for $n \geq 1$ we have the following formulas:*

**(a):** $\sum_{k=1}^{n} k(-1)^k W_{-k} = \frac{1}{2}((-1)^n ((n-1)W_{-n-1}-nW_{-n-2}+2(n+2)W_{-n-3})+W_1-2W_0)$.

**(b):** $\sum_{k=1}^{n} k(-1)^k W_{-2k} = \frac{1}{8}((-1)^n (2nW_{-2n+1}+(2n+3)W_{-2n}-2(2n+1)W_{-2n-1})+W_2-4W_0)$.

**(c):** $\sum_{k=1}^{n} k(-1)^k W_{-2k+1} = \frac{1}{8}((-1)^n ((2n+3)W_{-2n+1}-2(n+1)W_{-2n}+4nW_{-2n-1})-3W_1+2W_0)$.

From the last proposition, we have the following corollary which gives sum formulas of Jacobsthal-Padovan numbers (take $W_n = Q_n$ with $Q_0 = 1, Q_1 = 1, Q_2 = 1$).

COROLLARY 3.30. *For $n \geq 1$, Jacobsthal-Padovan numbers have the following properties:*

- **(a):** $\sum_{k=1}^{n} k(-1)^k Q_{-k} = \frac{1}{2}((-1)^n((n-1)Q_{-n-1} - nQ_{-n-2} + 2(n+2)Q_{-n-3}) - 1)$.
- **(b):** $\sum_{k=1}^{n} k(-1)^k Q_{-2k} = \frac{1}{8}((-1)^n(2nQ_{-2n+1} + (2n+3)Q_{-2n} - 2(2n+1)Q_{-2n-1}) - 3)$.
- **(c):** $\sum_{k=1}^{n} k(-1)^k Q_{-2k+1} = \frac{1}{8}((-1)^n((2n+3)Q_{-2n+1} - 2(n+1)Q_{-2n} + 4nQ_{-2n-1}) - 1)$.

Taking $Q_n = L_n$ with $L_0 = 3, L_1 = 0, L_2 = 2$ in the last proposition, we have the following corollary which presents sum formulas of Jacobsthal-Perrin numbers.

COROLLARY 3.31. *For $n \geq 1$, Jacobsthal-Perrin numbers have the following properties:*

- **(a):** $\sum_{k=1}^{n} k(-1)^k L_{-k} = \frac{1}{2}((-1)^n((n-1)L_{-n-1} - nL_{-n-2} + 2(n+2)L_{-n-3}) - 6)$.
- **(b):** $\sum_{k=1}^{n} k(-1)^k L_{-2k} = \frac{1}{8}((-1)^n(2nL_{-2n+1} + (2n+3)L_{-2n} - 2(2n+1)L_{-2n-1}) - 10)$.
- **(c):** $\sum_{k=1}^{n} k(-1)^k L_{-2k+1} = \frac{1}{8}((-1)^n((2n+3)L_{-2n+1} - 2(n+1)L_{-2n} + 4nL_{-2n-1}) + 6)$.

From the last proposition, we have the following corollary which gives sum formulas of adjusted Jacobsthal-Padovan numbers (take $L_n = K_n$ with $K_0 = 0, K_1 = 1, K_2 = 0$).

COROLLARY 3.32. *For $n \geq 1$, adjusted Jacobsthal-Padovan numbers have the following properties:*

- **(a):** $\sum_{k=1}^{n} k(-1)^k K_{-k} = \frac{1}{2}((-1)^n((n-1)K_{-n-1} - nK_{-n-2} + 2(n+2)K_{-n-3}) + 1)$.
- **(b):** $\sum_{k=1}^{n} k(-1)^k K_{-2k} = \frac{1}{8}(-1)^n(2nK_{-2n+1} + (2n+3)K_{-2n} - 2(2n+1)K_{-2n-1})$.
- **(c):** $\sum_{k=1}^{n} k(-1)^k K_{-2k+1} = \frac{1}{8}((-1)^n((2n+3)K_{-2n+1} - 2(n+1)K_{-2n} + 4nK_{-2n-1}) - 3)$.

Taking $K_n = M_n$ with $M_0 = 3, M_1 = 1, M_2 = 3$ in the last proposition, we have the following corollary which presents sum formulas of modified Jacobsthal-Padovan numbers.

COROLLARY 3.33. *For $n \geq 1$, modified Jacobsthal-Padovan numbers have the following properties:*

- **(a):** $\sum_{k=1}^{n} k(-1)^k M_{-k} = \frac{1}{2}((-1)^n((n-1)M_{-n-1} - nM_{-n-2} + 2(n+2)M_{-n-3}) - 5)$.
- **(b):** $\sum_{k=1}^{n} k(-1)^k M_{-2k} = \frac{1}{8}((-1)^n(2nM_{-2n+1} + (2n+3)M_{-2n} - 2(2n+1)M_{-2n-1}) - 9)$.
- **(c):** $\sum_{k=1}^{n} k(-1)^k M_{-2k+1} = \frac{1}{8}((-1)^n((2n+3)M_{-2n+1} - 2(n+1)M_{-2n} + 4nM_{-2n-1}) + 3)$.

