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The Banach Fixed Point Theorem for mappings in general $(<\mathcal{R}, R_0>, \varphi)$ -spaces

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Abstract.

The paper includes theorem giving the sufficient condition to the existence of a fixed point for mappings in arbitrary set equipped with the family of binary reflexive and symmetric relations satisfying some conditions. The result obtained is a generalization of the main theorem from [7].

Introduction

Let (X, d) be a complete metric space. The contraction ([1], [2]) is a mapping $F : X \to X$ such that there exists $L \in]0, 1[$ for which $d(Fx, Fy) \leq L \cdot d(x, y)$, for all $x, y \in X$. The well known Banach Fixed Point Theorem formulated for (X, d) ([1], [2]) reads as follows. If $F : X \to X$ is a contraction, then there exists exactly one fixed point, i.e. the solution of $Fx = x$. The Banach Fixed Point Theorem is an important tool in mathematical analysis and has been investigated under various conditions and developed in different directions ([3], [4], [5], [6], [8], [9],[10]). In the paper [7] it was defined the notion of $(\mathcal{R}, \varphi)$-*space* with Banach Fixed Point Theorem in it. In the presented paper is given a generalization of definitions and the main theorem from [7].

1 Notations, definitions, lemma

Let X, T be the arbitrary sets, $\varphi : T \to T$ be a fixed bijection and R_0 be an equivalence relation in X, so $R_0 \supseteq I := \{(x, x) : x \in X\}$. Moreover, let $\mathcal{R} = \{R_t\}_{t\in T}$ be a family of binary reflexive and symmetric relations in X forming a chain such that $\bigcup_{t\in T} R_t = X \times X$, $\bigcap_{t\in T} R_t = R_0$. Additionally we suppose that the family $\mathcal{R}$ satisfies the following condition of φ-transitivity

$$\forall t \in T\ \forall x, y, z \in X : \left[(x, y) \in R_t, (y, z) \in R_t \Rightarrow\ (x, z) \in R_{\varphi(t)}\right].$$

The examples of such families are given in the next part of the paper.

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Definition 1.1. A set X with the family of relations described above will be called in the sequel by *general* $(<\mathcal{R}, R_0>, \varphi)$*-space*.

Below is defined the notion of convergent sequence in $(<\mathcal{R}, R_0>, \varphi)$-space X.

Definition 1.2. We say that a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of general $(<\mathcal{R}, R_0>, \varphi)$-space X is $\mathcal{R}$*-convergent to* $x_0 \in X$ in X, which is denoted by $\lim_{n\to\infty} x_n \overset{\mathcal{R}}{=} x_0$ (or $x_n \overset{\mathcal{R}}{\to} x_0$), if and only if

$$\forall t \in T\ \exists N(t) \in \mathbb{N}\ \forall n \geq N(t): \quad (x_n, x_0) \in R_t.$$

In this case x_0 will be called $\mathcal{R}$-limit.

Lemma 1.3. *Each sequence in general $(<\mathcal{R}, R_0>, \varphi)$-space X has at most one $\mathcal{R}$-limit with respect to R_0, i.e. for every such limits $x_0, x_0^{'}$ we have $(x_0, x_0^{'}) \in R_0$.*

Proof. Let us suppose "ad absurdum" that a sequence $(x_n)_{n\in\mathbb{N}}$ has two distinct $\mathcal{R}$-limits $x_0 \neq x_0^{'}$ and $(x_0, x_0^{'}) \notin R_0$. Since $\bigcap_{t\in T} R_t = R_0$ then there exists $\bar{t} \in T$ for which $(x_0, x_0^{'}) \notin R_{\bar{t}}$. Let us take $t_0 \in T$ such that $\varphi(t_0) = \bar{t}$. By $\mathcal{R}$-convergence of the sequence $(x_n)_{n\in\mathbb{N}}$ to x_0 and to $x_0^{'}$ we have $(x_n, x_0) \in R_{t_0}$ and $(x_n, x_0^{'}) \in R_{t_0}$, for sufficiently large $n \in \mathbb{N}$. Hence, by supposition of φ-transitivity, we get the contradiction $(x_0, x_0^{'}) \in R_{\varphi(t_0)} = R_{\bar{t}}$. □

Definition 1.4. A sequence $(x_n)_{n\in\mathbb{N}}$ of elements of general $(<\mathcal{R}, R_0>, \varphi)$-space X is $\mathcal{R}$*-Cauchy sequence* if and only if

$$\forall t \in T\ \exists N(t) \in \mathbb{N}\ \forall n, m \geq N(t): \quad (x_n, x_m) \in R_t.$$

Definition 1.5. A general $(<\mathcal{R}, R_0>, \varphi)$-space X is called *complete* if and only if every $\mathcal{R}$-Cauchy sequence is $\mathcal{R}$-convergent in X.

The next definition are the generalization of definitions from [7].

