Journal of Progressive Research in Mathematics(JPRM)
ISSN: 2395-0218

SCITECH
RESEARCH ORGANISATION

Volume 13, Issue 3
Published online: June 15, 2018|

Journal of Progressive Research in Mathematics
www.scitecresearch.com/journals

An Optimal Class of Eighth-Order Iterative Methods Based on King's Method

T.GH. Al-Harbi[1], I.A. Al-Subaihi[2*],

[1,2]Department of Mathematics, Faculty of Science, Taibah University, Saudi Arabia.
Email: [1]t.gh1@hotmail.com, [2*]alsubaihi@hotmail.com.

Abstract

This paper based on King's fourth order methods. A class of eighth-order methods is presented for solving simple roots of nonlinear equations. The class is developed by combining King's fourth-order method and Newton's method as a third step using the forward divided difference and multiplication of three weight function. Some numerical comparisons have been considered to show the performance of the proposed method.

Keywords: Iterative method; optimal method; nonlinear equations; order of convergence.

1. Introduction

To find a simple root of nonlinear equations is the most and oldest problems in numerical analysis. In scientific departments, a need arises to solve nonlinear equations.

$$f(x) = 0, \qquad (1)$$

where $f: D \subset \mathbb{R} \to \mathbb{R}$ for an open interval D.

The Newton's method (NM) is the famous method to solve the nonlinear equation, see for example [1-16].

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \qquad (2)$$

Newton's method is optimal quadratic convergence [1-16]. The efficiency index (EI) can be defined by $\delta^{1/\alpha}$, such that δ is the order of the method and α is number of total function and its derivative evaluations per iteration [7, 14]. According to the optimality, optimal order is given by $2^{\alpha-1}$ [9]. The efficiency index of the method (2), EI = $2^{1/2} \approx 1.4142$.

In recent years there are many methods of the optimal three-step eighth-order to solve nonlinear equations, developed in [1-6]. A family of two steps is proposed by King [2], given by

$$w_n = x_n - \frac{f(x)}{f'(x)},$$
$$z_n = w_n - \frac{f(x) + \beta f(w)}{f(x) + (\beta - 2) f(w)}. \qquad (3)$$

Where $\beta \in \mathbb{R}$, this family is optimal fourth-order of convergence, and has efficiency index $4^{1/3} \approx 1.5874$.

2. The method and convergence analysis

Our aim is to develop a scheme that improves the order of convergence of King's methods (3). Next theorem will be an initial used to develop a new class of families of optimal eight-order of convergence.

Theorem 1.[11]: Let $\varphi_1(x), \varphi_2(x), \ldots, \varphi_s(x)$ be iterative functions with the orders $p_1, p_2, \ldots, p_s$, respectively. Then the composition of iterative functions $\varphi_1\left(\varphi_2(\ldots(\varphi_s(x))\ldots)\right)$, defines the iterative method of the order $p_1 p_2 \cdots p_s$.

Using theorem 1, and adding the Newton's method as a third, to King's method (3)

$$\begin{aligned} w_n &= x_n - \frac{f(x)}{f'(x)}, \\ z_n &= w_n - \frac{f(x) + \beta f(w)}{f(x) + (\beta - 2) f(w)} \frac{f(w)}{f'(x)}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(z_n)}. \end{aligned} \tag{4}$$

The method (4) has IE = $8^{1/5} \approx 1.5157$, and is not optimal. To reduce the numbers of functions evaluation of method (4) to four, to a develop family of optimal eighth-order of convergence methods. Replacing $f'(z_n)$ to $\frac{f[x_n,z_n]\, f[w_n,z_n]}{f[x_n,w_n]}$, using forward divided difference, where:

$f[w_n,z_n] = \frac{f(z_n)-f(w_n)}{z_n - w_n}$, $f[x_n,z_n] = \frac{f(z_n)-f(x_n)}{z_n - x_n}$, $f[x_n,w_n] = \frac{f(x_n)-f(w_n)}{x_n - w_n}$.

