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QUASI-LINEAR EVOLUTION AND ELLIPTIC EQUATIONS

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Abstarct.

In this article we use a new type of nonlinear elliptic operators $A^p : W_1^p\left(R^l, d^l x\right) \to W_{-1}^{p}\left(R^l, d^l x\right)$ that are associated with left side of elliptic equation and studied their properties. We draw up the form, that is associated with non-linear elliptic operator $A^p : W_1^p\left(R^l, d^l x\right) \to W_{-1}^{p}\left(R^l, d^l x\right)$, studying the properties this operator by means of form.

We proved some a priori estimates which are theorems about properties of solutions under certain conditions on the function that forming this equation. We proved the existence of solution of quasi-linear evolution equation with singular coefficients in $R^l, l > 2$ space by Galerkin method and showed that a given equation has a solution in the Sobolev space.

Keywords: differential form; parabolic equations; evolution equations; a priori estimate; weak solution; singular coefficients.

1 Introduction

The main objects of article are obtaining some a priori estimates and existence of weak solution in certain functional set provided that system coefficients belonging to given functional classes.

We consider quasi-linear parabolic differential system in divergence form in whole Euclidean space $R^l, l > 2$:

$$\frac{\partial}{\partial t}u^k + \lambda u^k - \sum_{i,j=1,\dots,l} \frac{\partial}{\partial x_i}\left(a_{ij}(t,x,\vec{u})\frac{\partial}{\partial x_j}u^k\right) + b^k(t,x,\vec{u},\nabla\vec{u}) = f^k(t,x), \quad k = 1,\dots,N \qquad (1)$$

with initial condition

$$u^k(0,x) = \left(u^1{}_0,\dots,u^N{}_0\right),$$

where is the unknown vector-function $u^k(t,x)=(u^1,....,u^N)$, $(t,x)\in[0,\infty)\times R^l, l>2$, $\lambda>0$ is real number and $f(t,x)=f^k=(f^1,...,f^N)$ is given function. Here $b(t,x,u,\nabla u)=b^k(t,x,\vec{u},\nabla\vec{u})$ is vector - function of four variables. Measurable matrix $a_{ij}(t,x,u)$ dimension $l\times l$ satisfies ellipticity condition: $\exists\nu$: $0<\nu<\infty$ and executed the following inequality $\nu I\le a(t,x,u)$, for almost all $t\in[t,T]$, $x\in R^l$, that

$$\nu(\vec{u})\sum_{i=1}^{l}\xi_i^2\le\sum_{ij=1,...,l}a_{ij}(t,x,\vec{u})\xi_i\xi_j\le\mu(\vec{u})\sum_{i=1}^{l}\xi_i^2\quad\forall\xi\in R^l \tag{2}$$

We will call weak solutions of quasi-linear differential system of parabolic type in $W_1^p(R^l,d^lx)$ the element $u(t,x)$ that almost all $t\in[0,T]$ satisfies integral identity:

$$\begin{gathered}\langle u(\tau),v(\tau)\rangle|_0^t+\int_0^t\left(-\langle u(\tau),\partial_t v(\tau)\rangle+\lambda\langle u(\tau),v(\tau)\rangle\right)d\tau+\\+\int_0^t\left\langle\sum_{i,j=1,...,l}a_{ij}\frac{\partial}{\partial x_j}u,\frac{\partial}{\partial x_i}v\right\rangle d\tau+\int_0^t\langle b,v\rangle d\tau=\int_0^t\langle f,v\rangle d\tau\end{gathered}, \tag{3}$$

and for any vector - element $v\in W_{1,0}^q$.

Where $\langle u,v\rangle=\sum_{i=1,...,N}\langle u_i,v_i\rangle\ \forall u\in L^p(R^l)\forall v\in L^q(R^l)$ and $\|u\|_{L^p(R^l)}=\left\langle\sum_{i=1,...,N}|u_i|^p\right\rangle^{\frac{1}{p}}=\left(\sum_{i=1,...,N}\langle|u_i|^p\rangle\right)^{\frac{1}{p}}$.

Usually, $u^k(t,x)=(u^1,....,u^N)$, $(t,x)\in[0,\infty)\times R^l, l>2$ is ordered set of N elements certain elements of functional space, such as $u_i\in W_m^p(R^l,d^lx), i=1,...,N$.

There are many versions of theorems of functional embedment almost all related to the establishment of specific functional estimations. If $k>m$ i $1\le p<q<\infty$, $(k-m)p<l$ and the equality $\frac{1}{q}=\frac{1}{p}-\frac{k-m}{l}$, then embedment spaces $W_k^p(R^l)\subset W_m^q(R^l)$ is continuous.

Conjugated to space $W_m^p(R^l,d^lx)$ is space $W_{-m}^q(R^l,d^lx)$, which by definition can be entered as a space of linear functionals on the linear space $W_m^p(R^l,d^lx)$. Let $p+q=pq$ then

$$\langle f,g\rangle\le\|f\|_{L^p(R^l)}\|g\|_{L^q(R^l)}\le\frac{\varepsilon^p}{p}\|f\|^p_{L^p(R^l)}+\frac{1}{\varepsilon^q q}\|g\|^q_{L^q(R^l)}$$

where $f\in L^p(R^l), g\in L^q(R^l), \varepsilon>0$ and

$$\begin{gathered}\langle f,f|f|^{p-2}\rangle=\|f\|_{L^p(R^l)}\left\|f|f|^{p-2}\right\|_{L^q(R^l)}=\\=\frac{1}{p}\|f\|^p_{L^p(R^l)}+\frac{1}{q}\left\|f|f|^{p-2}\right\|^{p-1}_{L^q(R^l)}=\|f\|^p_{L^p(R^l)}.\end{gathered}$$

We consider the conditions under which we study and parabolic system (1): first, $b(t,x,y,z)$ is measurable function and its arguments $b\in L^1_{loc}(R^l)$; second, function $b(t,x,y,z)$ satisfies estimation $|b(t,x,u,\nabla u)|\le\mu_1(t,x)|\nabla u|+\mu_2(t,x)|u|+\mu_3(t,x)$ where $\mu_1^2\in PK_\beta(A)$, $\mu_2\in PK_\beta(A)$, $\mu_3\in L^p(R^l)$ and

$\|u\|_{L^p(R^l)} = \left\langle \sum_{i=1,\dots,N} |u_i|^p \right\rangle^{\frac{1}{p}} = \left(\sum_{i=1,\dots,N} \left\langle |u_i|^p \right\rangle \right)^{\frac{1}{p}}$, $\quad \|u\|^{p-1}{}_{L^p(R^l)} = \left\langle \sum_{i=1,\dots,N} |u_i|^p \right\rangle^{\frac{p-1}{p}} \quad = \left\langle \sum_{i=1,\dots,N} \left(|u_i|^{\frac{p}{q}} \right)^q \right\rangle^{\frac{1}{q}} = \left\| |u|^{p-1} \right\|_{L^q(R^l)}$,

$|\nabla \vec{u}|^p = \sum_{i=1,\dots,N} \sum_{k=1,\dots l} \left| \frac{\partial}{\partial x_k} u_i \right|^p$ and $\|u\|^p = \left\langle \sum_{i=1}^{N} u^i u^i |u^i|^{p-2} \right\rangle \equiv \left\langle \sum_{i=1}^{N} u^i u^i |u|^{p-2} \right\rangle$. Similarly, the growth of $b(t,x,y,z)$ $|b(t,x,u,\nabla u) - b(t,x,v,\nabla v)| \le \mu_4(t,x)|\nabla(u-v)| + \mu_5(t,x)|u-v|$, where $\mu_4^2 \in PK_\beta(A)$, $\mu_5 \in PK_\beta(A)$.

