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The spectrum of $P^{(5)}_{(k,10-k)}$-designs

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Abstract

Given an hypergraph $H^{(h)}$, uniform of rank h, an $H^{(h)}$-*design* [or also a *design of type* $H^{(h)}$] of order v is a pair $\Sigma = (X, \mathcal{B})$, where X is a set of cardinality v and $\mathcal{B}$ is a collection of hypergraphs, all isomorphic to $H^{(h)}$, such that every h-subset of X is an edge of exactly one hypergraph $H^{(h)} \in \mathcal{B}$. An *hyperpath* $P^{(h)}_2$ is an uniform hypergraph, having two non disjoint edges. In this paper we determine the spectrum of hyperpath-designs of type $P^{(5)}_2$, in the case that hyperedges have 3 or 4 vertices in common and formulate a conjecture about the cases $k = 1, 2$.

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1 Introduction

Let $K^{(h)}_v = (X, \mathcal{E})$ be the complete hypergraph, uniform of rank h, defined in the vertex set $X = \{x_1, x_2, ..., x_v\}$, for $v \geq h$. This means that $\mathcal{E} = \mathcal{P}_h(X)$, the collection of all the h-subsets of X.
Let $H^{(h)}$ be a subhypergraph of $K^{(h)}_v$. An $H^{(h)}$-*design* [or also *a design of type* $H^{(h)}$], having *order* v, is a pair $\Sigma = (X, \mathcal{B})$, where X is a finite set of cardinality v, whose elements are called *vertices* , and $\mathcal{B}$ is a collection of hypergraphs, also called *blocks*, all isomorphic to $H^{(h)}$, with the condition that every h-subset Y of X is an edge of exactly one hypergraph $H^{(h)} \in \mathcal{B}$. An $H^{(h)}$-*design*, of order v, is also called an $H^{(h)}$-*decomposition* of $K^{(h)}_v$ [1].
In what follows, we will indicate by $Sp(H^{(h)})$ the *spectrum* of the correspondent $H^{(h)}$-designs, i.e. the set of all integers v such that there exist $H^{(h)}$-designs of order v.
Observe that:

- Among all the graphs, there is exactly one path with two edges and it is known as P_3. If x, y, z are the vertices of a path P_3 and the edges are $\{x, y\}, \{y, z\}$, we will indicate it by $[x, (y), z]$.
- Among all the uniform hypergraphs of rank 3, there are exactly two *hyperpaths* with two edges. The number of vertices can be 4 or 5. A $P^{(3)}(2, 4)$ will be the hyperpath having vertices a, b, c, d and edges $\{a, b, c\}, \{b, c, d\}$, and it will be indicate by $[a, (b, c), d]$. A $P^{(3)}(1, 5)$ will be the hyperpath having vertices a, b, c, d, e and edges $\{a, b, c\}, \{c, d, e\}$, and it will be indicated by $[a, b, (c), d, e]$.
- Among all the hypergraphs uniform of rank 4, there are exactly three *hyperpaths* with two edges. The number of vertices can be 5 or 6 or 7. A $P^{(4)}(3, 5)$ will be the hyperpath having vertices a, b, c, d, e and edges $\{a, b, c, d\}$, $\{b, c, d, e\}$, and it will be indicate by $[a, (b, c, d), e]$. A $P^{(4)}(2, 6)$ will be the hyperpath having vertices a, b, c, d, e, f and edges $\{a, b, c, d\}, \{c, d, e, f\}$, and it will be indicate by $[a, b, (c, d), e, f]$. A $P^{(4)}(1, 7)$ will be the hyperpath having vertices a, b, c, d, e, f, g and edges $\{a, b, c, d\}, \{d, e, f, g\}$, and it will be indicate by $[a, b, c, (d), e, f, g]$.

For $h = 2$, $H^{(2)}$ is a graph G and G-designs have been studied in the recent past by many authors.

