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On direct product pure-1-2-3 subgroups in abelian group $G_n \times G_m$

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Abstract.

In this paper, we shall define new subgroups which are called $pure-1-2-3$ in abelian groups $G_n \times G_m$ for all $n,m \in N$ which are a family of pure subgroups.

In [1], [2] H.M.A.Abdullah gave the some general properties of $pure-1-2-3$ in abelian group G, but here, we shall prove more than properties os this subgroups in Mod $G_n \times G_m$, which are not valid for pure subgroups.

Keywords: Subgroup; Abelian group; Direct product; Pure; Pure-1; Pure-2; Pure-3.

1. Introduction

we shall use following definitions, to get the results.

Definition 1.1 *A subgroup* S *of* G *said to be* $pure-1$ *in* G *if for all* $x \in S$ *and for all prime* p, $px = x$.

Remark 1.2

1. If S is $pure-1$ in G then $\forall x \in S$ we have $p^{nk}|_x \ n,k \in Z^+$ in S.
2. Every $pure-1$ subgroup is $pure$ in G.
3. Every $pure-1$ subgroup is divisible.

Definition 1.3.

Let A,B *are* $pure-1$ *subgroups in* G_n, G_m. *Then we shall called a direct product* $A \times B$ *is* $pure-1$ *in* $G_n \times G_m$ *if* A *and* B *are* $pure-1$ *in* G_n, G_m. *Which means that* $\forall x = (a,b) \in A \times B$ $\quad a \in A, b \in B$ then $p(a,b) = (pa, pb) = (a,b)$.

Now, we are ready to show some results relating to the $direct - pure-1$.

Theorem 1.4 *Let* $A_1 \times B_1$ *and* $A_2 \times B_2$ *are two direct product* $pure-1$ *in* $G_n \times G_m$.

Then

1. $A_1 \times B_1 \bigcap A_2 \times B_2$ is $pure-1$ in $G_n \times G_m$.
2. $A_1 \times B_1 + A_2 \times B_2$ is $pure-1$

Proof.

1. Let $x \in A_1 \times B_1 \bigcap A_2 \times B_2$ so $x \in A_1 \times B_1$ is $pure-1$, thus $\forall p$

$p(a,b) = x = (a,b) \in A_1 \times B_1 \quad a \in A_1, b \in B_1$.

And $A_2 \times B_2$ is $pure-1$, then $p(a_0,b_0) = x = (a_0,b_0) \in A_2 \times B_2$.

Therefore, we get

$p(a,b) = p(a_0,b_0) = x = (a,b) = (a_0,b_0)$

Thus, $a = a_0 \quad and\ x = (a,b)\ b = b_0$

so $px = p(a,b) = (a,b) = x \in A_1 \times B_1 \bigcap A_2 \times B_2$.

Thus, $A_1 \times B_1 \bigcap A_2 \times B_2$ is $pure-1$.

2. Let x be any element belong to $A_1 \times B_1 + A_2 \times B_2$

So, $x = ((a_1,b_1),(a_2,b_2)) \quad (a_1,b_1) \in A_1 \times B_1, (a_2,b_2) \in A_2 \times B_2$

but, both $A_1 \times B_1$ and $A_2 \times B_2$ are $pure-1$. Thus $\forall p$

$$p(a_1,b_1) = (a_1,b_1) \tag{1.1}$$

$$p(a_2,b_2) = (a_2,b_2) \tag{1.2}$$

By (1.1) and (1.2) we get

$p(a_1,b_1) + p(a_2,b_2) = ((a_1,b_1) + (a_2,b_2)) = x$

$p((a_1,b_1) + (a_2,b_2)) = x$, but $(a_1,b_1) + (a_2,b_2) \in A_1 \times B_1 + A_2 \times B_2$

So, $px = x$.We obtain

$A_1 \times B_1 + A_2 \times B_2$ is $pure-1$.

Theorem 1.5 *If* $A \times B$ *is any* $pure-1$ *in* $G_n \times G_m$, *and* $H \times K$ *any subgroup of* $G_n \times G_m$, *then* $A \times B \bigcap H \times K$ is $pure$ in $G_n \times G_m$.

Proof.

Let x any element in $A \times B \bigcap H \times K$. So $x \in A \times B\ and\ x \in H \times K$. Since $A \times B$ is a $pure-1$ in $G_n \times G_m$, so $\forall p$ (p is prime number)

$px = x$. But $x \in A \times B \bigcap H \times K$.

Hence, $A \times B \bigcap H \times K$ is $pure-1$.