The condition on coefficients of evolution system. Let matrix $a_{ij}(t,x,u)$ is measurable dimension $l \times l$ and satisfies ellipticity condition: $\exists \nu$: $0 < \nu < \infty$ executed following inequality $\nu I \le a(t,x,u)$, for almost all $t \in [t,T]$, $x \in R^l$.

We consider the conditions under which we study and parabolic system (1): 1. Here $b(t,x,y,z)$ is measurable function and its arguments $b \in L^1_{loc}(R^l)$; 2. Function $b(t,x,y,z)$ satisfies almost everywhere, almost all $t \in [0,T]$:

$$|b(t,x,u,\nabla u)| \le \mu_1(t,x)|\nabla u| + \mu_2(t,x)|u| + \mu_3(t,x). \tag{4}$$

In the condition (4) function $\mu_1^2 \in PK_\beta(A)$, $\mu_2 \in PK_\beta(A)$, function $\mu_3 \in L^p(R^l)$. 3. The growth function $b(t,x,y,z)$ almost everywhere satisfies the condition almost all $t \in [0,T]$:

$$|b(t,x,u,\nabla u) - b(t,x,v,\nabla v)| \le \mu_4(t,x)|\nabla(u-v)| + \mu_5(t,x)|u-v| \tag{5}$$

where $\mu_4^2 \in PK_\beta(A)$, $\mu_5 \in PK_\beta(A)$.

Coulomb potential is satisfied these conditions.

2. Quasi-linear elliptic operator

We study the elliptic system in space R^l:

$$\lambda u^k - \sum_{i,j=1,\dots,l} \frac{\partial}{\partial x_i} \left(a_{ij}(x,\vec{u}) \frac{\partial}{\partial x_j} u^k \right) + b^k(x,\vec{u},\nabla\vec{u}) = f^k, \quad k = 1,\dots,N$$

where $\vec{u}(x)$ is unknown function and $f(x) = f^k = (f^1,\dots,f^N)$ is given function. Here $b(x,u,\nabla u) = b^k(x,u^k,\nabla u^k)$ is given function of three variables. Measurable matrix $a_{ij}(x,u)$ dimension $l \times l$ satisfies ellipticity condition: $\exists \nu$: $0 < \nu < \infty$ and using the following inequality $\nu I \le a(x,u)$, for almost all $x \in R^l$, scilicet $\quad \nu(\vec{u}) \sum_{i=1}^{l} \xi_i^2 \le \sum_{ij=1,\dots,l} a_{ij}(x,\vec{u}) \xi_i \xi_j \quad \forall \xi \in R^l$.

If we define the function $\|u\|_{W_m^p} = \left(\sum_{i=1,\dots,N} \left(\|u_i\|_{L^p}^p + \sum_{1 \le |s| \le m} \left\| D^s u_i \right\|^p \right) \right)^{\frac{1}{p}}$, we can instead elliptic system in space R^l consider the elliptic equation

$$\lambda u - \frac{\partial}{\partial x_i}\left(a_{ij}(x,u)\frac{\partial}{\partial x_j}u \right) + b(x,u,\nabla u) = f\ , \qquad (6)$$

where $u(x)$ is the unknown function and $f(x) = f$ is given function.

Definition 1. *We call weak solution in* $W_1^p(R^l, d^l x)$ *an element* $u(x)$ *which satisfies the integral identity:*

$$\lambda\langle u,v\rangle + \left\langle \sum_{i,j=1,\dots,l} a_{ij}\frac{\partial}{\partial x_j}u, \frac{\partial}{\partial x_i}v \right\rangle + \langle b,v\rangle = \langle f,v\rangle, \qquad (7)$$

for any element $v \in W_{1,0}^q(R^l, d^l x)$. Equation (4) is a scalar integral identity. Based on this definition we build the form $h_\lambda^p : W_1^p \times W_1^q \to R$:

$$h_\lambda^p(u,v) \equiv \lambda\langle u,v\rangle + \langle \nabla v \circ a \circ \nabla u\rangle + \langle b(x,u,\nabla u), v\rangle, \qquad (8)$$

which will assume correctly defined (conditions on the coefficients specify below) for all items $u \in W_1^p(R^l, d^l x), v \in W_1^q(R^l, d^l x)$.

If we have elliptic system then we define $\langle u,v\rangle = \sum_{i=1,\dots,N}\langle u_i, v_i\rangle\ \forall u \in L^p(R^l)\forall v \in L^q(R^l)$.

Function $b(x,y,z)$ is measurable function and its arguments $b \in L_{loc}^1(R^l)$; $b(x,y,z)$ almost everywhere satisfies the inequality:

$$\left| b(x,u,\nabla u) \right| \le \mu_1(x)\left|\nabla u\right| + \mu_2(x)\left|u\right| + \mu_3(x). \qquad (9)$$

where functions $\mu_1^2 \in PK_\beta(A)$, $\mu_2 \in PK_\beta(A)$, function $\mu_3 \in L^p(R^l)$. The growth function $b(x,y,z)$ almost everywhere satisfies the condition:

$$\left| b(x,u,\nabla u) - b(x,v,\nabla v) \right| \le \mu_4(x)\left|\nabla(u-v)\right| + \mu_5(x)\left|u-v\right|, \qquad (10)$$

where $\mu_4^2 \in PK_\beta(A)$, $\mu_5 \in PK_\beta(A)$.

We estimate form (8), which made up the equation (6) for the conjugate element $u|u|^{p-2}$:

$$\left| h_\lambda^p(u, u|u|^{p-2}) \right| = \left| \lambda\left\langle u, u|u|^{p-2} \right\rangle + \left\langle \nabla\left(u|u|^{p-2}\right) \circ a \circ \nabla u \right\rangle + \left\langle b(x,u,\nabla u), u|u|^{p-2} \right\rangle \right| \le$$

$$\le \lambda\|w\|^2 + \frac{4(p-1)}{p^2}\langle \nabla w \circ a \circ \nabla w\rangle + \left\langle \mu_1(x)|\nabla u| + \mu_2(x)|u| + \mu_3(x), |u|^{p-1} \right\rangle \le$$

$$\le \lambda\|w\|^2 + \frac{4(p-1)}{p^2}\langle \nabla w \circ a \circ \nabla w\rangle + \frac{2}{p}\langle \mu_1|\nabla w|, |w|\rangle +$$

$$+ \beta\langle \nabla w \circ a \circ \nabla w\rangle + c(\beta)\|w\|^2 + \|\mu_3\|\|u\|^{p-1},$$

where by the definition used vector function $w = u|u|^{\frac{p-2}{2}}$, according $\nabla w = \frac{p}{2}|u|^{\frac{p-2}{2}}\nabla u$ we have the estimation:

$$\left\langle \mu_1|\nabla u|,|u|^{p-1}\right\rangle = \left\langle \mu_1|u|^{\frac{p-2}{2}}|\nabla u|,|u|^{\frac{p}{2}}\right\rangle \le \frac{2}{p}\left\langle \mu_1|\nabla w|,|w|\right\rangle,$$

$$\left\langle \mu_2(x),w^2\right\rangle \le \beta\left\langle \nabla w\circ a\circ\nabla w\right\rangle + c(\beta)\|w\|^2,$$

$$\left\langle \mu_3(x),|u|^{p-1}\right\rangle \le \|\mu_3\|\left\||u|^{p-1}\right\| = \|\mu_3\|\|u\|^{p-1},$$