Definition 1.6. For general $(<\mathcal{R}, R_0>, \varphi)$-space X, a mapping $f : X \to X$ is called $<\mathcal{R}, R_0>$*-contraction* if the following condition is satisfied for all $x, y \in X$

$$\forall t \in T: \quad \left[(x, y) \in R_t \ \Rightarrow \left(\exists \bar{t} \in T: \ (f(x), f(y)) \in R_{\bar{t}} \text{ and } R_{\bar{t}} \subsetneq R_t\right)\right].$$

and

$$\text{if } (x, y) \in R_0 \text{ then } (f(x), f(y)) \in R_0.$$

Definition 1.7. A general $(<\mathcal{R}, R_0>, \varphi)$-space X is called *strong* if and only if the intersection of every sequence $(R_{t_n})_{n\in\mathbb{N}}$ of shrinking relations equals R_0 and

$$\forall t_1, t_2 \in T: \ R_{t_1} \subsetneq R_{t_2} \ \Rightarrow \ R_{\varphi(t_1)} \subsetneq R_{\varphi(t_2)} \tag{1.1}$$

and for every $n \in \mathbb{N}, n \geq 3$, for all $t_1, t_2, ..., t_{n-1} \in T$ if $R_{t_{n-1}} \subsetneq R_{t_{n-2}} \subsetneq ... \subsetneq R_{t_1}$ then

$$(x_1, x_2) \in R_{t_1}, (x_2, x_3) \in R_{t_2}, ..., (x_{n-1}, x_n) \in R_{t_{n-1}} \ \Rightarrow \ (x_1, x_n) \in R_{\varphi(t_1)} \tag{1.2}$$

for all $x_1, ..., x_n \in X$.

Definition 1.8. A point $x^*$is called R_0-*fixed point* of f if $(x^*, f(x^*)) \in R_0$ (Evidently if $R_0 = I$ then I-fixed point is the classical fixed point of f).

2 Main theorem

Theorem 2.1. *Let X be a general, strong and complete $(< \mathcal{R}, R_0 >, \varphi)$-space and let the mapping $f : X \to X$ be an $< \mathcal{R}, R_0 >$-contraction. The above suppositions imply the existing of exactly one (with respect to R_0) R_0-fixed point x^* of f, so $(x^*, f(x^*)) \in R_0$.*

Proof. Choose $x_0 \in X$. Define the iterative sequence

$$x_n = f(x_{n-1}), \quad \text{for } n \in \mathbb{N},$$

so $x_1 = f(x_0)$, $x_2 = f(x_1) = f^2(x_0)$, ... , $x_{n-1} = f(x_{n-2}) = f^{n-1}(x_0)$, $x_n = f(x_{n-1}) = f^n(x_0), \ldots$. Let $n, m \in \mathbb{N}, n < m$. We have

$$(x_n, x_m) = (f^n(x_0), f^m(x_0)).$$

Let us assume that $(x_0, x_1) \in R_{t_0}$. Since the mapping f is $< \mathcal{R}, R_0 >$-contraction then we have

$$(x_1, x_2) = (f(x_0), f(x_1)) \in R_{t_1} \text{ and } R_{t_1} \subsetneq R_{t_0},$$

$$(x_2, x_3) = (f(x_1), f(x_2)) \in R_{t_2} \text{ and } R_{t_2} \subsetneq R_{t_1}.$$

Continuing

$$(x_{n-1}, x_n) = (f(x_{n-2}), f(x_{n-1})) \in R_{t_{n-1}} \text{ and } R_{t_{n-1}} \subsetneq R_{t_{n-2}},$$

$$(x_n, x_{n+1}) = (f(x_{n-1}), f(x_n)) \in R_{t_n} \text{ and } R_{t_n} \subsetneq R_{t_{n-1}},$$

$$\ldots$$

$$(x_{m-1}, x_m) = (f(x_{m-2}), f(x_{m-1})) \in R_{t_{m-1}} \text{ and } R_{t_{m-1}} \subsetneq R_{t_{m-2}},$$

From the above by (1.2)

$$(x_n, x_m) \in R_{\varphi(t_n)},$$

which means -by (1.1)- that $(x_n)_{n \in \mathbb{N}}$ is an $\mathcal{R}$-Cauchy sequence. Let x^* be its $\mathcal{R}$-limit. We have

$$\forall t \in T : (x_n, x^*) = (f(x_{n-1}), x^*) \in R_t, \quad \text{for all sufficiently large } n \in \mathbb{N}.$$

Similarly,

$$\forall t \in T : (x_{n-1}, x^*) \in R_t, \quad \text{for all sufficiently large } n \in \mathbb{N}$$

and since f is a $< \mathcal{R}, R_0 >$-contraction then we have also

$$\forall t \in T : (f(x_{n-1}), f(x^*)) \in R_t, \quad \text{for all sufficiently large } n \in \mathbb{N}.$$