To reduce the number of function evaluation from $\alpha = 5$ to $\alpha = 4$, equivalent construction of weighted functions, to increase the convergence third step will be multiply by H, K and B.

$$\begin{aligned} w_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= w_n - \frac{f(x_n) + \beta f(w_n)}{f(x_n) + (\beta - 2) f(w_n)} \frac{f(w_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - \{H(s_1) \cdot K(s_2) \cdot B(s_3)\} \frac{f(z_n) f[x_n,w_n]}{f[w_n,z_n] f[x_n,z_n]} \end{aligned} \tag{5}$$

where $H(s_1), K(s_2), B(s_3)$, are three real-valued weight functions, and

$$s_1 = \frac{f(w)}{f(x)},\ s_2 = \frac{f(z)}{f(w)},\ s_3 = \frac{f(z)}{f(x)}. \tag{6}$$

The weight functions H, K and B should be chosen such that the order of convergence of method (5) arrives at an optimal level of eight. The following theorem we prove that method (5) has an optimal eighth-order of convergence under conditions for the weighted functions.

Theorem 2. Let γ in an open interval D be a simple root of a sufficiently differentiable function f : D $\subseteq R \to R$. an open interval D. If x_0 is sufficiently close to γ then the family of iterative methods (1) has an optimal eighth-order of convergence when

$H(0)=1,\ H'(0) = H''(0) = 0,\ H'''(0) = -12\beta,\ |H^{(4)}(0)| < \infty$

$K(0) = 1,\ K'(0) = 0,\ |K''(0)| < \infty,\ B(0) = 1,\ B'(0) = 1.$

Proof: Let γ be a simple zero of equation (1) and Let $e_n = x_n - \gamma$ be the error at the nth iteration.

Expanding $f(x)$ about γ by Taylor expansion, we have

$$f(x) = f'(\gamma)(e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + c_7 e_n^7 + c_8 e_n^8 + c_9 e_n^9). \tag{7}$$

Where $c_k = \frac{f^{(k)}(\gamma)}{k! f'(\gamma)}$, $k = 2,3, \ldots$.

$$f'(x) = f'(\gamma)(1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + 6c_6 e_n^5 + 7c_7 e_n^6 + 8c_8 e_n^7 + 9c_9 e_n^8). \tag{8}$$

Dividing (7) by (8), gives us

$$\frac{f(x)}{f'(x)} = e_n + (-c_2)e_n^2 + (2c_2^2 - 2c_3)e_n^3 + \ldots + (-64c_2^7 + \ldots + 135c_2c_3^3 - 44c_2^2c_6 - 118c_2c_3c_5 - 64c_2c_4^2 - 75c_3^2c_4 + 19c_2c_7 + 27c_3c_6 + 31c_4c_5 - 7c_8)e_n^8 + O(e_n^9). \tag{9}$$

Substituting the last equation (9), into first step of (5) , we have:

$$w_n = \gamma + c_2 e_n^{\,2} + (-2c_2^2 + 2c_3)e_n^{\,3} + \ldots + (64c_2^7 - 304c_2^5c_3 + 176c_2^4c_4 + \ldots + 75c_3^2c_4 - 19c_2c_7 - 27c_3c_6 - 31c_4c_5 + 7c_8)e_n^{\,8} + O(e_n^{\,9}). \quad (10)$$

Expanding $f(w_n)$ about γ to get

$$f(w_n) = f'(\gamma)[(-2c_2^2 + 2c_3)e_n^{\,3} + \ldots + (144c_2^7 - 552c_2^5c_3 + 297c_2^4c_4 + \ldots + 73c_2c_4^2 + c_2^4 + 75c_3^2c_4 - 19c_2c_7 - 27c_3c_6 - 31c_4c_5 + 7c_8)e_n^{\,8}] + O(e_n^{\,9}). \quad (11)$$

