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Two Elementary Analytic Functions and Their Relationship with Hardy and Bergman Spaces

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Abstract

In this paper, we start by proving that the function $f(z) = 1/(1-z)$, which is holomorphic in the open unit disc U centred at the origin, is an element of a Hardy space $H^p, p > 0$, if and only if $p < 1$. Here we give a new proof for a known result. Moreover, the present work provides two different new proofs for one of the implications mentioned above. One proves that the same function is an element of a Bergman space $A^p, p > 0$, if and only if $p < 1$. This is the first completely new result of this work. From these theorems we deduce the behavior of the function $g(z) = \frac{1}{\sqrt{1-z^2}}$ in the half – open disc $U_+ \coloneqq \{z \in \mathbb{C};\ |z| < 1, Re(z) > 0\}$. Although the assertions claimed above refer to complex analytic functions, and the involved spaces are function spaces of analytic complex functions, the proofs from below are based on results and methods of real analysis.

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1. Introduction

Hardy spaces and Bergman spaces are intensively studied in the literature. For basic related definitions and properties of these spaces see the works [1] - [6]. Dyadic Hardy spaces generated by rearrangement invariant spaces are discussed in the papers [7] – [9]. Extremal problems in Hardy and Bergman spaces are studied in [6], respectively in [10]. For other aspects on these spaces (especially on weighted Bergman spaces) see [11] - [14]. We recall the definition of Hardy spaces (called H^p − spaces). To define them, we need to define firstly the integral means of an analytic function f in the unit disc $U = \{z \in \mathbb{C};\ |z| < 1\}$. For

$$0 < p < \infty, 0 < r < 1,$$

the integral mean of f is

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^p d\theta\right)^{1/p}$$

For $p = \infty$, we define $M_\infty(r,f) = \max_{0\le\theta<2\pi}\left|f\left(re^{i\theta}\right)\right|$. Similar definitions can be given for harmonic functions. If f is a fixed analytic or harmonic function and p is fixed, then $M_p(r,f)$ is an increasing function of r. For $0 < p \le \infty$, we say that a function f is in the Hardy space H^p if f is analytic in U and

$\|f\|_{H^p} := \lim_{r\to 1-} M_p(r,f) < \infty$. We call $\|\cdot\|_{H^p}$ the H^p- norm. For $1 \le p \le \infty$ it defines a true norm and the corresponding space H^p is a Banach space (see [1]).

Bergman spaces are closely related to Hardy spaces. The main difference is that in definition of Bergman spaces, double - integral means are involved. Namely, denote by σ the normalized Lebesgue area measure over U, so that $\sigma(U) = 1$ ($d\sigma = \frac{1}{\pi} dxdy$). Then, for $0 < p < \infty$, the Bergman space $A^p(U)$ or simply A^p consists in all functions f which are analytic in the unit disc, such that

$$\|f\|_{A^p} = \left(\int_U |f|^p \, d\sigma\right)^{1/p} < \infty$$

In other words, $f \in A^p$ if f is analytic in the unit open disc and is in L^p for Lebesgue area measure on the unit disc. We call $\|\cdot\|_{A^p}$ the A^p- norm. For $1 \le p < \infty$ it is a true norm.

Here are two main sharp properties on these spaces: $H^p \subset A^p$; if $p < q$, then $H^q \subset H^p$. The former property one proves by means of using polar coordinates, while the latter one is a consequence of Hőlder's inequality.

One of the purposes of the present work is to give new proofs for necessary and sufficient conditions on p such that the function f defined by (1) from below to be an element of H^p. The main purpose is to prove that the same condition is necessary and sufficient for the fact that the same function be an element of the space A^p. The answer is the same, for both two spaces: the necessary and sufficient condition is $p < 1$. Finally, the third aim of the paper is to derive from the results mentioned above the behavior of the function $g(z) = \frac{1}{\sqrt{1-z^2}}$ in the open half-disc $U_+ := \{z \in \mathbb{C};\ |z| < 1, Re(z) > 0\}$. The answer is given in Theorem 2.3.