Taking $r = 1, s = 0, t = 1$ in Theorem 3.1, we obtain the following proposition.

PROPOSITION 3.34. *If $r = 1, s = 0, t = 1$ then for $n \geq 1$ we have the following formulas:*

- **(a):** $\sum_{k=1}^{n} k(-1)^k W_{-k} = \frac{1}{9}((-1)^n((6n+2)W_{-n-1} - (3n+1)W_{-n-2} + (3n+4)W_{-n-3}) + 2W_2 - W_1 - 5W_0)$.
- **(b):** $\sum_{k=1}^{n} k(-1)^k W_{-2k} = (-1)^n(-W_{-2n+1} + (n+1)W_{-2n} - (n-1)W_{-2n-1}) - W_2 + 2W_1 - W_0$.
- **(c):** $\sum_{k=1}^{n} k(-1)^k W_{-2k+1} = (-1)^n(nW_{-2n+1} - (n-1)W_{-2n} - W_{-2n-1}) + W_2 - W_1 - W_0$.

From the last proposition, we have the following corollary which gives sum formulas of Narayana numbers (take $W_n = N_n$ with $N_0 = 0, N_1 = 1, N_2 = 1$).

COROLLARY 3.35. *For $n \geq 1$, Narayana numbers have the following properties:*

**(a):** $\sum_{k=1}^{n} k(-1)^k N_{-k} = \frac{1}{9}((-1)^n ((6n+2) N_{-n-1} - (3n+1) N_{-n-2} + (3n+4) N_{-n-3}) + 1)$.

**(b):** $\sum_{k=1}^{n} k(-1)^k N_{-2k} = (-1)^n (-N_{-2n+1} + (n+1) N_{-2n} - (n-1) N_{-2n-1}) + 1$.

**(c):** $\sum_{k=1}^{n} k(-1)^k N_{-2k+1} = (-1)^n (nN_{-2n+1} - (n-1) N_{-2n} - N_{-2n-1})$.

Taking $W_n = U_n$ with $U_0 = 3, U_1 = 1, U_2 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Narayana-Lucas numbers.

Corollary 3.36. *For $n \geq 1$, Narayana-Lucas numbers have the following properties:*

**(a):** $\sum_{k=1}^{n} k(-1)^k U_{-k} = \frac{1}{9}((-1)^n ((6n+2) U_{-n-1} - (3n+1) U_{-n-2} + (3n+4) U_{-n-3}) - 14)$.

**(b):** $\sum_{k=1}^{n} k(-1)^k U_{-2k} = (-1)^n (-U_{-2n+1} + (n+1) U_{-2n} - (n-1) U_{-2n-1}) - 2$.

**(c):** $\sum_{k=1}^{n} k(-1)^k U_{-2k+1} = (-1)^n (nU_{-2n+1} - (n-1) U_{-2n} - U_{-2n-1}) - 3$.

From the last proposition, we have the following corollary which gives sum formulas of Narayana-Perrin numbers (take $W_n = H_n$ with $H_0 = 3, H_1 = 0, H_2 = 2$).

Corollary 3.37. *For $n \geq 1$, Narayana-Perrin numbers have the following properties:*

**(a):** $\sum_{k=1}^{n} k(-1)^k H_{-k} = \frac{1}{9}((-1)^n ((6n+2) H_{-n-1} - (3n+1) H_{-n-2} + (3n+4) H_{-n-3}) - 11)$.

**(b):** $\sum_{k=1}^{n} k(-1)^k H_{-2k} = (-1)^n (-H_{-2n+1} + (n+1) H_{-2n} - (n-1) H_{-2n-1}) - 5$.

**(c):** $\sum_{k=1}^{n} k(-1)^k H_{-2k+1} = (-1)^n (nH_{-2n+1} - (n-1) H_{-2n} - H_{-2n-1}) - 1$.

Taking $r = 1, s = 1, t = 2$ in Theorem 3.1, we obtain the following proposition.

Proposition 3.38. *If $r = 1, s = 1, t = 2$ then for $n \geq 1$ we have the following formulas:*

**(a):** $\sum_{k=1}^{n} k(-1)^k W_{-k} = \frac{1}{9}((-1)^n ((6n+1) W_{-n-1} - (3n-1) W_{-n-2} + 2(3n+5) W_{-n-3}) + W_2 + W_1 - 8W_0)$.

**(b):** $\sum_{k=1}^{n} k(-1)^k W_{-2k} = \frac{1}{25}((-1)^n ((5n-6) W_{-2n+1} + (5n+19) W_{-2n} - 2(10n+3) W_{-2n-1}) + 3W_2 + 3W_1 - 22W_0)$.