In view of (7), (8), (9) and (11), we get

$$z_n = \gamma + (2\beta c_2^3 + c_2^3 - c_2c_3)e_n^{\,4} + \ldots + (2\beta^5 c_2^7 + \ldots + 20c_2^2c_6 + 68c_2c_3c_5 + 37c_2c_4^2 + 50c_3^2c_4 - 5c_2c_7 - 13c_3c_6 - 17c_4c_5)\, e_n^{\,8} + O(e_n^{\,9}). \quad (12)$$

From (12), we have

$$f(z_n) = f'(\gamma)[(2\beta c_2^3 + c_2^3 - c_2c_3)e_n^{\,4} + \ldots + (-144\beta^2c_2^2c_3c_4 - 612\beta c_2^2c_3c_4 + 96\beta c_2c_3c_5 + \ldots + 50c_3^2c_4 - 5c_2c_7 - 13c_3c_6 - 17c_4c_5)e_n^{\,8}] + O(e_n^{\,9}) \quad (13)$$

From (7), (10)-(13), it can be easily to found

$$f[x_n, w_n] = f'(\gamma)[(1 + c_2e_n + \ldots + (92c_2^2c_3c_4 - 30c_2c_3c_5 + \ldots + 7c_3c_6 + 7c_4c_5 + c_8 - 32c_2^7)e_n^{\,7}] + O(e_n^{\,8}) \quad (14)$$

$$f[w_n, z_n] = f'(\gamma)\ [1 + c_2^2e_n^{\,2} + \ldots + (-18c_2c_3c_4 - 82\beta c_2^4c_3 + \ldots + 4c_3^3 + 5c_2c_6 + 26c_2^6)e_n^{\,6}] + O(e_n^{\,7}). \quad (15)$$

$$f[x_n, z_n] = f'(\gamma)[1 + c_2e_n + c_3e_n^{\,2} + \ldots + (72c_2^2c_3c_4 + \ldots + 30c_2c_3^3 - 4c_2^2c_6 - 8c_2c_4^2 - 9c_3^2c_4 + c_8 - 20c_2^7)e_n^{\,7}] + O(e_n^{\,8}) \quad (16)$$

By expanding $H(s_1),\ K(s_2),\ B(s_3)$ using Taylor expansion, we have

$$H(s_1) = H(0) + H'(0)s_1 + \frac{1}{2}H''(0)s_1^{\,2} + \frac{1}{6}H'''(0)s_1^{\,3} + \frac{1}{24}H^{(4)}(0)s_1^{\,4} + \ldots + O(s_1^{\,9}). \quad (17)$$

$$K(s_2) = K(0) + K'(0)s_2 + \frac{1}{2}K''(0)s_2^{\,2} + \frac{1}{6}K'''(0)s_2^{\,3} + \ldots + O(s_2^{\,9}). \quad (18)$$

$$B(s_3) = B(0) + B'(0)s_3 + \ldots + O(s_3^{\,9}). \quad (19)$$

Finally, using (12)-(16) and the conditions

$H(0) = 1,\ H'(0) = H''(0) = 0,\ H'''(0) = -12\beta,\ |H^{(4)}(0)| < \infty$

$K(0) = 1, K'(0) = 0, B(0) = 1,\ B'(0) = 1,\ |K''(0)| < \infty,$

we obtain the error expression

$$e_{n+1} = \gamma + \left(-\frac{1}{12}\beta c_2^7 H^{(4)}(0) - \frac{1}{2}c_2^7 K''(0) + \ldots + 4c_2^3c_3^2\right)e_n^{\,8} + O(e_n^{\,9}). \quad (20)$$

Which indicates that the order of convergence of the family (5) is exactly eighth for any value of $\beta \in R$. This completes the proof.