Consider the function

$$f(z) = \frac{1}{1-z} = 1 + z + z^2 + \cdots z^{n-1} + \cdots, \qquad z \in \mathbb{C}, \qquad |z| < 1 \# (1)$$

The convergence of the geometric series in the right hand side is pointwise on $U := \{z;\ |z| < 1\}$ and absolutely and uniformly convergent on any closed disc centered at the origin, of radius $r \in (0,1)$ (or, equivalently, on any compact contained in U). The behavior of the function g on the open unit half disc U_+ is briefly discussed as well (see the above comments). The rest of the paper is organized as follows. Section 2 is devoted to the proof of the three results and related remarks. Section 3 concludes the paper.

2. The results

The next theorem answers the question: on what conditions on $p \in (0, \infty]$ is the function f defined by (1) an element of the Hardy space H^p? It is known [1] that the Hardy spaces H^p are defined for all $p \in (0, \infty]$, but for $p \in (0,1)$ they are not normed vector spaces, being just metric topological vector spaces (see [1] and [2]). The result of the next theorem is known (see [4] for a more general framework and [6], Proposition 1, for a proof based on real analysis). Our proof is new and the inequalities which it is based on are different to those from Proposition 1 of [6]. For one of the implications, two different proofs are given.

Theorem 2.1. *The function f given by* (1) *is an element of* H^p *if and only if* $p < 1$.

Proof. Let $p \in (0, \infty)$. To write the integral

$$I(r,p) = \int_{-\pi}^{\pi} \frac{1}{|1 - r\exp(i\theta)|^p} d\theta$$

under a more convenient form, we need the following computations

$$I(r,p)=\int_{-\pi}^{\pi}\frac{1}{(1-2r\cos(\theta)+r^2)^{p/2}}d\theta=2\int_{0}^{\pi}\frac{1}{(1-2r\cos(\theta)+r^2)^{p/2}}d\theta\Rightarrow$$

$$\lim_{r\uparrow 1}I(r,p)=2\lim_{r\uparrow 1}\int_{-1}^{1}\frac{1}{(1-2rt+r^2)^{p/2}(1-t^2)^{1/2}}dt,$$

by means the change of variable $\cos(\theta)=t$. On the other hand, one can write

$$\begin{aligned}&\int_{-1}^{1}\frac{1}{(1-2rt+r^2)^{p/2}(1-t^2)^{1/2}}dt=\\&\int_{-1}^{(1+r)/2}\frac{1}{(1-2rt+r^2)^{p/2}(1-t^2)^{1/2}}dt+\int_{(1+r)/2}^{1}\frac{1}{(1-2rt+r^2)^{p/2}(1-t^2)^{1/2}}dt\geq\\&\int_{-1}^{(1+r)/2}\frac{1}{(1-2rt+r^2)^{p/2}(1-t^2)^{1/2}}dt.\end{aligned}\qquad\#(2)$$

On the interval $[-1,(1+r)/2]$ we have that $1-2rt+r^2\leq 1-2t+1=2(1-t)$ (and equality occurs only at $t=(r+1)/2),$ so that the latter integral is greater than

$$\int_{-1}^{(1+r)/2}\frac{1}{(2(1-t))^{p/2}(1-t^2)^{1/2}}dt=\frac{1}{2^{p/2}}\int_{-1}^{(1+r)/2}\frac{1}{(1-t)^{(p+1)/2}(1+t)^{1/2}}dt\quad\#(3)$$

The latter integral in (3) is increasing with $r\uparrow 1$. Using the above remarks and passing through the limit in (2) and (3) (via Lebesgue monotone convergence theorem), one obtains for any $p\geq 1$:

$$\lim_{r\uparrow 1}\int_{-1}^{1}\frac{1}{(1-2rt+r^2)^{p/2}(1-t^2)^{1/2}}dt\geq\lim_{r\uparrow 1}\frac{1}{2^{p/2}}\int_{-1}^{(1+r)/2}\frac{1}{(1-t)^{(p+1)/2}(1+t)^{1/2}}dt=$$

$$=\frac{1}{2^{p/2}}\int_{-1}^{1}\frac{1}{(1-t)^{(p+1)/2}(1+t)^{1/2}}dt=\infty.$$