For $h = 3$, $H^{(3)}$-designs have been studied in [2], where the spectrum has been determined in some cases of hypergraphs $H^{(3)}$ with few edges. *Balanced* $H^{(3)}$-designs have been studied in [4]. Other general results can been found in [1].

For $h = 4$, $H^{(4)}$-designs have been studied in [3], where the spectrum has been determined for $P^{(4)}(3, 5)$-designs, $P^{(4)}(2, 6)$-designs and $P^{(4)}(1, 7)$-designs, where $P^{(4)}(u, 8 - u)$, for $u = 1, 2, 3$, is an hyperpath with two edges, i.e. an uniform hypergraph of rank 4, of order $8 - u$, with two edges having u vertices in common.
It is known that:

Theorem 1.1 *:* $Sp(P_3) = \{v \in N : v \equiv 0 \; or \; 1 \; mod \; 4, \; v \geq 4\}$.

Theorem 1.2 *:* $Sp(P^{(3)}(6 - u, u)) = \{v \in N : v \equiv 0 \; or \; 1 \; or \; 2 \; mod \; 4, \; v \geq u,$ *for* $u = 4, 5\}$.*[2]*

Some proof of the previous theorems can be found in [2] and also in [1]. For $P^{(4)}(3, 5)$, $P^{(4)}(2, 6)$, $P^{(4)}(1, 7)$, in [3] it is proved that:

Theorem 1.3 *:* $Sp(P^{(4)}(3, 5)) = Sp(P^{(4)}(2, 6)) = Sp(P^{(4)}(1, 7)) = \{v \in N : v \equiv 0 \; or \; 1 \; or \; 2 \; or \; 3 \; mod \; 8, \; v \geq 8\}$. *[3]*

In this paper we study the spectrum for $P^{(5)}(4, 6)$-*designs* $P^{(5)}(3, 7)$-*designs* and determine it completely. Further, we formulate a conjecture regarding

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$P^{(5)}(2,8)$-*designs* and $P^{(5)}(1,9)$-*designs*. The motivation for this research is to study the trend of the spectrum of $P^{(h)}(k,2h-k)$-designs. We will see that for $h=5$ the situation is similar to the case $h=3$, while in the cases $h=2, h=4$ the trend is the same.

In what follows, for a given $P^{(h)}_{(,k,2h-k)}$-design $\Sigma=(X,\mathcal{B})$, if $A=\{a_1,a_2,,...,a_p\}\subseteq X$ and $\{x,y,b_1,b_2,...,b_{h-p-1}\}\subseteq X$, then $[x,(A,b_1,b_2,...,b_{h-p-1}),y]$ will indicate the block of Σ having for h-edges $\{x,a_1,a_2,,...,a_p,b_1,b_2,...,b_{h-p-1}\}$ and $\{a_1,a_2,,...,a_p,b_1,b_2,...,b_{h-p-1},y\}$.

2 General results

In [3] the authors proved the following general results.

Theorem 2.1 *: If $\Sigma=(X,\mathcal{B})$ is a $P_2^{(h)}$-design of order v, for any h, then:*
1) $|\mathcal{B}|=\binom{v}{h}/2$;
2) *if $P_2^{(h)}=P^{(h)}(k,2h-k)$, then $v\geq 2h-k$.*

Theorem 2.2 - *If Σ is a $P_2^{(h)}(k,2h-k)$-design of order v, Γ a $P_2^{(h-1)}(k-1,2h-k-1)$-design of the same order v, then there exists a $P_2^{(h)}(k,2h-k)$-design Σ' of order $v'=v+1$, embedding Σ.*

3 Necessary conditions for $P_2^{(5)}$-designs

We have that:

Theorem 3.1 *: If $k=1,2,3,4$ and $\Sigma=(X,\mathcal{B})$ is a $P_2^{(5)}(k,10-k)$-design of order v, then v is even or $v\equiv 1$ or $3 \bmod 8$, , with always $v\geq 10-k$.*