Easily to show the following:

Theorem 1.6

1. If $\overline{A}\times\overline{B}$ is any subgroup of $pure-1$ subgroup $\breve{A}\times\breve{B}$ of $G_n\times G_m$. Then $\overline{A}\times\overline{B}$ is a $pure-1$.
2. If $A\times B$ is a $pure-1$ in $G_n\times G_m$ and $\overline{A}\times\overline{B}$ any subgroup of $\breve{A}\times\breve{B}$ then

$\frac{A\times B}{\overline{A}\times\overline{B}}$ is $pure-1$ in $\frac{G_n\times G_m}{\overline{A}\times\overline{B}}$

Proof.

1. Let $x\in\overline{A}\times\overline{B}$, so $x\in A\times B$ but $A\times B$ is $pure-1$ in $G_n\times G_m$, then $\forall p$, we have

$p(a,b)=(a,b)=x\in\overline{A}\times\overline{B}$ for some $a\in A, b\in B$.

So, $\overline{A}\times\overline{B}$ is $pure-1$.

2. Let $(a,b)+\overline{A}\times\overline{B}\in\frac{A\times B}{\overline{A}\times\overline{B}}$.

Since $(a,b)\in A\times B$ and $A\times B$ is a $pure-1$ in $G_n\times G_m$. Thus, we have $\forall p$, $P(a,b)=(a,b)$. Clearly $p(a,b)+\overline{A}\times\overline{B}=p((a,b)+\overline{A}\times\overline{B})=(a,b)+\overline{A}\times\overline{B}$. Thus, $\frac{\breve{A}\times\breve{B}}{\overline{A}\times\overline{B}}$ is $pure-1$ in $\frac{G_n\times G_m}{\overline{A}\times\overline{B}}$.

Definition 1.7 *A subgroup* $A\times B$ *is said to be* $pure-2$ *in* $G_n\times G_m$ *if* $\forall x, x\in A\times B$ and $\forall p, p$ is prime number, then $px=p(px)$.

Remark 1.8

1. It is clear that any $pure-1$ is a $pure-2$.
2. $\forall x, x\in A\times B$ and if $A\times B$ is $pure-2$ in $G_n\times G_m$, then we are ready to prove the following results of $pure-2$.

Theorem 1.9 *Any* $pure-2$ *subgroup of Torsion-free group is* $pure-1$.

Proof.

Let $A|timesB$ be any $pure-2$ in Torsion-free $G_n\times G_m$ and $x\in A\times B$, so $\forall p, px=p(px)$, put $x=(a,b)\in A\times B$, thus $p(a,b)=p(p(a,b))$, which implies that $p(a,b)-p(p(a,b))=(0,0)$ so $p((a,b)-(p(a,b)))=(0,0)$. But $G_n\times G_m$ is a Torsion-free. Then $P(a,b)=(a,b)$, we obtain the result.

Theorem 1.10 *Any* $pure-2$ *subgroup of Torsion-free group is*

1. $pure$.
2. $pure-1$.

Proof.

1. Let $A\times B$ be any $pure-2$ in $G_n\times G_m$, we claim that $A\times B$ is a neat in $G_n\times G_m$.

Let $x \in A \times B$, and suppose that $\forall p, p|_x$ in $G_n \times G_m$, so $x \in p_{G_n \times G_m} \bigcap_{A\times B}$. Thus,

$$x = p(g_n, g_m) \quad for some\, g_n \in G_n, g_m \in G_m. \tag{1.3}$$

Since $A \times B$ is a $pure-2$, so we can write $px = p(px)$,

so $p(p(g_n, g_m)) = pp(x) \Rightarrow p^2(g_n, g_m) = p^2 x$ but $G_n \times G_m$ is Torsion-free we get $(g_n, g_m) = x$ by (1.3) we can obtain $x = (g_n, g_m) = p(g_n, g_m) \in p(A \times B)$.

Which means that $p|_x \in A \times B, by$ ([3] $p.q2$) We get the result.

2. Let $A \times B$ be any $pure-2$ in a Torsion-free $G_n \times G_m$ for all $x \in A \times B$, so for all prime p we have $px = p(px)$, and let $x = (a,b)$ for some $a \in A, b \in B$ then, $p(a,b) = p(p(a,b))$, But $G_n \times G_m$ is a Torsion-free group. Thus, $p((a,b) - p(a,b)) = 0$ which implies that $(a,b) = p(a,b))$ and we we get . But $\forall (a,b) \in A \times B$ and for all prime p, $p|_{(a,b)}$ in $A \times B$. Moreover, $A \times B$ is a Neat in $G_n \times G_m$.