Thus $1\leq p<\infty\Rightarrow f\notin H^p$. On the other hand, obviously $f\notin H^{\infty}$, since

$$\lim_{r\uparrow 1}\frac{1}{|1-r\exp(0\cdot i\theta)|}=\infty,$$

Thus $f\in H^p\Rightarrow p<1$. Conversely, we now show that $f\in H^p$ for all $p\in(0,1)$. Observe that the two degree polynomial $P_{2,t}(r)=1-2rt+r^2$ has as unique global minimum point at $r_0=t$. Thus, we have $P_{2,t}(r)\geq P_{2,t}(t)\geq 1-t^2, r\in\mathbb{R}$. In particular, it results

$$\begin{aligned}&\lim_{r\uparrow 1}\int_{-1}^{1}\frac{1}{(1-2rt+r^2)^{p/2}(1-t^2)^{1/2}}dt\leq\int_{-1}^{1}\frac{1}{(1-t^2)^{p/2}(1-t^2)^{1/2}}dt=\\&\int_{-1}^{1}\frac{1}{(1-t^2)^{(p+1)/2}}dt<\infty,\end{aligned}$$

since $(p+1)/2<1$. Now the proof is complete. □

Second proof for the "if" part. Let rewrite $f(z)=1/(1-z)$, $z=re^{i\theta}$, $|z|=r<1$, $\theta\in(-\pi,\pi]$. This leads to

$$f(r,\theta)=\frac{1}{1-r\cos(\theta)-ir\sin(\theta)}=\frac{1-r\cos(\theta)}{1-2r\cos(\theta)+r^2}+ir\frac{\sin(\theta)}{1-2r\cos(\theta)+r^2}=$$
$$=f_1(r,\theta)+if_2(r,\theta),$$

where $f_1=\operatorname{Re} f$, $f_2=\operatorname{Im} f$. Evidently, one has: $\lim_{r\uparrow 1} f_1(r,\theta)=\frac{1}{2}$, $\theta\neq 0$. Hence on the unit circle f_1 is an element of L^p, $\forall p>0$. Passing to the imaginary part, one obtains

$$\lim_{r\uparrow 1} f_2(r,\theta)=\frac{\sin(\theta)}{2(1-\cos(\theta))}=\frac{1}{2}\operatorname{ctg}\left(\frac{\theta}{2}\right),\quad \theta\neq 0,$$

We compute

$$\lim_{\theta\to 0}\frac{\frac{\cos(\theta/2)}{\sin(\theta/2)}}{\frac{2}{\theta}}=1,$$

in order to apply the comparison to the limit criterion for improper integrals. The function $1/\theta$ is an element of $L^p((-\pi,\pi))$ for $0<p<1$. Observe that the origin is the unique singular point of f_2 from the closure of its domain of definition written above. Consequently, f_2 is in L^p, $\forall p\in(0,1)$. It results that both real and imaginary parts of f are in L^p for any $p\in(0,1)$ (and $r=1$), hence $f\in H^p$ for all $p\in(0,1)$. This concludes the second proof of this part. □

Remark 2.1. Notice that in the first part of the previous proof we have used the inequality

$$1-2rt+r^2\le 2(1-t),\; r\in[-1,(1+r)/2]$$

in which equality occurs if and only if $t=(r+1)/2$. See the proof for details. Thus we have an applied elementary optimization problem.