3. The Concrete Iterative Methods

In what follows, we give some iterative forms of scheme (5)

Method1: choosing

$H(s_1) = 1 + (-2\beta s_1^{\,3}) + (ds_1^{\,a+1})$, *where*, $a \geq 3,\ a,\ d,\ \beta \in R$,

$K(s_2) = 1 + (t\, s_2^{\,b})$, $b > 1,\ t \in R$,

$B(s_3) = 1 + s_3 + (m\, s_3^{\,u})$, *where*, $u > 1,\ u,\ m \in R$, it can easily be seen that the functions satisfies conditions of Theorem 2 . We get another eighth order methods (TSM1), given by

$$w_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = w_n - \frac{f(x_n)+\beta f(w_n)}{f(x_n)+(\beta-2)f(w_n)}\frac{f(w_n)}{f'(x_n)},$$

$$x_{n+1}= z_n-\{t\, s_2{}^b(d\, m\, s_1{}^{a+1}s_3{}^u + d\, s_3\, s_1{}^{a+1} + d\, s_1{}^{a+1} - 2\,\beta\, m\, s_1{}^3 s_3{}^u + m\, s_3{}^u - 2\,\beta\, s_1{}^3 - 2\,\beta\, s_1{}^3 s_3 + s_3 + 1) + d\, s_1{}^{a+1}(m\, s_3{}^u + s_3 + 1) - 2\,\beta\, s_1{}^3(m\, s_3{}^u + s_3 + 1) + m\, s_3{}^u + s_3 + 1\}\frac{f(z_n)f[x_n,w_n]}{f[w_n,z_n]f[x_n,z_n]} \quad (21)$$

Method 2 : Let

$H(s_1) = 1 + (-2\beta s_1{}^3)$, $\quad \beta \in R$,
$K(s_2) = 1 + s_2{}^j\, e^{s_2}$, $\quad j \in R$,
$B(s_3) = \sin(s_3) + \cos(s_3)$,

it can easily be seen that the functions satisfies conditions of Theorem 2 .

Hence we get a family of eighth-order of method , (TSM2),

$$w_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = w_n - \frac{f(x_n)+\beta f(w_n)}{f(x_n)+(\beta-2)f(w_n)}\frac{f(w_n)}{f'(x_n)},$$

$$x_{n+1}= z_n-\{s_2{}^j\, e^{s_2}\{\{sin(s_3)+cos(s_3)\}\{-2\beta\, s_1{}^3+1\}\} + \{sin(s_3)+cos(s_3)\}\{-2\beta\, s_1{}^3+1\}\}\frac{f(z_n)f[x_n,w_n]}{f[w_n,z_n]f[x_n,z_n]}. \quad (22)$$

Method 3: For the function H, K and B defined by

$H(s_1) = 1 + (-2\beta s_1{}^3 e^{s_1})$, $\quad \beta \in R$,
$K(s_2) = 1 + s_2{}^4$,
$B(s_3) = 1 + \sin(s_3)$.

Another new eighth order methods (TSM3), given by

$$w_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = w_n - \frac{f(x_n)+\beta f(w_n)}{f(x_n)+(\beta-2)f(w_n)}\frac{f(w_n)}{f'(x_n)},$$

$$x_{n+1}= z_n-\{-2\,\beta s_1{}^3 e^{s_1}(s_2{}^4 + s_2{}^4 sin(s_3) + sin(s_3) + 1) + (s_2{}^4 + 1)(sin(s_3)+1)\}\frac{f(z_n)f[x_n,w_n]}{f[w_n,z_n]f[x_n,z_n]}. \quad (23)$$

Method 4: let

$H(s_1) = \cos(s_1) + \frac{s_1{}^2}{2} + (-2\beta s_1{}^3)$, $\quad \beta \in R$,
$K(s_2) = 1 + t\, s_2{}^l + 2s_2{}^q$, $\quad l,q > 1$, $\quad l,q,t \in R$,
$B(s) = 1 + s_3$.