By the above theorems we get the main results:

Theorem 1.11 *Let $G_n \times G_m$ be a Torsion-free group then the following statements are equivalents:*

1. $A \times B$ is a $pure-1$;
2. $A \times B$ is a $pure-2$;
3. $A \times B$ is a $pure$.

Theorem 1.12 *If $A \times B$ is $pure-2$ and a Neat subgroup of $G_n \times G_m$ then $A \times B$ is $pure$.*

Proof.

Since $A \times B$ is a Neat in $G_n \times G_m$, then we have $p(G_n \times G_m) \bigcap A \times B = p(A \times B), \forall p, p$ is prime number. we shall prove the statement by induction.

So let $\forall p$ and $\forall k \in Z^+$

$$P^k G_n \times G_m \bigcap A \times B = P^k (A \times B) \text{ is true.}$$

We have to show that

$$P^k G_n \times G_m \bigcap A \times B = P^{k+1}(A \times B).$$

The induction $P^{k+1}(A \times B) \subseteq P^{k+1} G_n \times G_m \bigcap A \times B$ is obvious.

Remained to show that $P^{k+1} G_n \times G_m \bigcap A \times B \subseteq P^{k+1}(A \times B)$.

Let $(a,b) \in P^{k+1} G_n \times G_m \bigcap A \times B$ and we may write (a,b) in the form $(a,b) = P^{k+1}(g_n, g_m)$ for some $(g_n, g_m) \in G_n \times G_m$. Thus, $(a,b) = P^k (p(g_n, g_m)) \in A \times B \bigcap P^k G_n \times G_m$ but $A \times B$ is a Neat. Thus,

$$(a,b) = P^k (P(g_n, g_m)) \in A \times B \bigcap P^k G_n \times G_m = P^k (A \times B).$$

Therefore,

$(a,b) = P^k (a_0, b_0)$ for some $(a_0, b_0) \in A \times B$. Since $A \times B$ is $pure-2$, then we get

$(a,b) = P^k(a_0,b_0) = P^{k-1}(P(a_0,b_0)) = P^{k-1}(P(P(a_0,b_0))) = P^{k+1}(a_0,b_0) \in P^{k+1}A\times B$. Assuming $k \geq 1$. So $A\times B \bigcap P^{k+1}(G_n \times G_m) = P^{k+1}(A\times B)$. Consequently $A\times B$ is a $pure$

Now we shall prove some properties of $pure-2$ subgroups, which are valid in $pure-1$.

Theorem 1.13 *If* $A\times B$ *is a* $pure-2$ *subgroup of* $G_n \times G_m$ *then:*

1. $A\times B \bigcap C\times D$ is also $pure-2$ for any subgroup $C\times D$ of $G_n \times G_m$.
2. If $A\times B$ is a $pure-2$ subgroup of $G_n \times G_m$ then any subgroup of $A\times B$ is also $pure-2$.
3. The sum of any two $pure-2$ subgroups of $G_n \times G_m$ will be $pure-2$.
4. If $A\times B$ is a $pure-2$ subgroup of $A\times B$, then $A\times B/c\times D$ is $pure-2$.

Definition 1.14 *A subgroup* $A\times B$ *of* $G_n \times G_m$ *is called* $pure-3$ *if* $(\forall p))$, $(\forall k \in Z^+$ *and* $\forall (a,b) \in A\times B$

$P^k(a,b) = P^k(P(a,b)) = P^{k+1}(a,b)$

Remark 1.15

1. Any $pure-2$ is $pure-3$.
2. If $k=1$, then a $pure-3$ is a $pure-2$.

Now, we are ready to prove the following results of $pure-3$ subgroups.

Theorem 1.16 *If* $A\times B$ *is* $pure-3$ *and Neat subgroup of* $G_n \times G_m$*, then* $A\times B$ *is a* $pure$.

Proof.

Since $A\times B$ is $pure-3$, then we have $\forall p$, $P(G_n \times G_m)\bigcap A\times B = P(A\times B)$.

We will prove the statement by induction. Let $(\forall p),(\forall k)$, $P^k(G_n \times G_m)\bigcap A\times B = P(A\times B)$ is true, so we have to showing that

$$P^{k-1}(G_n \times G_m)\bigcap A\times B = P^{k+1}(A\times B).$$

Let $(a,b) = P^{k+1}(g_n, g_m)$ for some $(g_n, g_m) \in G_n \times G_m$. Thus,

$$(a,b) = P^{k+1}(g_n, g_m) \in A\times B \bigcap P^k(G_n \times G_m) = P_k(A\times B).$$

Therefore, $(a,b) = P^k(a_0,b_0)$ for some $(a_0,b_0) \in A\times B$. Since $A\times B$ is $pure-3$, then $(a,b) = P^k(a_0,b_0) = P^k(P(a_0,b_0)) \in P^{k+1}(A\times B)$.