Corollary 2.1. We have

$$\lim_{r\uparrow 1}\int_{-\pi}^{\pi}\left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} r^{n+m}e^{i(n-m)\theta}\right)^{1/2} d\theta=\infty.$$

Proof. By the first part of the preceding proof, the relation $f\notin H^1$ holds. This can be rewritten as

$$\infty=\lim_{r\uparrow 1}\int_{-\pi}^{\pi}|f(re^{i\theta})|\,d\theta=\lim_{r\uparrow 1}\int_{-\pi}^{\pi}(f\bar{f})^{1/2}(re^{i\theta})d\theta=$$
$$=\lim_{r\uparrow 1}\int_{-\pi}^{\pi}\left(\left(\sum_{n=0}^{\infty} r^n e^{in\theta}\right)\left(\sum_{m=0}^{\infty} r^m e^{-im\theta}\right)\right)^{1/2} d\theta=$$
$$\lim_{r\uparrow 1}\int_{-\pi}^{\pi}\left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} r^{n+m}e^{i(n-m)\theta}\right)^{1/2} d\theta.$$

This concludes the proof. □

Remark 2.2. From the first part of Theorem 1.1, we know that $f\notin H^2$. To determine the speed of convergence of $\|f\|_{H^2}$ to infinity, we compute

$$\| f_r \|_2^2 = \frac{1}{2\pi}\left(\sum_{n=0}^{\infty} r^{2n}\right)(2\pi) = \frac{1}{1-r^2}.$$

Hence one can write

$$\| f_r \|_2 \approx \frac{1}{\sqrt{2}\sqrt{1-r}}, \quad r \uparrow 1.$$

The following problem arises naturally: could we prove a similar result for the same function f, when the Hardy space H^p is replaced by the Bergman space A^p? The answer is affirmative. Namely, we prove the following new result.

Theorem 2.2. Let $p > 0$. *The function f given by* (1) *is an element of the Bergman space* A^p *if and only if* $p < 1$.

Proof. Observe that for each $p \in (0,1)$, we have $f \in H^p \subset A^p$, where the former relation follows from theorem 2.1 and the latter one results from the definition of the two spaces in question. Now let $p \geq 1$. Computing the integral which defines the p −power of the norm of f in the Bergman space in polar coordinates, as in the proof of theorem 2.1, we find

$$\|f\|_{A^p}^p = \int_U |f|^p \, d\sigma = \frac{1}{\pi}\int_U |f(x,y)|^p \, dxdy = \frac{1}{\pi}\int_{[0,1[\times]-\pi,\pi[} r\left|f\left(re^{i\theta}\right)\right|^p dr d\theta =$$

$$\frac{1}{\pi}\int_{[0,1[\times]-\pi,\pi[} r \cdot \frac{1}{|1-re^{i\theta}|^p} dr d\theta = \frac{1}{\pi}\int_{[0,1[\times]-\pi,\pi[} r \cdot \frac{1}{(1-2r\cos(\theta)+r^2)^{p/2}} dr d\theta =$$

$$\frac{2}{\pi}\int_{[0,1[\times]-1,1[} r \cdot \frac{1}{(1-2rt+r^2)^{p/2}(1-t^2)^{1/2}} dr dt.$$

(where $r = r, t = \cos(\theta)$). Notice that we can write

$$[0,1[\times]-1,1[= \bigcup_n B_n, B_n := \left\{(r,t);\, 0 \le r \le r_n (<1), -1 < t \le \frac{r+1}{2}\right\}, n \in \mathbb{N},$$

where $r_n \uparrow 1, n \to \infty$. Following the proof of theorem 2.1, we have already remarked that on B_n one has

$$\frac{1}{(1-2rt+r^2)^{p/2}} \geq \frac{1}{\left(2(1-t)\right)^{p/2}},$$

because of $t \le \frac{r+1}{2}$. Since the sequence $(r_n)_n$ is increasing to 1, the sequence $(B_n)_n$ is also increasing with respect to the inclusion order relation. All the functions under the integral sign being positive, Lebesgue monotone convergence theorem can be applied. It results:

$$\|f\|_{A^p}^p = \frac{2}{\pi}\int_{[0,1[\times]-1,1[} \lim_{n\to\infty} \chi_{B_n}(r,t) \frac{r}{(1-2rt+r^2)^{p/2}(1-t^2)^{1/2}} dr dt =$$

$$\frac{2}{\pi}\lim_{n\to\infty}\int_{[0,1[\times]-1,1[} \chi_{B_n}(r,t) \frac{r}{(1-2rt+r^2)^{p/2}(1-t^2)^{1/2}} dr dt \geq$$