**(c):** $\sum_{k=1}^{n} k(-1)^k W_{-2k+1} = \frac{1}{25}((-1)^n ((10n+13) W_{-2n+1} - 3(5n+4) W_{-2n} + 2(5n-6) W_{-2n-1}) + 6W_2 - 19W_1 + 6W_0)$.

From the last proposition, we have the following corollary which gives sum formulas of third order Jacobsthal numbers (take $W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1$).

Corollary 3.39. *For $n \geq 1$, third order Jacobsthal numbers have the following properties:*

**(a):** $\sum_{k=1}^{n} k(-1)^k J_{-k} = \frac{1}{9}((-1)^n ((6n+1) J_{-n-1} - (3n-1) J_{-n-2} + 2(3n+5) J_{-n-3}) + 2)$.

**(b):** $\sum_{k=1}^{n} k(-1)^k J_{-2k} = \frac{1}{25}((-1)^n ((5n-6) J_{-2n+1} + (5n+19) J_{-2n} - 2(10n+3) J_{-2n-1}) + 6)$.

**(c):** $\sum_{k=1}^{n} k(-1)^k J_{-2k+1} = \frac{1}{25}((-1)^n ((10n+13) J_{-2n+1} - 3(5n+4) J_{-2n} + 2(5n-6) J_{-2n-1}) - 13)$.

Taking $J_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5$ in the last proposition, we have the following corollary which presents sum formulas of third order Jacobsthal-Lucas numbers.

Corollary 3.40. *For $n \geq 1$, third order Jacobsthal-Lucas numbers have the following properties:*

**(a):** $\sum_{k=1}^{n} k(-1)^k j_{-k} = \frac{1}{9}((-1)^n ((6n+1) j_{-n-1} - (3n-1) j_{-n-2} + 2(3n+5) j_{-n-3}) - 10)$.

**(b):** $\sum_{k=1}^{n} k(-1)^k j_{-2k} = \frac{1}{25}((-1)^n ((5n-6) j_{-2n+1} + (5n+19) j_{-2n} - 2(10n+3) j_{-2n-1}) - 26)$.

**(c):** $\sum_{k=1}^{n} k(-1)^k j_{-2k+1} = \frac{1}{25}((-1)^n ((10n+13) j_{-2n+1} - 3(5n+4) j_{-2n} + 2(5n-6) j_{-2n-1}) + 23)$.

From the last proposition, we have the following corollary which gives sum formulas of modified third order Jacobsthal-Lucas numbers (take $j_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3$).

Corollary 3.41. *For $n \geq 1$, modified third order Jacobsthal-Lucas numbers have the following properties:*

**(a):** $\sum_{k=1}^{n} k(-1)^k K_{-k} = \frac{1}{9}((-1)^n ((6n+1) K_{-n-1} - (3n-1) K_{-n-2} + 2(3n+5) K_{-n-3}) - 20)$.

**(b):** $\sum_{k=1}^{n} k(-1)^k K_{-2k} = \frac{1}{25}((-1)^n ((5n-6) K_{-2n+1} + (5n+19) K_{-2n} - 2(10n+3) K_{-2n-1}) - 54)$.

**(c):** $\sum_{k=1}^{n} k(-1)^k K_{-2k+1} = \frac{1}{25}((-1)^n ((10n+13) K_{-2n+1} - 3(5n+4) K_{-2n} + 2(5n-6) K_{-2n-1}) + 17)$.

Taking $K_n = Q_n$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2$ in the last proposition, we have the following corollary which presents sum formulas of third order Jacobsthal-Perrin numbers.

Corollary 3.42. *For $n \geq 1$, third order Jacobsthal-Perrin numbers have the following properties:*

**(a):** $\sum_{k=1}^{n} k(-1)^k Q_{-k} = \frac{1}{9}((-1)^n ((6n+1) Q_{-n-1} - (3n-1) Q_{-n-2} + 2(3n+5) Q_{-n-3}) - 22)$.

**(b):** $\sum_{k=1}^{n} k(-1)^k Q_{-2k} = \frac{1}{25}((-1)^n ((5n-6) Q_{-2n+1} + (5n+19) Q_{-2n} - 2(10n+3) Q_{-2n-1}) - 60)$.

**(c):** $\sum_{k=1}^{n} k(-1)^k Q_{-2k+1} = \frac{1}{25}((-1)^n ((10n+13) Q_{-2n+1} - 3(5n+4) Q_{-2n} + 2(5n-6) Q_{-2n-1}) + 30)$.

Taking $r = 2, s = 3, t = 5$ in Theorem 3.1, we obtain the following proposition.

Proposition 3.43. *If $r = 2, s = 3, t = 5$ then for $n \geq 1$ we have the following formulas:*

**(a):** $\sum_{k=1}^{n} k(-1)^k W_{-k} = \frac{1}{25}((-1)^n ((20n+9) W_{-n-1} - (10n-3) W_{-n-2} + 5(5n+11) W_{-n-3}) - W_2 + 8W_1 - 20W_0)$.