by Holder and Young estimations

$$\frac{2}{p}\left\langle \mu_1|\nabla w|,|w|\right\rangle \le \frac{2}{p}\|\mu_1 w\|\|\nabla w\|,$$

$$\|\mu_1 w\| = \left\langle (\mu_1 w)^2\right\rangle^{\frac{1}{2}} \le \left(\beta\left\langle \nabla w\circ a\circ\nabla w\right\rangle + c(\beta)\|w\|^2\right)^{\frac{1}{2}},$$

so

$$\frac{2}{p}\left\langle \mu_1|\nabla w|,|w|\right\rangle \le \frac{2}{p}\|\mu_1 w\|\|\nabla w\| = \frac{2}{p}\|\nabla w\|\left\langle (\mu_1 w)^2\right\rangle^{\frac{1}{2}} \le$$

$$\le \frac{2}{p}\|\nabla w\|\left(\beta\left\langle \nabla w\circ a\circ\nabla w\right\rangle + c(\beta)\|w\|^2\right)^{\frac{1}{2}} \le$$

$$\le \frac{1}{p}\left(\frac{1}{\varepsilon^2}\|\nabla w\|^2 + \varepsilon^2\left(\beta\left\langle \nabla w\circ a\circ\nabla w\right\rangle + c(\beta)\|w\|^2\right)\right).$$

Then we get estimation

$$\left|h_\lambda^p(u,u|u|^{p-2})\right| \le \lambda\|w\|^2 + \frac{4(p-1)}{p^2}\left\langle \nabla w\circ a\circ\nabla w\right\rangle +$$

$$+\frac{1}{p}\left(\frac{1}{\varepsilon^2}\|\nabla w\|^2 + \varepsilon^2\left(\beta\left\langle \nabla w\circ a\circ\nabla w\right\rangle + c(\beta)\|w\|^2\right)\right) +$$

$$+\beta\left\langle \nabla w\circ a\circ\nabla w\right\rangle + c(\beta)\|w\|^2 + \frac{\sigma^p}{p}\|\mu_3\|^p + \frac{1}{\sigma^q q}\|u\|^p,$$

finally

$$\left|h_\lambda^p(u,u|u|^{p-2})\right| \le \left(\lambda + \left(\frac{\varepsilon^2}{p}+1\right)c(\beta) + \frac{1}{\sigma^q q}\right)\|w\|^2 +$$

$$+\left(\frac{4(p-1)}{p^2} + \frac{\beta\varepsilon^2}{p} + \beta\right)\left\langle \nabla w\circ a\circ\nabla w\right\rangle + \frac{1}{p}\frac{1}{\varepsilon^2}\|\nabla w\|^2 + \frac{\sigma^p}{p}\|\mu_3\|^p.$$

It is shown that $\|w\|^2_{L^2(R^l)} = \|u\|^p_{L^p(R^l)}$. Coefficient β depends on the data of (smoothness coefficient equation (μ_i)); coefficient $c(\beta)$ depends on β; coefficients ε i σ – arbitrary positive; coefficients ε selected based on the matrix a constants ellipticity, coefficient σ shifts affect the range of the value, it is less substantial.

So for every fixed $u \in W_1^p$ form $h_\lambda^p(u,v)$ is a continuous linear in $v \in W_1^q$ functional of W_1^q, and therefore each $u \in W_1^p$ is associated with an element conjugate to W_1^q space W_{-1}^p, so function exists that $A^p : W_1^p \to W_{-1}^p$. Operator $A^p : W_1^p \to W_{-1}^p$ takes the following action: $h_\lambda^p(u,v) = \langle A^p(u), v \rangle$. First suppose that the functions that form the equation quite smooth and prove the existence of solution Galerkin method with a special basis of uniqueness of the solution is the result of strict accretive operator generated by form, which is composed from equation. Then suppose that coefficients are measurable and by it cutting and smoothing we reduce the problem to the previous case. Next step is removal of the conditions of cutting and smoothing.

Theorem 1. *Weak solutions of equations (6) with conditions (9, 10) uniformly limited in* W_1^p.

Proof. We form an integral identity:

$$\lambda\langle u_k, \xi\rangle + \langle d\xi \circ a \circ du_k\rangle + \langle b(x, u_k, \nabla u_k), \xi\rangle \equiv \langle f, \xi\rangle,$$

We put $\xi = u_k |u_k|^{p-2}$, obtain:

$$\lambda\langle u_k, u_k |u_k|^{p-2}\rangle + \frac{4(p-1)}{p^2}\left\langle d\left(u_k |u_k|^{\frac{p-2}{2}}\right) \circ a \circ d\left(u_k |u_k|^{\frac{p-2}{2}}\right)\right\rangle +$$

$$+\left\langle b, u_k |u_k|^{p-2}\right\rangle \equiv \left\langle f, u_k |u_k|^{p-2}\right\rangle.$$

With conditions (9), using Young's and Holder inequalities , we find:

$$|\langle b, u_k |u_k|^{p-2}\rangle| \le$$

$$\le \frac{1}{p}\left(\frac{1}{\varepsilon^2}\|\nabla w\|^2 + \varepsilon^2\left(\beta\langle \nabla w \circ a \circ \nabla w\rangle + c(\beta)\|w\|^2\right)\right) +$$

$$+\beta\langle \nabla w \circ a \circ \nabla w\rangle + c(\beta)\|w\|^2 + \frac{\sigma^p}{p}\|\mu_3\|^p + \frac{1}{\sigma^q q}\|u_k\|^p,$$

Next, we get:

$$|\langle f, u_k |u_k|^{p-2}\rangle| \le \| f \|_p \| u_k |u_k|^{p-2}\|_q \le \| f \|_p \| u_k \|^{p-1}.$$

Then, using arguments similar to the previous one, we obtain

$$\| u_k \| + \| \nabla u_k \| \le c(\lambda, p, l, \lambda_0, N) \| f \|.$$

So, since $\| u_k \|_{W_1^p} < C$, where constant has depends on function coefficient (structure of equations), then because of weak compactness of space $W_1^p(R^l, d^l x)$ we find that there exists a subsequence $(u_{k'}(x))$, that is a property: $u_{k'} \xrightarrow[W_1^p]{} u_0$ weak i $A^p(u_{n'}) \xrightarrow[W_1^p]{} y$ weak. Show that $y = A^p(u_0) = f$. We form the integral identity:

$$\left\langle A^p(u_{k'}), v_i^* \right\rangle = \left\langle f, v_i^* \right\rangle, i = 1, ..., k'$$

and go to the limit $k' \to +\infty$. Then we obtain:

$$\lim_{n\to\infty} A^p(u_{k'}) = y = f,$$

the limit in $W_{-1}^p(R^l, d^l x)$ space.

We have:

$$\lim_{k'\to\infty} \left\langle A^p(u_{k'}) - A^p(v), (u_{k'} - v)|u_{k'} - v|^{p-2} \right\rangle =$$

$$= \left\langle y - A^p(v), (u_0 - v)|u_0 - v|^{p-2} \right\rangle \geq 0.$$

We put $v = u_0 - tz, t > 0, \quad z \in W_1^p(R^l, d^l x)$ and reducing both sides of the resulting inequality in t^{p-1}, obtain $\left\langle y - A^p(u_0 - tz), z|z|^{p-2} \right\rangle \geq 0.$

With semi-continuity of operator $A^p : W_1^p \to W_{-1}^p$, given the arbitrary element $z \in W_1^p(R^l, d^l x)$, obtain $y = A^p(u_0) = f$, ie for given initial data constructed sequence $\{u_{k'}\}$ and proved its convergence to the element $u_0 \in W_1^p(R^l, d^l x)$, therefore element $u_0 \in W_1^p(R^l, d^l x)$ will be solutions of the conditions mentioned above.