Consequently $A\times B$ is $pure$.

Now, we shall give the generalization of the Theorem 1.10.

Theorem 1.17 *Any* $pure-3$ *subgroup of Torsion-free* $G_n \times G_m$ *is* $pure$

Theorem 1.18 *Any* $pure-3$ *subgroup of a torsion-free abelian group* G *is* $pure-1$

Proof.

Let $A\times B$ be a $pure$ in a free $G_n \times G_m$, let $(a,b)\in A\times B$, then $(\forall p), k\in Z^+$ we have $P^k(a,b) = P^k(p(a,b))$ (Assume that $k \geq 1$). So $P^k((a,b)-P(a,b)) = 0$ and since G is torsion-free, then $P(a,b)-(a,b)=0$ So $(a,b) = P(a,b)$, we obtain that $A\times B$ is $pure-1$.

We know that the intersection of divisible ($pure$), (Neat) subgroup is not divisible ($pure-Neat$). But we shall show that if $A\times B$ and $A_1\times B_1$ are two $pure-3$ and Neat in $G_n \times G_m$ then $A\times B \bigcap A_1\times B_1$ is $pure$.

First we need the following lemma:

Lemma 1.19 *Let* $A\times B$ *be a* $pure-3$ *in* $G_n \times G_m$ *and* $A_1\times B_1$, *be any subgroup of* $G_n \times G_m$. *Then* $P^k(A\times B \bigcap A_1\times B_1) = P^k A\times B \bigcap P^k A_1\times B_1, (\forall p)\ (\forall k, k\in Z^+$.

Proof.

Let $(a,b)\in P^k A\times B \bigcap P^k A_1\times B_1$. Therefore, $(a,b) = P^k(a_0,b_0)$ for some $(a_0,b_0 \in A\times B$. Since $A\times B$ is $pure-3$. Then $P^k(a,b) = P^{2k}(a,b) = P^k(a_0,b_0)$. Then $(a_0,b_0) = P^k(a,b) \in P^k(A\times B \bigcap A_1\times B_1)$ we obtain the result.

Remark 1.20 *The above lemma is satisfied for* $pure-1-2$ *subgroups.*

Theorem 1.21 *Let* $A\times B$ *and* $A_1\times B_1$ *be two* $pure-3$ *and Neat in* $G_n \times G_m$. *Then* $A\times B \bigcap A_1\times B_1$.

Proof.

By using Theorem 1.16 we have $A\times B$ and $A_1\times B_1$ are $pure$. We need to prove $A\times B \bigcap A_1\times B_1$ is $pure$.

Let $(a,b)\in A\times B \bigcap A_1\times B_1$ and let us suppose that $P|^k\ (a,b)$ is solvable in $G_n \times G_m$. Then $(a,b) = P^k(g_n, g_m)$ for some $(g_n,g_m)\in G_n\times G_m$. Since $A\times B$ and $A_1\times B_1$ are $pure$. Then $P^k(a_0,b_0) = (a,b) = P^k(a_1,b_1)$ for some $(a_0,b_0)\in A\times B, (a_1,b_1)\in A_1\times B_1$ By lemma 1.19 we have

$P^k(A\times B) \bigcap P^k A_1\times B_1 = P^k(A\times B \bigcap A_1\times B_1)$ .Then $(a,b) = P^k(y,z)$ for some $(y,z)\in A\times B \bigcap A_1\times B_1$.

Consequently, $A\times B \bigcap A_1\times B_1$ is $pure$.

Now, we are showing the following results:

Theorem 1.22 *Let* $P(S \times M)$ *be a* $pure-2$ *in* $G_n \times G_m$. *Then* $P(S \times M)$ *is* $pure$.

Proof.

Claim $P(S \times M)$ is Neat. We have to show that $P(G_n \times G_m) \bigcap P(S \times M) = P(P(S \times M))$.

Let $(a,b) = P(g_n, g_m) = P(S,M) \in P(S \times M)$ since $P(S \times M)$ is $pure-2$, $P(S,M) = P(P(S,M))$, so $(a,b) = P(S,M) = P(P(S,M)) \in P(P(S \times M))$. Consequently $P(S \times M)$ is Neat. By theorems (1.12, 1.13) We obtain that $P(S \times M)$ is $pure$ in $G_n \times G_m$.

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