**(b):** $\sum_{k=1}^{n} k(-1)^k W_{-2k} = \frac{1}{625}((-1)^n ((75n-11) W_{-2n+1} - (50n-199) W_{-2n} - 5(100n+77) W_{-2n-1}) + 77W_2 - 143W_1 - 430W_0)$.

**(c):** $\sum_{k=1}^{n} k(-1)^k W_{-2k+1} = \frac{1}{625}((-1)^n ((100n+177) W_{-2n+1} - 11(25n+38) W_{-2n} + 5(75n-11) W_{-2n-1}) + 11W_2 - 199W_1 + 385W_0)$.

From the last proposition, we have the following corollary which gives sum formulas of 3-primes numbers (take $W_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = 2$).

Corollary 3.44. *For $n \geq 1$, 3-primes numbers have the following properties:*

**(a):** $\sum_{k=1}^{n} k(-1)^k G_{-k} = \frac{1}{25}((-1)^n ((20n+9) G_{-n-1} - (10n-3) G_{-n-2} + 5(5n+11) G_{-n-3}) + 6)$.

**(b):** $\sum_{k=1}^{n} k(-1)^k G_{-2k} = \frac{1}{625}((-1)^n ((75n-11) G_{-2n+1} - (50n-199) G_{-2n} - 5(100n+77) G_{-2n-1}) + 11)$.

**(c):** $\sum_{k=1}^{n} k(-1)^k G_{-2k+1} = \frac{1}{625}((-1)^n ((100n+177) G_{-2n+1} - 11(25n+38) G_{-2n} + 5(75n-11) G_{-2n-1}) - 177)$.

Taking $W_n = H_n$ with $H_0 = 3, H_1 = 2, H_2 = 10$ in the last proposition, we have the following corollary which presents sum formulas of Lucas 3-primes numbers.

Corollary 3.45. *For $n \geq 1$, Lucas 3-primes numbers have the following properties:*

**(a):** $\sum_{k=1}^{n} k(-1)^k H_{-k} = \frac{1}{25}((-1)^n ((20n+9) H_{-n-1} - (10n-3) H_{-n-2} + 5(5n+11) H_{-n-3}) - 54)$.

**(b):** $\sum_{k=1}^{n} k(-1)^k H_{-2k} = \frac{1}{625}((-1)^n ((75n-11) H_{-2n+1} - (50n-199) H_{-2n} - 5(100n+77) H_{-2n-1}) - 806)$.

**(c):** $\sum_{k=1}^{n} k(-1)^k H_{-2k+1} = \frac{1}{625}((-1)^n ((100n+177) H_{-2n+1} - 11(25n+38) H_{-2n} + 5(75n-11) H_{-2n-1}) + 867)$.

From the last proposition, we have the following corollary which gives sum formulas of modified 3-primes numbers (take $W_n = E_n$ with $E_0 = 0, E_1 = 1, E_2 = 1$).

Corollary 3.46. *For $n \geq 1$, modified 3-primes numbers have the following properties:*

**(a):** $\sum_{k=1}^{n} k(-1)^k E_{-k} = \frac{1}{25}((-1)^n ((20n+9) E_{-n-1} - (10n-3) E_{-n-2} + 5(5n+11) E_{-n-3}) + 7)$.

**(b):** $\sum_{k=1}^{n} k(-1)^k E_{-2k} = \frac{1}{625}((-1)^n ((75n-11) E_{-2n+1} - (50n-199) E_{-2n} - 5(100n+77) E_{-2n-1}) - 66)$.

**(c):** $\sum_{k=1}^{n} k(-1)^k E_{-2k+1} = \frac{1}{625}((-1)^n ((100n+177) E_{-2n+1} - 11(25n+38) E_{-2n} + 5(75n-11) E_{-2n-1}) - 188)$.

Taking $r = 5, s = 3, t = 2$ in Theorem 3.1, we obtain the following proposition.

Proposition 3.47. *If $r = 5, s = 3, t = 2$ then for $n \geq 1$ we have the following formulas:*

**(a):** $\sum_{k=1}^{n} k(-1)^k W_{-k} = \frac{1}{5}((-1)^n ((4n+3) W_{-n-1} + (n+3) W_{-n-2} + 2(n+1) W_{-n-3}) + W_2 - 5W_1 - 4W_0)$.

**(b):** $\sum_{k=1}^{n} k(-1)^k W_{-2k} = \frac{1}{625}((-1)^n (-(75n+38) W_{-2n+1} + (475n+199) W_{-2n} - 2(100n-91) W_{-2n-1}) - 91W_2 + 493W_1 + 74W_0)$.

**(c):** $\sum_{k=1}^{n} k(-1)^k W_{-2k+1} = \frac{1}{625}((-1)^n ((100n+9) W_{-2n+1} - 17(25n-4) W_{-2n} - 2(75n+38) W_{-2n-1}) + 38W_2 - 199W_1 - 182W_0)$.