The uniqueness of this solution follows from the properties of accretiveness of operator $A^p(\cdot)$. Indeed, let u_0, u_0' are two such solutions. Then, just equality:

$$\left\langle A^p(u_0), w \right\rangle = f, \quad \left\langle A^p(u_0'), w \right\rangle = f \quad \forall w \in W_1^q(R^l, d^l x),$$

that $\left\langle A^p(u_0) - A^p(u_0'), w \right\rangle = 0$.

Let $w = (u_0 - u_0')|u_0 - u_0'|^{p-2}$, so:

$$0 = \left\langle A^p(u_0) - A^p(u_0'), (u_0 - u_0')|u_0 - u_0'|^{p-2} \right\rangle \geq$$

$$\geq \lambda \|u_0 - u_0'\|_p^p + (p-1)\left\langle \nabla(u_0 - u_0') \circ a \circ \nabla(u_0 - u_0'), |u_0 - u_0'|^{p-2} \right\rangle -$$

$$-\left\langle \mu_4(x)|\nabla(u_0 - u_0')| + \mu_5(x)|u_0 - u_0'|, (u_0 - u_0')|u_0 - u_0'|^{p-2} \right\rangle \geq$$

$$\geq \left(\lambda - \frac{\varepsilon^2 c(\beta)}{p} - c(\beta) \right) \|w\|^2 +$$

$$+\left(\frac{4(p-1)}{p^2}-\beta\frac{\varepsilon^2}{p}-\frac{1}{p\varepsilon^2\nu}-\beta\right)\langle\nabla w\circ a\circ\nabla w\rangle\geq 0,$$

that equivalent equality $u_0 = u_0'$, that solutions of equations coincide.

Let the functions that form the quasi-linear equation (6) is measurable and satisfy the above conditions for the growth and power of the singularities.

Let $u(x)$ is Lebesgue measurable function at R^l. We denote by u_ϑ and s_ϑ cutting and support, respectively

$$u_\vartheta=\begin{cases}u-\vartheta, & u>\vartheta,\\ 0;| & u|\leq\vartheta,\\ u+\vartheta, & u<-\vartheta,\end{cases}$$

$$s_\vartheta(u)=\left\{x\in R^l:|u(x)|>\vartheta,\vartheta>0\right\}.$$

Let $a^m(x)$, $f^m(x)$, $b^m(x,y,z)$ are cutting for argument x functions $a(x)$, $f(x)$, $x\in R^l$, $b(x,y,z),(x,y,z)\in\{R^l\times R\times R^l\}$, respectively.

Consider the "smooth" approximation of functions $a^m(x)$; $b^m(x,y,z)$, $f^m(x)$ for argument x:

$$a^{n,m}(x)=\int_{R^l}\rho_n(x-t)\,a^m(t)dt=\rho_n*a^m,$$

where $\rho_n(t)$ is smooth integral approximation 1 in R^l.

We go to the limit is as follows initially removed smoothing, that go to the limit with $n\to\infty$ then remove the cut, that go to the limit with $m\to\infty$ (sequence of limits is important).

The equation of equations:

$$\lambda u-d\circ a^{m,n}\circ du+b^{m,n}(x,u,\nabla u)=f^{m,n},\lambda>0, \tag{11}$$

and form

$$h_\lambda^{p,mn}(u,v)=\lambda\langle u,v\rangle+\langle dv\circ a^{m,n}\circ du\rangle+\langle b^{m,n},v\rangle. \tag{12}$$

For every fixed vector $u\in W_1^p$ the form $h_\lambda^{p,mn}(u,v)$ is a continuous linear at $v\in W_1^q$ functional of W_1^q, and therefore each $u\in W_1^p$ is associated with an element conjugate to W_1^q space W_{-1}^p, there is mapping $A^{p,mn}:W_1^p\to W_{-1}^p$ generated by this form.

Theorem 2. *The equation (11) under condition (9, 10) has unique solution in* W_1^p.

Theorem 3. *Generalized solutions of equation (11) with the conditions (9) uniformly limited in* W_1^p.

Proof. We form the integral identity:

$$\lambda\langle u^{m,n},\xi\rangle+\langle d\xi\circ a^{m,n}\circ du^{m,n}\rangle+\langle b^{m,n}(x,u^{m,n},\nabla u^{m,n}),\xi\rangle\equiv\langle f^{m,n},\xi\rangle,$$

Let $\xi = u^{m,n} | u^{m,n} |^{p-2}$, we obtain:

$$\lambda \langle u^{m,n}, u^{m,n} | u^{m,n} |^{p-2} \rangle +$$

$$+ \frac{4(p-1)}{p^2} \left\langle d\left(u^{m,n} | u^{m,n} |^{\frac{p-2}{2}} \right) \circ a^{m,n} \circ d\left(u^{m,n} | u^{m,n} |^{\frac{p-2}{2}} \right) \right\rangle +$$

$$+ \left\langle b^{m,n}, u^{m,n} | u^{m,n} |^{p-2} \right\rangle \equiv \left\langle f^{m,n}, u^{m,n} | u^{m,n} |^{p-2} \right\rangle.$$

With conditions (9), using Holder and Young's inequality, we find:

$$| \langle b^{m,n}, u^{m,n} | u^{m,n} |^{p-2} \rangle | \leq$$

$$\leq \int_{R^l} \int_R | b^{m,n}(t) | \rho_n(x-t) u^{m,n}(x) | u^{m,n}(x) |^{p-2} \, dtdx \leq$$

$$\leq \int_{R^l} \int_R (\mu_1^m(t) \nabla u^{m,n}(x) +$$

$$+ \mu_2^m(t) u^{m,n}(x) + \mu_3^m(t)) \rho_n(x-t) \times \times u^{m,n}(x) | u^{m,n}(x) |^{p-2} \, dtdx \leq$$

$$\leq \int_{R^l} \int_R (\mu_1^m(t) | \nabla u^{m,n}(x) | \rho_n(x-t) u^{m,n}(x) | u^{m,n}(x) |^{p-2} \, dtdx +$$

$$+ \int_{R^l} \int_R (\mu_2^m(t) | u^{m,n}(x) | \rho_n(x-t) u^{m,n}(x) | u^{m,n}(x) |^{p-2} \, dtdx +$$

$$+ \int_{R^l} \int_R (\mu_3^m(t) \rho_n(x-t) u^{m,n}(x) | u^{m,n}(x) |^{p-2} \, dtdx \leq$$

$$\leq \frac{2}{p} \int_{R^l} \int_R (\mu_1^m(t) | \nabla W(x) | \rho_n(x-t) W(x) dtdx +$$

$$+ \int_{R^l} \int_R (\mu_2^m(t) W^2(x) \rho(x-t) dtdx + \int_{R^l} \int_R (\mu_3^m(t) \rho_n(x-t) u^{m,n}(x) | u^{m,n}(x) |^{p-2} \, dtdx,$$

estimate can be written:

$$\int_{R^l} \int_R (\mu_2^m(t) W^2(x) \rho(x-t)) dtdx =$$

$$\int_{R^l} \left(\int_R \mu_2^m(t) \left| \rho_n(x-t) \right| | W(x) |^2 \, dx \right) =$$

$$= \int_{R^l} \left(\int_R \mu_2^m(t) dt \right) \left| \rho_n(x) \right| W^2(x-t) dx =$$

$$= \int_{R^l} \left| \rho_n(x) \right| \int_R (\mu_2^m(t) W^2(x-t) dt) dx \leq \int_{R^l} \left| \rho_n(x) \right| (\beta \| \nabla W(x) \|_2^2 +$$

$$+ c(\beta) \| \nabla W(x) \|_2^2) dx = \beta \| \nabla W \|_2^2 + c(\beta) \| \nabla W \|_2^2,$$

we estimate integrals:

$$\int_{R^l} \int_R \mu_1^m(t) | \nabla W(x) | \rho_n(x-t) W(x) dtdx \leq$$

$$\leq \left(\int_{R^l} \int_R \rho_n(x-t) | \nabla W(x) |^2 \, dtdx \right)^{\frac{1}{2}} *$$

$$* \left(\int_{R^l} \int_R \left| \rho_n(x-t) \right| (\mu_1^m(t) | W(x) |)^2 dtdx \right)^{\frac{1}{2}},$$

where $W = u^{m,n} | u^{m,n} |^{\frac{p-2}{2}}$.