Proof. For $k=1,2,3,4$, if $\Sigma=(X,\mathcal{B})$ is a $P_2^{(5)}$-design of order v, then:

$$|\mathcal{B}|=\binom{v}{5}/2=\frac{v(v-1)(v-2)(v-3)(v-4)}{5\cdot4\cdot3\cdot2\cdot1}\cdot\frac{1}{2}.$$

From which, necessarily, it follows that or v is even or $v=8k+1$ or $v=8k+3$. Further, if $B\in\mathcal{B}$ and the two edges of B have k vertices in common for $k=1,2,3,4$, then $|B|=10-k$ and therefore $v\geq 10-k$. □

4 The spectrum of $P^{(5)}_{(4,6)}$-designs

In this section we determine all the positive integer v such that there exist $P^{(5)}_{(4,6)}$-designs of order v.

Theorem 4.1 - *There exist $P^{(5)}_{(4,6)}$-designs of order $v = 6$.*

Proof. Let $X = \{x_1, x_2, ..., x_8\}$. Consider the family $\mathcal{B}$ of the following hypergraphs:

$$[x_1, (X - \{x_1, x_2\}), x_2],$$
$$[x_3, (X - \{x_3, x_4\}), x_4],$$
$$[x_5, (X - \{x_5, x_6\}), x_6].$$

It is immediate to verify that $\Sigma = (X, \mathcal{B})$ is a $P^{(5)}_{(4,6)}$-design of order $v = 6$. □

Theorem 4.2 - *If $v \in Sp(P^{(5)}_{(4,6)}) \cap Sp(P^{(4)}_{(3,5))})$, then $v + 1 \in Sp(P^{(5)}_{(4,6)})$.*

Proof. *Construction $v \to v + 1$.*

Let $\Sigma_1 = (X, \mathcal{B}_1)$ be a $P^{(5)}_{(4,6)}$-design and let $\Sigma_2 = (X, \mathcal{B}_2)$ be a $P^{(4)}_{(3,5)}$-design, both of order v. Let $\infty \notin X$, $X' = X \cup \{\infty\}$. Define the following family Π of hypergraphs $P^{(5)}_{(4,6)}$:

$$\Pi = \{[x', (\infty, x_{j,1}, x_{j,2}, x_{j,3}), x''] : \quad [x', (x_{j,1}, x_{j,2}, x_{j,3}), x''] \in \mathcal{B}_2\}.$$

If $\mathcal{B} = \mathcal{B}_1 \cup \Pi$, then $\Sigma = (X', \mathcal{B})$ is a $P^{(5)}_{(4,6)}$-design of order $v + 1$.
Indeed, consider any $Y \subseteq X', |Y| = 5$. If $Y \subseteq X$, there exists exactly one block $B \in \mathcal{B}_1$ having Y as edge and also no block of Π has Y as edge. If $\infty \in Y$ and therefore $|Y \cap X| = 4$, then there exists exactly one block of $B \in \mathcal{B}_2$ containing $Y - \{\infty\}$ as edge in Σ_2 and therefore there exists exactly one block of Π containing Y as edge.
The statement is so proved. □

Theorem 4.3 - *If $v \in Sp(P^{(5)}_{(4,6)} \cap Sp(P^{(3)}_{(2,4)}$, v even, then $v + 2 \in Sp(P^{(5)}_{(4,6)}$.*

Proof. *Construction $v \to v + 2$.*
Let $\infty_1, \infty_2 \notin X$, $\infty_1 \neq \infty_2$, $X' = X \cup \{\infty_1, \infty_2\}$. Further, let

$\Sigma_1 = (X, \mathcal{B}_1$ be a $P^{(5)}_{(4,6)}$-design of order v;
$\Sigma_2 = (X, \mathcal{B}_2)$ be a $P^{(3)}_{(2,4)}$-design of order v.