From the above, by φ-transitivity

$$\forall t \in T: \ (x^*, f(x^*)) \in R_{\varphi(t)},$$

and considering that $\bigcap_{t\in T} R_{\varphi(t)} = R_0$ we get $f(x^*)R_0x^*$. Now, we will prove that this is the only one (with respect to R_0) R_0-fixed point. Let us suppose that $f(x_1^*)R_0x_1^*$, $f(x_2^*)R_0x_2^*$. By definition 1.6, we get easily

$$(x_1^*, f^n(x_1^*)) \in R_0, \text{ for all } n \in \mathbb{N}$$

and

$$(x_2^*, f^n(x_2^*)) \in R_0, \quad \text{for all } n \in \mathbb{N}.$$

Let $t_{s_1} \in T$ be such as $(x_1^*, x_2^*) \in R_{t_{s_1}}$. By the same definition 1.6 we get $(f(x_1^*), f(x_2^*)) \in R_{t_{s_2}}$ and $R_{t_{s_2}} \subsetneq R_{t_{s_1}}$, so

$$\forall n \in \mathbb{N} \ (f^n(x_1^*), f^n(x_2^*)) \in R_{t_{s_n}} \subsetneq R_{t_{s_{n-1}}}.$$

From the above $\forall n \in \mathbb{N}$ $(x_2^*, f^n(x_1^*)) \in R_{\varphi(t_{s_n})}$ and

$$\forall n \in \mathbb{N} \ (x_1^*, x_2^*) \in R_{\varphi^2(t_{s_n})} \quad \Rightarrow \quad (x_1^*, x_2^*) \in R_0,$$

and the proof of theorem is finished. □

3 Examples, Remark, Problem

Example 3.1. Let $X := \mathbb{R}$. Define $T := \mathbb{R}_+$ and $\varphi : T \to T$, $\varphi(t) := 2t$. Let $A := \{(2z, 2z+1), (2z+1, 2z) : \ z \in \mathbb{Z}\}$, $R_0 := I \cup A$ and

$$\forall t \in T: \ R_t := \{(x, y) \in X^2 : \ |x - y| \leq t\} \cup A. \tag{3.1}$$

Then $\mathbb{R}$ forms the complete $(<\mathcal{R}, R_0>, \varphi)$-space.

Example 3.2. Let $X := \mathbb{R}$, $T := \{..., 2^n, ..., 64, 32, 16, 8, 4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ..., \frac{1}{2^n}, ...\}$. We put $\varphi : T \to T$ as $\varphi(t) := 2t$. The set A and the relation R_0 are the same as in example 3.1. Let us define the family $\mathcal{R} = \{R_t\}_{t\in T}$ of reflexive and symmetric binary relations in X as in (3.1). The set X forms the complete and strong $(<\mathcal{R}, R_0>, \varphi)$-space.

Example 3.3. Let $X := \{f : \mathbb{R} \to \mathbb{R}\}$ be the set of functions such that $f(x) = 0$ for $x \in U$, where U is a neighborhood of zero. Let

$$T := \left\{2^{10}, 2^9, ..., 64, 32, 16, 8, 4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ..., \frac{1}{2^9}, \frac{1}{2^{10}}, \frac{1}{2^{11}}, ...\right\}.$$

We put $\varphi : T \to T$ as $\varphi(t) := t$. Let us define the family $\mathcal{R} = \{R_t\}_{t\in T}$ of binary reflexive and symmetric relations in X as follows

$$\forall t \in T: \ R_t := \{(f, g) \in X^2 : \ \forall x \in [-t, t] \ f(x) = g(x)\}.$$

Moreover

$$R_0 := R_{2^{10}} = \Big\{(f,g) \in X^2 :\ \forall x \in \Big[-2^{10}, 2^{10}\Big]\ f(x) = g(x)\Big\}.$$

The set X forms complete and strong $(<\mathcal{R}, R_0>, \varphi)$-space. One can observe that the mapping $\Phi : X \to X$ defined as follows

$$\Phi(f)(x) := \begin{cases} f\Big(\frac{1}{2}x\Big), & \text{for } x \in [-2^{10}, 2^{10}], \\ f(x), & \text{for } x \notin [-2^{10}, 2^{10}]. \end{cases}$$

is an $<\mathcal{R}, R_0>$-contraction. One can observe easily that every function of the form

$$f(x) = \begin{cases} 0, & \text{for } x \in [-2^{10}, 2^{10}], \\ g(x), & \text{for } x \notin [-2^{10}, 2^{10}]. \end{cases},$$

where $g(x)$ is an arbitrary function, is the fixed point of Φ. Evidently, for arbitrary two such functions f_1, f_2 we have $(f_1, f_2) \in R_0$.

Remark 3.4. One can observe easily that if we put $R_0 := I$ in theorem 2.1 we get the main theorem from [7].

Problem 3.5. Let me suggest to the readers as the problem. Find interesting examples of $(<\mathcal{R}, R_0>, \varphi)$-spaces, $<\mathcal{R}, R_0>$-contractions and other applications of the proved theorem.

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