Another new eighth order methods (TSM4), given by

$$w_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = w_n - \frac{f(x_n)+\beta f(w_n)}{f(x_n)+(\beta-2)f(w_n)}\frac{f(w_n)}{f'(x_n)},$$

$$x_{n+1}= z_n-\{2\,\beta s_1{}^3\{(-ts_2{}^l - 2s_2{}^q - 1)(s_3+1)\} + s_1{}^2\{(\frac{t s_2{}^l}{2} + s_2{}^q + \frac{1}{2} s_3 + 1)\} + cos(s_1)\{(ts_2{}^l + 2s_2{}^q + 1)(s_3+1)\}\}\frac{f(z_n)f[x_n,w_n]}{f[w_n,z_n]f[x_n,z_n]}. \quad (24)$$

Method 5: let

$H(s_1) = 1 + (-2\beta s_1{}^3)$, $\quad \beta \in R$,
$K(s_2) = \cos(s_2) + (t\, s_2{}^g)$, $\quad g > 1$, $g,t \in R$,
$B(s_3) = \cos(s_3) + s + (m\, s_3{}^\sigma)$, $\quad \sigma > 1$, $\quad \sigma, m \in R$.

A new eighth order methods (TSM5)

$$w_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$z_n = w_n - \frac{f(x_n) + \beta f(w_n)}{f(x_n) + (\beta - 2) f(w_n)} \frac{f(w_n)}{f'(x_n)},$$
$$x_{n+1} = z_n - \{2\beta s_1^3 (t s_2^g(-m s_3^\sigma - s_3 - \cos(s_3))) + (t s_2^g + \cos(s_2) - 2\beta s_1^3 \cos(s_2))(m s_3^\sigma + s_3 + \cos(s_3))\} \frac{f(z_n) f[x_n, w_n]}{f[w_n, z_n] f[x_n, z_n]}. \quad (25)$$

4. Numerical results

In this section, we present some numerical test to illustrate the efficiency of the new methods. We compared the performance of (TSM1-TSM5) of the new optimal eighth-order methods, with the Newton's method (NM), (2), King's method (KM), (3), and some optimal eighth order methods for example, (BWRM), proposed by Bi–Wu–Ren in [3], given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$z_n = y_n - \frac{2f(x_n) - f(y_n)}{2f(x_n) - 5f(y_n)} \frac{f(y_n)}{f'(x_n)},$$
$$x_{n+1} = z_n - \frac{f(x_n) + (2+\delta) f(z_n)}{f(x_n) + \delta f(z_n)} \frac{f(z_n)}{f[z_n, y_n] + (z_n - y_n) f[z_n, x_n, x_n]}, \quad (26)$$

where $\delta = 0$, method (LWM) proposed by Liu and Wang [10], given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$z_n = y_n - \frac{f(x_n)}{f(x_n) - 2f(y_n)} \frac{f(y_n)}{f'(x_n)},$$
$$x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)} \left[\left(\frac{f(x_n) - f(y_n)}{f(x_n) - 2f(y_n)} \right)^2 + \frac{f(z_n)}{f(y_n) - \mu f(z_n)} + \frac{4f(z_n)}{f(x_n) + \beta f(z_n)} \right], \quad (27)$$

where $\beta = 0$, $\mu = 1$, and method (SM) proposed by Sharma in [13], given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$z_n = y_n - \frac{f(x_n)}{f(x_n) - 2f(y_n)} \frac{f(y_n)}{f'(x_n)},$$
$$x_{n+1} = z_n - \left[1 + \frac{f(z_n)}{f(x_n)} + \left(\frac{f(z_n)}{f(x_n)} \right)^2 \right] \frac{f[x_n, y_n] f(z_n)}{f[x_n, z_n] f[y_n, z_n]}, \quad (28)$$

Table 1. Test functions

Functions	Roots
$f_1(x) = x^2 - (1-x)^{25}$	$\gamma = 0.143739259299754$
$f_2(x) = \cos(x) - x$	$\gamma = 0.739085133215161$
$f_3(x) = 10e^{-x^2} - 1$	$\gamma = 1.67963061042845$
$f_4(x) = \log(x^4 + x + 1) + xe^x$	$\gamma = 0.0$

All computations were done by MATLAB (R2017a), using 1000 digits floating point (digits:= 1000). The stopping criteria are

i. $|x_n - \gamma| \le 10^{-200}$,
ii. $|f(x_n)| \le 10^{-200}$.