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Define the following families of hypergraphs $P^{(5)}_{(4,6)}$:

$$\Pi_1 = \{[x', (\infty_1, \infty_2, x_{j,1}, x_{j,2}, x''] : \quad [x', (x_{j,1}, x_{j,2}, x''] \in \mathcal{B}_2\};$$

$$\Pi_2 = \{[\infty_1, (x_{j,1}, x_{j,2}, x_{j,3}, x_{j,4}), \infty_2] : \quad \{x_{j,1}, x_{j,2}, x_{j,3}, x_{j,4}\} \in \mathcal{P}_4(X)\}.$$

If $\mathcal{B} = \mathcal{B}_1 \cup \Pi_1, \Pi_2$, it is possible to verify that $\Sigma = (X', \mathcal{B})$ is a $P^{(5)}_{(4,6)}$-design of order $v + 2$, which proves the statement. □

Collecting together the results proved in the previous Theorems, it follows that:

Theorem 4.4 - *There exist $P^{(5)}_{(4,6)}$-designs if and only if:*
1) v is even, $v \geq 6$;
or
2) $v \equiv 1$ or $v \equiv 3$, mod 8.

Proof. The statement follows from Theorems 4.1, 4.3, considering that for $v \geq 8$, v even, $P^{(5)}_{(3,7}$-designs of order v there exist. The statement 2) follows from 1) and Theorem 4.2, considering that for $v = 8h$ and $v = 8h + 2$, for any $h \in \mathbb{N}$, $P^{(4)}_{(3,5)}$-designs of order v there exist. □

5 An extension to $P^{(5)}_{(3,7)}$-designs

By the same technique used for $P^{(5)}_{(4,6)}$-designs it is possible to determine the spectrum of $P^{(5)}_{(3,7)}$-designs.

Theorem 5.1 - *There exist $P^{(5)}_{(3,7)}$-designs of order $v = 8$.*

Proof. Let $X = \{1, 2, ..., 8\}$. Consider the family $\mathcal{B}$ of the following hypergraphs:

$$\begin{array}{llll}
[4,5,(1,2,3),6,8], & [3,7,(1,2,5),4,6], & [3,4,(1,2,7),6,8], & [2,6,(1,3,4),5,8], \\
[2,8,(1,3,5),4,6], & [2,7,(1,4,5),6,8], & [2,8,(1,4,7),3,5], & [3,7,(1,2,6),4,8], \\
[2,3,(1,7,8),4,6], & [1,8,(2,3,4),5,6], & [2,3,(6,7,8),4,5], & [1,2,(5,6,8),3,7], \\
[1,6,(2,3,5),7,8], & [3,8,(2,4,6),5,7], & [1,6,(2,4,7),5,8], & [3,7,(2,5,6),4,8], \\
[1,3,(6,7,8),2,5], & [1,5,(2,7,8),3,4], & [1,8,(3,4,6),5,7], & [1,7,(3,4,6),5,8], \\
[1,8,(3,4,7),2,6], & [1,5,(3,6,8),4,7], & [1,6,(4,5,7),3,8], & [1,7,(4,5,8),2,3], \\
[1,5,(6,7,8),2,4], & [1,6,(3,5,7),2,4], & [2,6,(1,5,7),3,8], & [1,4,(2,5,8),3,6].
\end{array}$$

It is immediate to verify that $\Sigma = (X, \mathcal{B})$ is a $P^{(5)}_{(4,6)}$-design of order $v = 8$.

Theorem 5.2 - *If* $v \in Sp(P^{(5)}_{(3,7)}) \cap Sp(P^{(4)}_{(2,6))})$, *then* $v+1 \in Sp(P^{(5)}_{(3,7)})$.

Proof. *Construction* $v \rightarrow v+1$.