From the last proposition, we have the following corollary which gives sum formulas of reverse 3-primes numbers (take $W_n = N_n$ with $N_0 = 0, N_1 = 1, N_2 = 5$).

Corollary 3.48. *For $n \geq 1$, reverse 3-primes numbers have the following properties:*

**(a):** $\sum_{k=1}^{n} k(-1)^k N_{-k} = \frac{1}{5}(-1)^n ((4n+3) N_{-n-1} + (n+3) N_{-n-2} + 2(n+1) N_{-n-3})$.

**(b):** $\sum_{k=1}^{n} k(-1)^k N_{-2k} = \frac{1}{625}((-1)^n (-(75n+38)N_{-2n+1} + (475n+199)N_{-2n} - 2(100n-91)N_{-2n-1}) + 38)$.

**(c):** $\sum_{k=1}^{n} k(-1)^k N_{-2k+1} = \frac{1}{625}((-1)^n ((100n+9)N_{-2n+1} - 17(25n-4)N_{-2n} - 2(75n+38)N_{-2n-1}) - 9)$.

Taking $N_n = S_n$ with $S_0 = 3, S_1 = 5, S_2 = 31$ in the last proposition, we have the following corollary which presents sum formulas of reverse Lucas 3-primes numbers.

Corollary 3.49. *For $n \geq 1$, reverse Lucas 3-primes numbers have the following properties:*

**(a):** $\sum_{k=1}^{n} k(-1)^k S_{-k} = \frac{1}{5}((-1)^n ((4n+3)S_{-n-1} + (n+3)S_{-n-2} + 2(n+1)S_{-n-3}) - 6)$.

**(b):** $\sum_{k=1}^{n} k(-1)^k S_{-2k} = \frac{1}{625}((-1)^n (-(75n+38)S_{-2n+1} + (475n+199)S_{-2n} - 2(100n-91)S_{-2n-1}) - 134)$.

**(c):** $\sum_{k=1}^{n} k(-1)^k S_{-2k+1} = \frac{1}{625}((-1)^n ((100n+9)S_{-2n+1} - 17(25n-4)S_{-2n} - 2(75n+38)S_{-2n-1}) - 363)$.

From the last proposition, we have the following corollary which gives sum formulas of reverse modified 3-primes numbers (take $S_n = U_n$ with $U_0 = 0, U_1 = 1, U_2 = 4$).

Corollary 3.50. *For $n \geq 1$, reverse modified 3-primes numbers have the following properties:*

**(a):** $\sum_{k=1}^{n} k(-1)^k U_{-k} = \frac{1}{5}((-1)^n ((4n+3)U_{-n-1} + (n+3)U_{-n-2} + 2(n+1)U_{-n-3}) - 1)$.

**(b):** $\sum_{k=1}^{n} k(-1)^k U_{-2k} = \frac{1}{625}((-1)^n (-(75n+38)U_{-2n+1} + (475n+199)U_{-2n} - 2(100n-91)U_{-2n-1}) + 129)$.

**(c):** $\sum_{k=1}^{n} k(-1)^k U_{-2k+1} = \frac{1}{625}((-1)^n ((100n+9)U_{-2n+1} - 17(25n-4)U_{-2n} - 2(75n+38)U_{-2n-1}) - 47)$.

**3.4. The Case $x = i$.** We now consider the complex case $x = i$ in Theorem 3.1. Taking $x = i, r = s = t = 1$ in Theorem 3.1, we obtain the following proposition.

Proposition 3.51. *If $r = s = t = 1$ then for $n \geq 1$ we have the following formulas:*

**(a):** $\sum_{k=1}^{n} k i^k W_{-k} = \frac{1}{4}(i^n((2-2in)W_{-n-1} + ((2-2i)n-2i)W_{-n-2} + (2n+2-2i)W_{-n-3}) - (2+2i)W_2 + (4+2i)W_1 - (2-4i)W_0)$.

**(b):** $\sum_{k=1}^{n} k i^k W_{-2k} = \frac{i}{8}(i^n(i(4n-4)W_{-2n+1} + (4i-4-8in)W_{-2n} + (4-4n+4i)W_{-2n-1}) - (4+4i)W_2 + (4+8i)W_1 + 8W_0)$.

**(c):** $\sum_{k=1}^{n} k i^k W_{-2k+1} = \frac{i}{8}(i^n(-4(in+1)W_{-2n+1} - 4((1-i)n-1)W_{-2n} + 4i(n-1)W_{-2n-1}) + 4iW_2 + 4(1-i)W_1 - 4(1+i)W_0)$.

From the above proposition, we have the following corollary which gives sum formulas of Tribonacci numbers (take $W_n = T_n$ with $T_0 = 0, T_1 = 1, T_2 = 1$).