Displayed in the Table2, the number of iterations denoted by (IT), the absolute value of the function $|f(x_n)|$, the absolute error, $|x_n - \gamma|$. Moreover, computational order of convergence (COC) approximated by [15]

$$\rho = \frac{ln|(x_{n+1} - \gamma)/(x_n - \gamma)|}{ln|(x_n - \gamma)/(x_{n-1} - \gamma)|}.$$

Remark: The weighted function in (TSM1), choosing $a = 3, d = t = m = 1, u = 2$, for (TSM2), $j = 1$, for (TSM4), $l = q = 2,\ t = 1$, for method (TSM5), choosing $t = m = 1, \sigma = g = 2$. The last four methods (TSM2-TSM5) had been tested at only one value for beta (β = 0) except the first one (TSM1) it had been tested at four different values for beta $(\beta = 0, 2, \frac{1}{4}, -1)$.

1. Conclusion

In this paper, a class optimal eighth-order iterative method for solving nonlinear equations has been developed. New proposed families are obtained by approximating $f'(z_n)$ using divided difference and the equivalent construction of weighted function. The new class of methods have $\text{IE} = 8^{1/4} \approx 1.682$, and four functions evaluation. Comparing other methods using numerical examples to explain the convergence of the new methods.

Table 2. Numerical comparison.

Method	IT	$\|f(x_n)\|$	$\|x_n - \gamma\|$	COC
$f_1(x) = x^2 - (1-x)^{25}, x_0 = 0.35$				
NM	10	3.32259e-325	3.73026e-325	2
KM	5	2.16538e-314	2.43107e-314	4
BWRM	4	2.19735e-617	2.46696e-617	8
SHM	4	0	0	6.4216
LWM	4	1.15158e-752	1.29287e-752	8
TSM1, $\beta = 0$	4	6.71351e-669	7.53725e-669	8
TSM1, $\beta = 2$	4	4.22806e-795	4.74683e-795	8
TSM1, $\beta = \frac{1}{4}$	4	9.4418e-705	1.06003e-704	8
TSM1, $\beta = -1$	4	6.59206e-390	7.4009e-390	8
TSM2	5	9.28842e-358	1.04281e-357	8
TSM3	5	7.88589e-545	8.85348e-545	8
TSM4	6	2.72499e-829	3.05934e-829	8
TSM5	5	2.75948e-654	3.09806e-654	8
$f_2(x) = \cos(x) - x,\ x_0 = 0.6$				
NM	8	3.00558e-379	1.79587e-379	2
KM	4	1.10351e-349	6.5936e-350	4
BWRM	3	8.27062e-721	4.94178e-721	8
SHM	3	9.49798e-674	5.67514e-674	8
LWM	3	2.57724e-626	1.53992e-626	8
TSM1, $\beta = 0$	3	4.04228e-689	2.4153e-689	8
TSM1, $\beta = 2$	3	1.742e-614	1.04086e-614	8
TSM1, $\beta = \frac{1}{4}$	3	1.79467e-704	1.07233e-704	8
TSM1, $\beta = -1$	3	7.31632e-766	4.37158e-766	8
TSM2	4	1.7944e-673	1.07217e-673	8
TSM3	4	1.01004e-673	6.03511e-674	8
TSM4	4	2.74121e-718	1.6379e-718	8
TSM5	4	2.08987e-688	3.49764e-688	8