Let $\Sigma_1 = (X, \mathcal{B}_1)$ be a $P^{(5)}_{(3,7)}$-design and let $\Sigma_2 = (X, \mathcal{B}_2)$ be a $P^{(4)}_{(2,6)}$-design, both of order v. Let $\infty \notin X$, $X' = X \cup \{\infty\}$. Define the following family Π of hypergraphs $P^{(5)}_{(3,7)}$:

$$\Pi = \{[x_{1,i}, x_{2,i}, (\infty, x_{j,1}, x_{j,2}), x_{3,i}, x_{4,i}] : [x_{1,i}, x_{2,i}, (x_{j,1}, x_{j,2}), x_{3,i}, x_{4,i}] \in \mathcal{B}_2\}.$$

If $\mathcal{B} = \mathcal{B}_1 \cup \Pi$, it is possible to verify that $\Sigma = (X', \mathcal{B})$ is a $P^{(5)}_{(3,7)}$-design of order $v+1$, which proves the statement. □

Theorem 5.3 - *If* $v \in Sp(P^{(5)}_{(3,7)} \cap Sp(P^{(4)}_{(3,5)} \cap Sp(P^{(3)}_{(1,5)}$, v *even, then* $v+2 \in Sp(P^{(5)}_{(3,7)}$.

Proof. *Construction* $v \rightarrow v+2$. Let $\infty_1, \infty_2 \notin X$, $\infty_1 \neq \infty_2$, $X' = X \cup \{\infty_1, \infty_2\}$. Further, let

$\Sigma_1 = (X, \mathcal{B}_1$ be a $P^{(5)}_{(3,7)}$-design of order v;
$\Sigma_2 = (X, \mathcal{B}_2)$ be a $P^{(4)}_{(3,5)}$-design of order v;
$\Sigma_3 = (X, \mathcal{B}_3)$ be a $P^{(3)}_{(1,5)}$-design of order v.

Define the following families of hypergraphs $P^{(5)}_{(3,7)}$:

$$\Pi_1 = \{[[x_{1,i}, x_{2,i}, (\infty_1, \infty_2, x_{j,1}), x_{i,3}, x_{i,4}] : [x_{1,i}, x_{2,i}, (x_{j,1}), x_{3,i}, x_{4,i}] \in \mathcal{B}_3\};$$

$$\Pi_2 = \{[\infty_1, x_{i,1}, (x_{j,1}, x_{j,2}, x_{j,3}), x_{i,2}, \infty_2] : [x_{i,1}, (x_{j,1}, x_{j,2}, x_{j,3}), x_{i,2}] \in \mathcal{B}_2\}.$$

If $\mathcal{B} = \mathcal{B}_1 \cup \Pi_1, \Pi_2$, it is possible to verify that $\Sigma = (X', \mathcal{B})$ is a $P^{(5)}_{(3,7)}$-design of order $v+2$, which proves the statement. □

Collecting together the previous Theorems, it follows that:

Theorem 5.4 - *There exist* $P^{(5)}_{(3,7)}$-*designs if and only if:*

1) v *is even,* $v \geq 8$;

or

2) $v \equiv 1$ *or* $v \equiv 3$, *mod* 8.

Proof. The statement 1) follows from Theorems 5.1, 5.3, considering that $P^{(4)}_{(3,5)}$-designs of orer v exist for every v even, $v \geq 6$. The statement 2) follows from Theorems 5.1, 5.2, considering that $P^{(4)}_{(3,5)}$-designs and $P^{(3)}_{(1,5)}$-designs of order v exist for any $h \in \mathbb{N}$ and $v = 8h, v = 8h+2$. □

6 Conjecture for $k = 1$ and $k = 2$

It seems very difficult to determine the spectrum of $P^{(5)}_{(k,10-k)}$-designs for $k = 1, 2$, i.e. for $P^{(5)}_{(1,9)}$-designs and $P^{(5)}_{(2,8)}$-designs. It is opinion of the authors that in these cases the spectrum is similar to the one already found for $k = 4$ and $k = 3$.

Conjecture: *For* $k = 1, 2$, *there exist* $P^{(5)}(k, 10-k)$*-designs of order* v *if and only if*: $v \geq 10 - k$ *and*
$v \equiv 1 \ mod\ 8$ *or* $v \equiv 3 \ mod\ 8$.

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