Then again used Young inequality and form- bounded μ_1^2, we have:

$$\int_R \left(\int_{R^l} \rho_n (x-t) W^2(x) dx (\mu_1^m(t))^2 dt \right) =$$

$$= \int_{R^l} \left(\int_R (\mu_1^m(t))^2 dt \right) \rho_n(x) | W(x-t)|^2 \, dx =$$

$$= \int_{R^l} \rho_n(x) \int_R (\left(\mu_1^m(t) \right)^2 | W(x-t)|^2 \, dt) dx \le$$

$$\le \int_{R^l} \rho_n(x)\_(\beta \| \nabla W(x-t) \|_2^2 +$$

$$+ c(\beta) \| W(x) \|_2^2) dx = \beta \| \nabla W \|_2^2 + c(\beta) \| W \|_2^2 .$$

Then

$$\int_{R^l} \int_R (\mu_3^m(t) \rho_n (x-t) u^{m,n}(x) | u^{m,n}(x) |^{p-2} \, dt dx \le$$

$$\le \| \mu_3 \|_p \left\| u^{m,n}(x) | u^{m,n}(x) |^{p-2} \right\|_q = \| \mu_3 \|_p \left\| u^{m,n}(x) \right\|^{p-1}$$

Using the properties of approximation of function f we obtain:

$$| \langle f^{m,n}, u^{m,n} | u^{m,n} |^{p-2} \rangle | \le \| f^{m,n} \|_p \| u^{m,n} | u^{m,n} |^{p-2} \|_q \le$$

$$\le \| f^n \|_p \| u^{m,n} \|^{p-1} \le \| f \|_p \| u^{m,n} \|_p^{p-1} .$$

Then, using Young's inequality, for sufficiently large m, n and considerations such as the previous one, obtain $\| u^{m,n} \| + \| \nabla u^{m,n} \| \le c(\lambda, p, l, \lambda_0, N) \| f \|_p$.

3. A priori estimations for quasi-linear parabolic differential equations

We study the parabolic equation in whole space $R^l, l > \mathbf{2}$:

$$\frac{\partial}{\partial t} u + \lambda u - \sum_{i,j=1,\dots,l} \frac{\partial}{\partial x_i} \left(a_{ij}(t,x,u) \frac{\partial}{\partial x_j} u \right) + b(t,x,u,\nabla u) = f(t,x),$$

so by definition of weak solution, we have the integral identity

$$\langle u(\tau), v(\tau) \rangle |_{\mathbf{0}}^t + \int_{\mathbf{0}}^t \left(- \langle u(\tau), \partial_t v(\tau) \rangle + \lambda \langle u(\tau), v(\tau) \rangle \right) d\tau + \int_{\mathbf{0}}^t \left\langle \sum_{i,j=1,\dots,l} a_{ij} \frac{\partial}{\partial x_j} u, \frac{\partial}{\partial x_i} v \right\rangle d\tau + \int_{\mathbf{0}}^t \langle b, v \rangle d\tau = \int_{\mathbf{0}}^t \langle f, v \rangle d\tau$$

for all $u(t,x) \in W_{\mathbf{1,0}}^p$, $t \in [\mathbf{0}, T]$ and for all $v \in W_{\mathbf{1,0}}^q$.

We rewrite identity for $t \in [\mathbf{0}, T]$ in the form of

$$\langle u(\tau), v(\tau) \rangle |_{\mathbf{0}}^t + \int_{\mathbf{0}}^t \left(- \langle u(\tau), \partial_\tau v(\tau) \rangle + \left\langle \sum_{i,j=1,\dots,l} a_{ij} \frac{\partial}{\partial x_j} u, \frac{\partial}{\partial x_i} v \right\rangle \right) d\tau = \int_{\mathbf{0}}^t \langle f, v \rangle d\tau - \int_{\mathbf{0}}^t \left(\lambda \langle u(\tau), v(\tau) \rangle \right) d\tau - \int_{\mathbf{0}}^t \langle b, v \rangle d\tau.$$ Let

element $v(\tau) = u |u|^{p-2} (\tau)$ and we estimate

$$\left\langle u(\tau), u|u|^{p-2}(\tau)\right\rangle\Big|_0^t + \lambda\int_0^t \left|\left\langle u(\tau), u|u|^{p-2}(\tau)\right\rangle\right| d\tau +$$
$$+\int_0^t \left(-\left\langle u(\tau), \partial_\tau\left(u|u|^{p-2}(\tau)\right)\right\rangle + \left\langle \sum_{i,j=1,\dots,l} a_{ij}\frac{\partial}{\partial x_j}u, \frac{\partial}{\partial x_i}\left(u|u|^{p-2}(\tau)\right)\right\rangle\right) d\tau \le$$
$$\le \int_0^t \left|\left\langle f, \left(u|u|^{p-2}(\tau)\right)\right\rangle\right| d\tau + \int_0^t \left\langle \mu_1(t,x)|\nabla u| + \mu_2(t,x)|u| + \mu_3(t,x), u|u|^{p-2}(\tau)\right\rangle d\tau.$$

Further, we estimate each term on the right side separately

$$\left|\left\langle f, u|u|^{p-2}(\tau)\right\rangle\right| \le \|f\|\left\|u|u|^{p-2}(\tau)\right\| \le \frac{\sigma^p}{p}\|f\|^p + \frac{1}{q\sigma^q}\left\|u|u|^{p-2}(\tau)\right\|^q,$$

$$\left\langle \sum_{i,j=1,\dots,l} a_{ij}\frac{\partial}{\partial x_j}u, \frac{\partial}{\partial x_i}\left(u|u|^{p-2}(\tau)\right)\right\rangle = \frac{4(p-1)}{p^2}\left\langle \nabla\left(u|u|^{\frac{p-2}{2}}(\tau)\right)\circ a\circ\nabla\left(u|u|^{\frac{p-2}{2}}(\tau)\right)\right\rangle,$$

we use denoting of function $w = u|u|^{\frac{p-2}{2}}(\tau)$ and respectively $\nabla w = \frac{p}{2}|u|^{\frac{p-2}{2}}\nabla u$.