Method	IT	$\|f(x_n)\|$	$\|x_n-\gamma\|$	COC
$f_3(x)=10e^{-x^2}-1, x_0=1.4$				
NM	9	1.14863e-358	4.15583e-359	2
KM	5	1.21826e-672	4.40776e-673	4
BWRM	3	2.362e-362	8.54593e-363	8
SHM	3	9.08643e-318	3.28755e-318	8
LWM	-	fails	-	-
TSM1, $\beta=0$	3	2.08835e-332	7.55582e-333	8
TSM1, $\beta=2$	3	4.18088e-422	1.51268e-422	8
TSM1, $\beta=\frac{1}{4}$	3	6.06631e-336	2.19484e-336	8
TSM1, $\beta=-1$	3	1.52893e-364	5.5318e-365	8
TSM2	4	1.51868e-317	5.49473e-318	8
TSM3	4	1.73424e-317	6.27462e-318	8
TSM4	4	2.90408e-381	1.05072e-381	8
TSM5	4	5.87805e-332	2.12673e-332	8
$f_4(x)=\log(x^4+x+1)+xe^x, x_0=0.25$				
NM	8	6.57994e-276	3.28997e-276	2
KM	4	2.03822e-227	1.01911e-227	4
BWRM	3	1.53876e-428	7.69378e-429	8
SHM	3	9.33255e-441	4.66628e-441	8
LWM	3	1.57085e-388	7.85424e-389	8
TSM1, $\beta=0$	3	3.52917e-426	1.76459e-426	8
TSM1, $\beta=2$	3	1.005e-466	5.025e-467	8
TSM1, $\beta=\frac{1}{4}$	3	2.22892e-438	1.11446e-438	8
TSM1, $\beta=-1$	3	1.89319e-387	9.46593e-388	8
TSM2	4	1.16514e-440	5.82572e-441	8
TSM3	4	4.91806e-441	2.45903e-441	8
TSM4	4	2.94173e-417	1.47087e-417	8
TSM5	4	1.70735e-426	8.53677e-427	8

References

[1] I. A. Al-subaihi, A. J. Alqarni. Higher-Order Iterative Methods for Solving Nonlinear Equations, Life Science Journal, 11, 12, pp. 85-91, 2014.

[2] Al-Harbi, I.A. Al-Subaihi. Family of Optimal Eighth-Order of Convergence for Solving Nonlinear Equations, 2015.

[3] W. Bi, Q. Wu, H. Ren, A new family of eighth-order iterative methods for solving nonlinear equations, Appl. Math.**214**, pp.236–245, 2009.

[4] C. Chun, B.Neta, An analysis of a new family of eighth-order optimal methods, Appl. Math. Comput. 245(2014), 86–107.

[5] C. Lee, Chun, M.Y. A new optimal eighth-order family of iterative methods for the solution of nonlinear equations. Appl. Math. Comput., 223: 506–519 (2013).

[6] A. Cordero, J. R. Torregrosa, and M. P. Vassileva, "Three-step iterative methods with optimal eighthorder convergence," *Journal of Computational and Applied Mathematics*, vol. 235, no. 10, pp. 3189–3194, 2011.

[7] W. Gautschi, Numerical Analysis, An Introduction,Birkhauser, Barton, Mass, USA, 1997.

[8] R. F. King, "A family of fourth order methods for nonlinear equations," SIAM Journal on Numerical Analysis, vol. 10, pp. 876–879, 1973.

[9] H. T. Kung and J. F. Traub, "Optimal order of one-point and multipoint iteration," Journal of the Association for Computing Machinery, **21**, pp. 643–651, 1974 .

[10] L. Liu, X. Wang, Eighth-order methods with high efficiency index for solving nonlinear equations, Applied Mathematics and Computation, **215**, pp.3449-3454, 2010.

[11] S. Miodrag Petkovic, Ljiljana D. Petkovic, Families Of Optimal Multipoint Method For Solving Nonlinear Equations: A SURVEY, Appl. Anal. Discrete Math. **4**, pp.1-22, 2010.

[12] A.M. Ostrowski, Solution of Equations in Euclidean and Banach Spaces, Academic Press, New York, 1960.

[13] R. Sharma, Sharma, A new family of modified Ostrowski's methods with accelerated eighth order convergence, Numerical Algorithms, **54**, pp. 445-458, 2010.
[14] J. Traub, Iterative methods for the solution of equations, American Mathematical Soc, 1982.
[15] S. Weerakoon, G. I. Fernando. A variant of Newton's method with accelerated third-order convergence, Appl. Math. Lett., **17**, 8, pp.87-93, 2000 .
[16] Zhanlav, V. Ulziibayar. Modified King's Methods with Optimal Eighth-order of Convergence and High Efficiency Index, American Journal of Computational and Applied Mathematics, 2016.