Corollary 3.52. *For $n \geq 1$, Tribonacci numbers have the following properties:*

**(a):** $\sum_{k=1}^{n} ki^{k}T_{-k}=\frac{1}{4}(i^{n}((2-2in)T_{-n-1}+((2-2i)\,n-2i)\,T_{-n-2}+(2n+2-2i)T_{-n-3})+2)$.

**(b):** $\sum_{k=1}^{n} ki^{k}T_{-2k}=\frac{i}{8}(i^{n}(i\,(4n-4)\,T_{-2n+1}+(4i-4-8in)T_{-2n}+(4-4n+4i)T_{-2n-1})+4i)$.

**(c):** $\sum_{k=1}^{n} ki^{k}T_{-2k+1}=\frac{i}{8}(i^{n}(-4\,(in+1)\,T_{-2n+1}-4\,((1-i)\,n-1)\,T_{-2n}+4i\,(n-1)\,T_{-2n-1})+4)$.

Taking $W_n=K_n$ with $K_0=3,K_1=1,K_2=3$ in the last proposition, we have the following corollary which presents sum formulas of Tribonacci-Lucas numbers.

Corollary 3.53. *For $n\geq 1$, Tribonacci-Lucas numbers have the following properties:*

**(a):** $\sum_{k=1}^{n} ki^{k}K_{-k}=\frac{1}{4}(i^{n}((2-2in)K_{-n-1}+((2-2i)\,n-2i)\,K_{-n-2}+(2n+2-2i)K_{-n-3})-8+8i)$.

**(b):** $\sum_{k=1}^{n} ki^{k}K_{-2k}=\frac{i}{8}(i^{n}(i\,(4n-4)\,K_{-2n+1}+(4i-4-8in)K_{-2n}+(4-4n+4i)K_{-2n-1})+16-4i)$.

**(c):** $\sum_{k=1}^{n} ki^{k}K_{-2k+1}=\frac{i}{8}(i^{n}(-4\,(in+1)\,K_{-2n+1}-4\,((1-i)\,n-1)\,K_{-2n}+4i\,(n-1)\,K_{-2n-1})-8-4i)$.

From the last proposition, we have the following corollary which gives sum formulas of Tribonacci-Perrin numbers (take $W_n=M_n$ with $M_0=3,M_1=0,M_2=2$).

Corollary 3.54. *For $n\geq 1$, Tribonacci-Perrin numbers have the following properties:*

**(a):** $\sum_{k=1}^{n} ki^{k}M_{-k}=\frac{1}{4}(i^{n}((2-2in)M_{-n-1}+((2-2i)\,n-2i)\,M_{-n-2}+(2n+2-2i)M_{-n-3})-10+8i)$.

**(b):** $\sum_{k=1}^{n} ki^{k}M_{-2k}=\frac{i}{8}(i^{n}(i\,(4n-4)\,M_{-2n+1}+(4i-4-8in)M_{-2n}+(4-4n+4i)M_{-2n-1})+16-8i)$.

**(c):** $\sum_{k=1}^{n} ki^{k}M_{-2k+1}=\frac{i}{8}(i^{n}(-4\,(in+1)\,M_{-2n+1}-4\,((1-i)\,n-1)\,M_{-2n}+4i\,(n-1)\,M_{-2n-1})-12-4i)$.

Taking $W_n=U_n$ with $U_0=1,U_1=1,U_2=1$ in the last proposition, we have the following corollary which presents sum formulas of modified Tribonacci numbers.

Corollary 3.55. *For $n\geq 1$, modified Tribonacci numbers have the following properties:*

**(a):** $\sum_{k=1}^{n} ki^{k}U_{-k}=\frac{1}{4}(i^{n}((2-2in)U_{-n-1}+((2-2i)\,n-2i)\,U_{-n-2}+(2n+2-2i)U_{-n-3})+4i)$.

**(b):** $\sum_{k=1}^{n} ki^{k}U_{-2k}=\frac{i}{8}(i^{n}(i\,(4n-4)\,U_{-2n+1}+(4i-4-8in)U_{-2n}+(4-4n+4i)U_{-2n-1})+8+4i)$.

**(c):** $\sum_{k=1}^{n} ki^{k}U_{-2k+1}=\frac{i}{8}(i^{n}(-4\,(in+1)\,U_{-2n+1}-4\,((1-i)\,n-1)\,U_{-2n}+4i\,(n-1)\,U_{-2n-1})-4i)$.

From the last proposition, we have the following corollary which gives sum formulas of modified Tribonacci-Lucas numbers (take $W_n=G_n$ with $G_0=4,G_1=4,G_2=10$).

Corollary 3.56. *For $n\geq 1$, modified Tribonacci-Lucas numbers have the following properties:*

**(a):** $\sum_{k=1}^{n} ki^{k}G_{-k}=\frac{1}{4}(i^{n}((2-2in)G_{-n-1}+((2-2i)\,n-2i)\,G_{-n-2}+(2n+2-2i)G_{-n-3})-12+4i)$.