$$\left\langle \mu_1|\nabla u|, |u|^{p-1}\right\rangle = \left\langle \mu_1|u|^{\frac{p-2}{2}}|\nabla u|, |u|^{\frac{p}{2}}\right\rangle \le \frac{2}{p}\left\langle \mu_1|\nabla w|, |w|\right\rangle,$$

$$\left\langle \mu_2(x), w^2\right\rangle \le \beta\left\langle \nabla w\circ a\circ\nabla w\right\rangle + c(\beta)\|w\|^2,$$

$$\left\langle \mu_3(x), |u|^{p-1}\right\rangle \le \|\mu_3\|\left\||u|^{p-1}\right\| = \|\mu_3\|\|u\|^{p-1},$$

next use the Hölder and Young estimates $\frac{2}{p}\left\langle \mu_1|\nabla w|, |w|\right\rangle \le \frac{2}{p}\|\mu_1 w\|\|\nabla w\|$,

$$\|\mu_1 w\| = \left\langle (\mu_1 w)^2\right\rangle^{\frac{1}{2}} \le \left(\beta\left\langle \nabla w\circ a\circ\nabla w\right\rangle + c(\beta)\|w\|^2\right)^{\frac{1}{2}},$$

so

$$\frac{2}{p}\left\langle \mu_1|\nabla w|, |w|\right\rangle \le \frac{2}{p}\|\mu_1 w\|\|\nabla w\| = \frac{2}{p}\|\nabla w\|\left\langle (\mu_1 w)^2\right\rangle^{\frac{1}{2}} \le$$
$$\le \frac{2}{p}\|\nabla w\|\left(\beta\left\langle \nabla w\circ a\circ\nabla w\right\rangle + c(\beta)\|w\|^2\right)^{\frac{1}{2}} \le$$
$$\le \frac{1}{p}\left(\frac{1}{\varepsilon^2}\|\nabla w\|^2 + \varepsilon^2\left(\beta\left\langle \nabla w\circ a\circ\nabla w\right\rangle + c(\beta)\|w\|^2\right)\right)$$
$$\le \frac{1}{p}\left(\frac{1}{\varepsilon^2}\left\langle \nabla w\circ a\circ\nabla w\right\rangle + \varepsilon^2\left(\beta\left\langle \nabla w\circ a\circ\nabla w\right\rangle + c(\beta)\|w\|^2\right)\right).$$

Then we get estimations

$$\int_0^t \left(\left\langle \partial_\tau u(\tau), \left(u|u|^{p-2}(\tau) \right) \right\rangle + \left\langle \sum_{i,j=1,\dots,l} a_{ij} \frac{\partial}{\partial x_j} u, \frac{\partial}{\partial x_i} \left(u|u|^{p-2}(\tau) \right) \right\rangle \right) d\tau$$

$$+\lambda \int_0^t \left| \left\langle u(\tau), u|u|^{p-2}(\tau) \right\rangle \right| d\tau + \le$$

$$\le \int_0^t \left| \frac{\sigma^p}{p} \|f\|^p + \frac{1}{q\sigma^q} \left\| u|u|^{p-2}(\tau) \right\|^q \right| d\tau +$$

$$+\int_0^t \left(\frac{1}{p} \left(\frac{1}{\varepsilon^2} \|\nabla w\|^2 + \varepsilon^2 \left(\beta \langle \nabla w \circ a \circ \nabla w \rangle + c(\beta) \|w\|^2 \right) \right)(\tau) \right) d\tau +$$

$$+\int_0^t \left(\beta \langle \nabla w \circ a \circ \nabla w \rangle + c(\beta) \|w\|^2 + \frac{1}{\gamma^q q} \|w\|^2 + \frac{\gamma^p}{p} \|\mu_3\|^p \right) d\tau.$$

We use the equality $\left\langle u(\tau), u|u|^{p-2}(\tau) \right\rangle |_0^t = p \int_0^t \left\langle \partial_\tau u(\tau), \left(u|u|^{p-2}(\tau) \right) \right\rangle d\tau$ that is true for almost all real values t

$$\frac{1}{p} \|w\|^2 |_0^t + 4 \frac{p-1}{p^2} \int_0^t \left(\langle \nabla w \circ a \circ \nabla w \rangle \right) d\tau + \lambda \int_0^t \|w\|^2 d\tau \le$$

$$\le \int_0^t \left| \frac{\sigma^p}{p} \|f\|^p + \frac{1}{q\sigma^q} \left\| u|u|^{p-2}(\tau) \right\|^q \right| d\tau +$$

$$+\int_0^t \left(\frac{1}{p} \left(\frac{1}{\varepsilon^2} \|\nabla w\|^2 + \varepsilon^2 \left(\beta \langle \nabla w \circ a \circ \nabla w \rangle + c(\beta) \|w\|^2 \right) \right)(\tau) \right) d\tau +$$

$$\int_0^t \left(\beta \langle \nabla w \circ a \circ \nabla w \rangle + c(\beta) \|w\|^2 + \frac{1}{\gamma^q q} \|w\|^2 + \frac{\gamma^p}{p} \|\mu_3\|^p \right) d\tau.$$

Because of $\left\| u|u|^{p-2}(\tau) \right\|^q = \left\langle |u|^{(p-1)q} \right\rangle = \|u\|^p = \|w\|^2$, then we obtain inequality

$$\frac{1}{p} \|w\|^2 |_0^t + 4 \frac{p-1}{p^2} \int_0^t \left(\langle \nabla w \circ a \circ \nabla w \rangle \right) d\tau + \lambda \int_0^t \|w\|^2 d\tau \le$$

$$\le \int_0^t \left| \frac{\sigma^p}{p} \|f\|^p + \frac{1}{q\sigma^q} \left\| u|u|^{p-2}(\tau) \right\|^q \right| d\tau +$$

$$+\int_0^t \left(\frac{1}{p} \left(\frac{1}{\varepsilon^2} \|\nabla w\|^2 + \varepsilon^2 \left(\beta \langle \nabla w \circ a \circ \nabla w \rangle + c(\beta) \|w\|^2 \right) \right)(\tau) \right) d\tau +$$

$$\int_0^t \left(\beta \langle \nabla w \circ a \circ \nabla w \rangle + c(\beta) \|w\|^2 + \frac{1}{\gamma^q q} \|w\|^2 + \frac{\gamma^p}{p} \|\mu_3\|^p \right) d\tau.$$

Let $p = 2$ then

$$\frac{1}{2} \|u\|^2 |_0^t + \int_0^t \langle \nabla u \circ a \circ \nabla u \rangle d\tau + \lambda \int_0^t \|u\|^2 d\tau \le$$

$$\le \left(\frac{1}{2\sigma^2} + \frac{c(\beta)}{2} \varepsilon^2 + c(\beta) + \frac{1}{2\gamma^2} \right) \int_0^t \|u\|^2 d\tau +$$

$$+\left(\frac{1}{2} \left(\frac{1}{\varepsilon^2} + \beta\varepsilon^2 \right) + \beta \right) \int_0^t \langle \nabla u \circ a \circ \nabla u \rangle d\tau +$$

$$+\int_0^t \frac{\sigma^2}{2} \|f\|^2 d\tau + \int_0^t \frac{\gamma^2}{2} \|\mu_3\|^2 d\tau.$$

If we put that $\varepsilon^2=\dfrac{1}{\sqrt{\beta}}$ then $\dfrac{1}{2}\left(\dfrac{1}{\varepsilon^2}+\beta\varepsilon^2\right)+\beta=\sqrt{\beta}+\beta=\sqrt{\beta}\left(1+\sqrt{\beta}\right)$ and then

$$\frac{1}{2}\|u\|^2\,|_0^t+\int_0^t\langle\nabla u\circ a\circ\nabla u\rangle d\tau+\lambda\int_0^t\|u\|^2\,d\tau\le\left(\frac{1}{\sqrt{\beta}}+\frac{c(\beta)}{2\sqrt{\beta}}+c(\beta)\right)\int_0^t\|u\|^2\,d\tau+$$

$$+\sqrt{\beta}\left(1+\sqrt{\beta}\right)\int_0^t\langle\nabla u\circ a\circ\nabla u\rangle d\tau+\frac{\sqrt{\beta}}{2}\int_0^t\|f\|^2\,d\tau+\frac{\sqrt{\beta}}{2}\int_0^t\|\mu_3\|^2\,d\tau.$$