$$\frac{2}{\pi 2^{p/2}} \lim_{n\to\infty} \int_{[0,1[\times]-1,1[} N_{B_n}(r,t) \frac{r}{(1-t)^{p/2}(1-t^2)^{1/2}} drdt =$$

$$\frac{2}{\pi 2^{p/2}} \int_{[0,1[\times]-1,1[} \lim_{n} N_{B_n}(r,t) \frac{r}{(1-t)^{p/2}(1-t^2)^{1/2}} drdt =$$

$$\frac{2}{\pi 2^{p/2}} \int_{[0,1[\times]-1,1[} \frac{r}{(1-t)^{(p+1)/2}(1+t)^{1/2}} drdt = \frac{1}{\pi 2^{p/2}} \int_{-1}^{1} \frac{dt}{(1-t)^{(p+1)/2}(1+t)^{1/2}} = +\infty$$

Hence for $p \geq 1$, we have that $f \notin A^p$. This concludes the proof. □

In the sequel, we study the behavior of the function

$$g: U_+ \coloneqq \{z \in \mathbb{C};\ |z| < 1, Re(z) > 0\} \to \mathbb{C}, g(z) = \frac{1}{\sqrt{1-z^2}}, z \in U_+$$

related to the limits of the means

$$\|g\|_{H^p(U_+)} \coloneqq \lim_{\rho \uparrow 1} \left(\frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \left| g\left(\rho e^{i\theta_1}\right) \right|^p d\theta_1 \right)^{1/p}, \ \|g\|_{A^p(U_+)} \coloneqq \left(\int_{U_+} |g|^p \, d\sigma \right)^{1/p},$$

$$d\sigma = \frac{2}{\pi} dxdy$$

From the previous results, we deduce the following theorem.

Theorem 2.3. (i) *We have* $\|g\|_{H^{p_1}(U_+)} < \infty$ *if and only if* $p_1 < 2$.
(ii) $\|g\|_{A^{p_1}(U_+)} < \infty$ *if and only if* $p_1 < 2$.
Proof. (i) We have

$$\|g\|_{H^{p_1}} < \infty \Leftrightarrow \lim_{\rho \uparrow 1} \int_{-\pi/2}^{\pi/2} \frac{1}{|1-\rho^2(\cos(2\theta_1)+i\sin(2\theta_1)|^{p_1/2}} d\theta_1 < \infty \Leftrightarrow$$

$$\lim_{\rho \uparrow 1} \int_{-\pi/2}^{\pi/2} \frac{1}{\left(1+\rho^4-2\rho^2\cos(2\theta_1)\right)^{p_1/4}} d\theta_1 < \infty \Leftrightarrow \lim_{r \uparrow 1} \int_{0}^{\pi} \frac{1}{\left(1+r^2-2r\cos(\theta)\right)^{p_1/4}} d\theta < \infty$$

According to the proof of Theorem 2.1, the latter relation is equivalent to $f \in H^{p_1/2}$, where f is given by (1). But this last relation is true if and only if $p_1/2 < 1$, i. e. $p_1 < 2$ (cf. Theorem 2.1).
(ii) This assertion follows using the computation and the result proved at point (i). Passing to polar coordinates ρ, θ_1 in the double integral from the definition of $\|g\|_{A^p(U_+)}$, and making the change of variables $r = \rho^2 \in]0,1[, \theta = 2\theta_1 \in]-\pi,\pi[$, the conclusion follows. The necessary and sufficient condition for the convergence of that integral is $\frac{p_1}{2} < 1$. See also the proof of Theorem 2.2. □

3. Conclusion

The results claimed in the Abstract have been proved using the tools of real analysis: inequalities, Lebesgue monotone convergence theorem and appropriate changes of variables in integrals. We have given two different new proofs for one of the implications of Theorem 2.1. The proof of the converse implication is also new. One gives complete proofs for the main new results of theorems 2.2 and 2.3. The interested reader might prove similar theorems for other holomorphic functions, eventually over other regions of the complex plane.

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