**(b):** $\sum_{k=1}^{n} ki^{k}G_{-2k}=\frac{i}{8}(i^{n}(i\,(4n-4)\,G_{-2n+1}+(4i-4-8in)G_{-2n}+(4-4n+4i)G_{-2n-1})+8-8i)$.

**(c):** $\sum_{k=1}^{n} ki^{k}G_{-2k+1}=\frac{i}{8}(i^{n}(-4\,(in+1)\,G_{-2n+1}-4\,((1-i)\,n-1)\,G_{-2n}+4i\,(n-1)\,G_{-2n-1})+8i)$.

Taking $W_n=H_n$ with $H_0=4,H_1=2,H_2=0$ in the last proposition, we have the following corollary which presents sum formulas of adjusted Tribonacci-Lucas numbers.

COROLLARY 3.57. *For $n \geq 1$, adjusted Tribonacci-Lucas numbers have the following properties:*

**(a):** $\sum_{k=1}^{n} ki^{k}H_{-k} = \frac{1}{4}(i^{n}((2-2in)H_{-n-1} + ((2-2i)n-2i)H_{-n-2} + (2n+2-2i)H_{-n-3}) + 20i)$.

**(b):** $\sum_{k=1}^{n} ki^{k}H_{-2k} = \frac{i}{8}(i^{n}(i(4n-4)H_{-2n+1} + (4i-4-8in)H_{-2n} + (4-4n+4i)H_{-2n-1}) + 40 + 16i)$.

**(c):** $\sum_{k=1}^{n} ki^{k}H_{-2k+1} = \frac{i}{8}(i^{n}(-4(in+1)H_{-2n+1} - 4((1-i)n-1)H_{-2n} + 4i(n-1)H_{-2n-1}) - 8 - 24i)$.

Corresponding sums of the other third order linear sequences can be calculated similarly when $x = i$.

## References


[1] Bruce, I., A modified Tribonacci sequence, Fibonacci Quarterly, 22(3), 244–246, 1984.

[2] Catalani, M., Identities for Tribonacci-related sequences, arXiv:math/0209179, 2012.

[3] Choi, E., Modular Tribonacci Numbers by Matrix Method, Journal of the Korean Society of Mathematical Education Series B: Pure and Applied. Mathematics. 20(3), 207–221, 2013.

[4] Elia, M., Derived Sequences, The Tribonacci Recurrence and Cubic Forms, Fibonacci Quarterly, 39 (2), 107-115, 2001.

[5] Frontczak, R., Sums of Tribonacci and Tribonacci-Lucas Numbers, International Journal of Mathematical Analysis, 12 (1), 19-24, 2018.

[6] Gökbaş, H., Köse, H., Some Sum Formulas for Products of Pell and Pell-Lucas Numbers, Int. J. Adv. Appl. Math. and Mech. 4(4), 1-4, 2017.

[7] Hansen., R.T., General Identities for Linear Fibonacci and Lucas Summations, Fibonacci Quarterly, 16(2), 121-28, 1978.

[8] Koshy, T., Fibonacci and Lucas Numbers with Applications, A Wiley-Interscience Publication, New York, 2001.

[9] Koshy, T., Pell and Pell-Lucas Numbers with Applications, Springer, New York, 2014.

[10] Lin, P. Y., De Moivre-Type Identities For The Tribonacci Numbers, Fibonacci Quarterly, 26, 131-134, 1988.

[11] Parpar, T., k'ncı Mertebeden Rekürans Bağıntısının Özellikleri ve Bazı Uygulamaları, Selçuk Üniversitesi, Fen Bilimleri Enstitüsü, Yüksek Lisans Tezi, 2011.

[12] Pethe, S., Some Identities for Tribonacci sequences, Fibonacci Quarterly, 26(2), 144–151, 1988.

[13] Scott, A., Delaney, T., Hoggatt Jr., V., The Tribonacci sequence, Fibonacci Quarterly, 15(3), 193–200, 1977.

[14] Shannon, A.G, Horadam, A.F., Some Properties of Third-Order Recurrence Relations, The Fibonacci Quarterly, 10 (2), 135-146, 1972.

[15] Shannon, A., Tribonacci numbers and Pascal's pyramid, Fibonacci Quarterly, 15(3), pp. 268 and 275, 1977.

[16] Soykan, Y., On Summing Formulas For Generalized Fibonacci and Gaussian Generalized Fibonacci Numbers, Advances in Research, 20(2), 1-15, 2019.

[17] Soykan, Y., Corrigendum: On Summing Formulas For Generalized Fibonacci and Gaussian Generalized Fibonacci Numbers, 2020. https://www.researchgate.net/publication/337063487_Corrigendum_On_Summing_Formulas _For_Generalized_Fibonacci_and_Gaussian_Generalized_Fibonacci_Numbers.

[18] Soykan, Y., Summing Formulas For Generalized Tribonacci Numbers, Universal Journal of Mathematics and Applications, 3(1), 1-11, 2020. ISSN 2619-9653, DOI: https://doi.org/10.32323/ujma.637876

[19] Soykan, Y., Matrix Sequences of Tribonacci and Tribonacci-Lucas Numbers, arXiv:1809.07809v1 [math.NT] 20 Sep 2018.