Hölder smoothness and continuity of generalized solutions of (1). We will call limited generalized solution of equation (1) the element $u(t,x)\in V_{1,0}^2$ such that $vrai\max|u(t,x)|<\infty$, and which satisfies the integral identity

$$\langle u(\tau),v(\tau)\rangle\,|_0^t+\int_0^t\left(-\langle u(\tau),\partial_t v(\tau)\rangle+\lambda\langle u(\tau),v(\tau)\rangle\right)d\tau+\int_0^t\left\langle\sum_{i,j=1,\dots,l}a_{ij}\frac{\partial}{\partial x_j}u,\frac{\partial}{\partial x_i}v\right\rangle d\tau+\int_0^t\langle b,v\rangle d\tau=\int_0^t\langle f,v\rangle d\tau$$

for all $t\in[0,T]$ and for any function $v\in W_{1,0}^2$ and $vrai\max|v(t,x)|<\infty$, $t\in[0,T]$.

We use conditions and obtain estimation for all $t\in[0,T]$ and for any function $v\in W_{1,0}^2$ and $vrai\max|v(t,x)|<\infty$, $t\in[0,T]$

$$\langle u(\tau),v(\tau)\rangle\,|_0^t+\int_0^t\left(-\langle u(\tau),\partial_\tau v(\tau)\rangle+\left\langle\sum_{i,j=1,\dots,l}a_{ij}\frac{\partial}{\partial x_j}u,\frac{\partial}{\partial x_i}v\right\rangle\right)d\tau\le$$

$$\le\int_0^t\langle f,v\rangle d\tau-\int_0^t\left(\lambda\langle u(\tau),v(\tau)\rangle\right)d\tau+$$

$$+\int_0^t\langle\mu_1(t,x)|\nabla u|+\mu_2(t,x)|u|+\mu_3(t,x),\,v(\tau)\rangle d\tau.$$

Let $u(t,x)$ is a generalized solution, we define $v_{\bar h}(t,x)$ is averaging of function $v(t,x)$ by formula

$$v_{\bar h}(t,x)=\frac{1}{h}\int_{t-h}^{t}v(\tau,x)d\tau\,,\quad u_h(t,x)=\frac{1}{h}\int_t^{t+h}u(\tau,x)d\tau$$

we transform

$$-\int_0^T\langle u\partial_t v_{\bar h}\rangle dt=-\int_0^T\langle u_h\partial_t v\rangle dt=\int_0^T\langle\partial_t u_h v\rangle dt\ ,$$

because

$$\int_0^T u(t)v_{\bar h}(t)dt=\int_0^{T-h}u_h(t)v(t)dt$$

where the function $v(t,x)$ identically equal zero in the interval $t\le 0$ i $T\ge t\ge T-h$.

Remark. We can change the order averaging and differentiation in x.

We rewrite as

$$\int_0^{T-h} \left(\langle \partial_\tau u_h, v \rangle + \lambda \langle u_h, v \rangle \right) d\tau +$$
$$+\int_0^{T-h} \left(\left\langle \left(\sum_{i,j=1,\dots,l} a_{ij} \frac{\partial}{\partial x_j} u \right)_h, \frac{\partial}{\partial x_i} v \right\rangle + \langle b_h, v \rangle \right) d\tau = \int_0^{T-h} \langle f_h, v \rangle d\tau.$$

The last equality holds for any element $v \in W_{1,0}^2$, therefore we put in $v = u_h$, by integrating at t and go to the limit at $h \to 0$, then we get

$$\frac{1}{2}\langle u,u\rangle |_0^t + \int_0^t \left(\left\langle \sum_{i,j=1,\dots,l} a_{ij} \frac{\partial}{\partial x_j} u, \frac{\partial}{\partial x_i} u \right\rangle + \langle b,u \rangle \right) d\tau + \lambda \int_0^t \|u\|^2 d\tau =$$
$$= \int_0^t \langle f,u \rangle d\tau.$$

Since, we have for any element $v \in V_{1,0}^2$ integrals

$$\int_0^{T-h} \left(\left\langle \left(\sum_{i,j=1,\dots,l} a_{ij} \frac{\partial}{\partial x_j} u \right)_h, \frac{\partial}{\partial x_i} v \right\rangle + \langle b_h, v \rangle \right) d\tau, \quad \int_0^{T-h} \langle f_h, v \rangle d\tau$$

limit at $h \to 0$ to $\int_0^T \left(\left\langle \sum_{i,j=1,\dots,l} a_{ij} \frac{\partial}{\partial x_j} u, \frac{\partial}{\partial x_i} v \right\rangle + \langle b,v \rangle \right) d\tau$, $\int_0^T \langle f,v \rangle d\tau$, and thus it is true for $v = u$.

Using (6) and introduced the denote for any $t_1, t_2 \in [h, T-h]$ we write

$$\int_{t_1}^{t_2} \left(\langle \partial_\tau u_h, v \rangle + \lambda \langle u_h, v \rangle \right) d\tau +$$
$$+\int_{t_1}^{t_2} \left(\left\langle \sum_{i,j=1,\dots,l} a_{ij} \frac{\partial}{\partial x_j} u, \frac{\partial}{\partial x_i} v_{\bar{h}} \right\rangle + \langle b, v_{\bar{h}} \rangle \right) d\tau = \int_{t_1}^{t_2} \langle f_h, v \rangle d\tau,$$

if we put $v = u_h$ where $u(t,x) \equiv u^k(t,x) = \max[u(t,x) - k, 0]$, and we denote the set of points in space $R^l, l > 2$, $P_k(t) = \{x \in R^l : u(t,x) > k, t \in [0,T]\}$, and a set of points $P_k(t) = \{(t,x) \in [0,\tau] \times R^l : u(\tau,x) > k, \tau \in [0,T], l > 2\}$, then we get

$$\frac{1}{2}\|u^k(t)\|_{P_k(t)}^2 + \int_0^t \langle \nabla u^k \circ a \circ \nabla u^k \rangle_{P_k(t)} d\tau + \lambda \int_0^t \|u^k\|_{P_k(t)}^2 d\tau \le$$
$$\le \left(\frac{1}{\sqrt{\beta}} + \frac{c(\beta)}{2\sqrt{\beta}} + c(\beta) \right) \int_0^t \|u\|_{P_k(t)}^2 d\tau +$$
$$+\sqrt{\beta}\left(1 + \sqrt{\beta}\right) \int_0^t \langle \nabla u \circ a \circ \nabla u \rangle_{P_k(t)} d\tau +$$
$$+\frac{\sqrt{\beta}}{2} \int_0^t \|f\|_{P_k(t)}^2 d\tau + \frac{\sqrt{\beta}}{2} \int_0^t \|\mu_3\|_{P_k(t)}^2 d\tau.$$

We use elementary estimation $(a+b)^2 \le 2(a^2 + b^2)$, we get

$$\int_0^t \|u\|_{P_k(t)}^2 d\tau \le 2\left(\|u-k\|_{P_k(t)}^2 + k^2 \int_0^t mes\, P_k(\tau)\, d\tau \right).$$

Lemma 1. *Let element* $u \in V_0^2$ *satisfies the following identity*

$$\int_0^T \left(-\langle u, \partial_\tau \varphi\rangle + \lambda \langle u, \varphi\rangle\right) d\tau +$$
$$+\int_0^T \left(\left\langle \sum_{i,j=1,\ldots,l} a_{ij} \frac{\partial}{\partial x_j} u, \frac{\partial}{\partial x_i}\varphi \right\rangle + \langle b, \varphi\rangle\right) d\tau = \int_0^T \langle f, \varphi\rangle d\tau,\ f \in L^2$$

where φ *is arbitrary element of the space* $W_{1,0}^2\left([0, T]\times R^l\right)$, *while element* $u \in V_0^2$ *belong to space* $V_{1,\frac{1}{2}}^2\left([0,T]\times R^l\right)$.