[20] Soykan, Summation Formulas For Generalized Tetranacci Numbers, Asian Journal of Advanced Research and Reports, 7(2), 1-12, 2019. doi.org/10.9734/ajarr/2019/v7i230170.

[21] Soykan, Y., Linear Summing Formulas of Generalized Pentanacci and Gaussian Generalized Pentanacci Numbers, Journal of Advanced in Mathematics and Computer Science, 33(3), 1-14, 2019.

[22] Soykan, Y., Sum Formulas For Generalized Fifth-Order Linear Recurrence Sequences, Journal of Advances in Mathematics and Computer Science, 34(5), 1-14, 2019, Article no.JAMCS.53303, ISSN: 2456-9968, DOI: 10.9734/JAMCS/2019/v34i530224.

[23] Soykan, Y., On Summing Formulas of Generalized Hexanacci and Gaussian Generalized Hexanacci Numbers, Asian Research Journal of Mathematics, 14(4), 1-14, 2019, Article no.ARJOM.50727.

[24] Soykan,Y., On Summing Formulas for Horadam Numbers, Asian Journal of Advanced Research and Reports 8(1): 45-61, 2020, DOI: 10.9734/AJARR/2020/v8i130192.

[25] Soykan, Y., Generalized Fibonacci Numbers: Sum Formulas, Journal of Advances in Mathematics and Computer Science, 35(1), 89-104, 2020, DOI: 10.9734/JAMCS/2020/v35i130241.

[26] Soykan, Y., On Four Special Cases of Generalized Tribonacci Sequence: Tribonacci-Perrin, modified Tribonacci, modified Tribonacci-Lucas and adjusted Tribonacci-Lucas Sequences, Journal of Progressive Research in Mathematics, 16(3), 3056-3084, 2020.

[27] Soykan, Y., On Generalized Third-Order Pell Numbers, Asian Journal of Advanced Research and Reports, 6(1): 1-18, 2019.

[28] Soykan Y., On Generalized Padovan Numbers, MathLAB Journal, In Print.

[29] Soykan, Y., Generalized Pell-Padovan Numbers, Asian Journal of Advanced Research and Reports, 11(2), 8-28, 2020. DOI: 10.9734/AJARR/2020/v11i230259

[30] Soykan Y., On Generalized Narayana Numbers, Int. J. Adv. Appl. Math. and Mech. 7(3), 43-56, 2020, (ISSN: 2347-2529).

[31] Soykan, Y., A Study on Generalized Jacobsthal-Padovan Numbers, Earthline Journal of Mathematical Sciences, 4(2), 227-251, 2020. https://doi.org/10.34198/ejms.4220.227251

[32] Polatlı, E.E., Soykan, Y., On Generalized Third-Order Jacobsthal Numbers, Submitted.

[33] Soykan, Y., On Generalized Grahaml Numbers, Journal of Advances in Mathematics and Computer Science, 35(2), 42-57, 2020. DOI: 10.9734/JAMCS/2020/v35i230248.

[34] Soykan Y., On Generalized Reverse 3-primes Numbers, Journal of Scientific Research and Reports, 26(6), 1-20, 2020. DOI: 10.9734/JSRR/2020/v26i630267

[35] Soykan Y., Generalized Tribonacci Numbers: Summing Formulas, Int. J. Adv. Appl. Math. and Mech. 7(3), 57-76, 2020.

[36] Soykan, Y., On Sum Formulas for Generalized Tribonacci Sequence, Journal of Scientific Research & Reports, 26(7), 27-52, 2020. ISSN: 2320-0227, DOI: 10.9734/JSRR/2020/v26i730283

[37] N.J.A. Sloane, The on-line encyclopedia of integer sequences, http://oeis.org/

[38] Spickerman, W., Binet's formula for the Tribonacci sequence, Fibonacci Quarterly, 20, 118–120, 1982.

[39] Yalavigi, C.C., A Note on 'Another Generalized Fibonacci Sequence', The Mathematics Student. 39, 407–408, 1971.

[40] Yalavigi, C. C., Properties of Tribonacci numbers, Fibonacci Quarterly, 10(3), 231–246, 1972.

[41] Yilmaz, N., Taskara, N., Tribonacci and Tribonacci-Lucas Numbers via the Determinants of Special Matrices, Applied Mathematical Sciences, 8(39), 1947-1955, 2014.

[42] Marcellus E. Waddill, Using Matrix Techniques to Establish Properties of a Generalized Tribonacci Sequence (in Applications of Fibonacci Numbers, Volume 4, G. E. Bergum et al., eds.). Kluwer Academic Publishers. Dordrecht, The Netherlands: pp. 299-308, 1991.

[43] Waddill, M. E., The Tetranacci Sequence and Generalizations, Fibonacci Quarterly, 9-20, 1992.