Space $V_{1,\frac{1}{2}}^2\left([0,T]\times R^l\right)$ is subset of space $W_{1,0}^2\left([0,T]\times R^l\right)$ consisting of all elements of continuous at t in norm $L^2\left(R^l\right)$ with the norm

$$\|u\|_V = \max_{t\in[0,T]} \|u(t)\| + \|\nabla_i u\|_{[0,T]\times R^l}$$

and the condition

$$\int_0^{T-p} \left\langle \frac{1}{h}\left|u(t+h,\cdot)-u(t,\cdot)\right|^2\right\rangle dt \xrightarrow{h\to 0} 0.$$

Proof. We put $\varphi_{\bar h}(t,x) = \frac{1}{h}\int_{t-h}^{t} \varphi(\tau,x)d\tau$ for any $\varphi \in W_{1,0}^2\left([0,T]\times R^l\right)$, then we have

$$\int_0^T \left(-\langle u_h, \partial_\tau \varphi\rangle + \lambda \langle u_h, \varphi\rangle\right) d\tau +$$
$$+\int_0^T \left(\left\langle \left(\sum_{i,j=1,\ldots,l} a_{ij} \frac{\partial}{\partial x_j} u\right)_h, \frac{\partial}{\partial x_i}\varphi \right\rangle + \langle b_h, \varphi\rangle\right) d\tau = \int_0^T \langle f_h, \varphi\rangle d\tau,$$

and we denote $\varphi(t,x) = \chi(t)\psi(x)$, where $\chi(t)$ - smooth function of the time variable, $\psi \in W_{1,0}^2\left(R^l\right)$. Then we

$$\int_{-\infty}^{\infty} \left(-\partial_\tau \chi(\tau)\langle u_h, \psi\rangle + \lambda\chi(\tau) \langle u_h, \psi\rangle\right) d\tau +$$
$$+\int_{-\infty}^{\infty} \chi(\tau)\left(\left\langle \left(\sum_{i,j=1,\ldots,l} a_{ij} \frac{\partial}{\partial x_j} u\right)_h, \frac{\partial}{\partial x_i}\psi \right\rangle + \langle b_h, \psi\rangle\right) d\tau = \int_{-\infty}^{\infty} \chi(\tau)\langle f_h, \psi\rangle d\tau,$$

so

$$\partial_\tau \langle u_h, \psi\rangle + \lambda \langle u_h, \psi\rangle +$$
$$+\left\langle \left(\sum_{i,j=1,\ldots,l} a_{ij} \frac{\partial}{\partial x_j} u\right)_h, \frac{\partial}{\partial x_i}\psi \right\rangle + \langle b_h, \psi\rangle = \langle f_h, \psi\rangle\ \forall \psi \in W_{1,0}^2\left(R^l\right),$$

and

$$\langle \partial_\tau u_h, \psi\rangle + \lambda \langle u_h, \psi\rangle +$$
$$+\left\langle \left(\sum_{i,j=1,\ldots,l} a_{ij} \frac{\partial}{\partial x_j} u\right)_h, \frac{\partial}{\partial x_i}\psi \right\rangle + \langle b_h, \psi\rangle = \langle f_h, \psi\rangle\ \forall \psi \in W_{1,0}^2\left(R^l\right),$$

and then for any positive h_1, h_2 , we have

$$\left\langle \partial_\tau u_{h_1} - \partial_\tau u_{h_2}, \psi \right\rangle + \lambda \left\langle u_{h_1} - u_{h_2}, \psi \right\rangle +$$
$$+ \left\langle \left(\sum_{i,j=1,\dots,l} a_{ij} \frac{\partial}{\partial x_j} u \right)_{h_1} - \left(\sum_{i,j=1,\dots N} a_{ij} \frac{\partial}{\partial x_j} u \right)_{h_2}, \frac{\partial}{\partial x_i} \psi \right\rangle + \left\langle b_{h_1} - b_{h_2}, \psi \right\rangle = \left\langle f_{h_1} - f_{h_2}, \psi \right\rangle \forall \psi \in W^2_{1,0}\left(R^l\right),$$

we put $\psi = u_{h_1} - u_{h_2}$ then we get

$$\frac{1}{2} \partial_\tau \left\| u_{h_1} - u_{h_2} \right\|^2 + \lambda \left\| u_{h_1} - u_{h_2} \right\|^2 + \left\langle \left(\sum_{i,j=1,\dots,l} a_{ij} \frac{\partial}{\partial x_j} u \right)_{h_1} - \left(\sum_{i,j=1,\dots,l} a_{ij} \frac{\partial}{\partial x_j} u \right)_{h_2}, \frac{\partial}{\partial x_i} \left(u_{h_1} - u_{h_2} \right) \right\rangle +$$
$$+ \left\langle b_{h_1} - b_{h_2}, u_{h_1} - u_{h_2} \right\rangle = \left\langle f_{h_1} - f_{h_2}, u_{h_1} - u_{h_2} \right\rangle,$$

we are integrating at time variable and we get

$$\frac{1}{2} \left\| u_{h_1} - u_{h_2} \right\|^2 \Big|_{t_1}^{t_2} + \lambda \int_{t_1}^{t_2} \left\| u_{h_1} - u_{h_2} \right\|^2 d\tau +$$
$$+ \int_{t_1}^{t_2} \left\langle \left(\sum_{i,j=1,\dots,l} a_{ij} \frac{\partial}{\partial x_j} u \right)_{h_1} - \left(\sum_{i,j=1,\dots,l} a_{ij} \frac{\partial}{\partial x_j} u \right)_{h_2}, \frac{\partial}{\partial x_i} \left(u_{h_1} - u_{h_2} \right) \right\rangle d\tau +$$
$$+ \int_{t_1}^{t_2} \left\langle b_{h_1} - b_{h_2}, u_{h_1} - u_{h_2} \right\rangle d\tau = \int_{t_1}^{t_2} \left\langle f_{h_1} - f_{h_2}, u_{h_1} - u_{h_2} \right\rangle d\tau, \quad t_1, t_2 \in [0, T].$$

We turn to the limit at $h_1 \to 0, h_2 \to 0$, we have

$$\left\| u_{h_1} - u_{h_2} \right\| + \left\| \frac{\partial}{\partial x_i} \left(u_{h_1} - u_{h_2} \right) \right\| + \left\| \left(\sum_{i,j=1,\dots,l} a_{ij} \frac{\partial}{\partial x_j} u \right)_{h_1} - \left(\sum_{i,j=1,\dots,l} a_{ij} \frac{\partial}{\partial x_j} u \right)_{h_2} \right\| +$$
$$+ \left\| b_{h_1} - b_{h_2} \right\| + \left\| f_{h_1} - f_{h_2} \right\| \xrightarrow{h_1, h_2 \to 0} 0.$$

Then, we put $\psi(x) = \Delta_h u \equiv u(t+h, x) - u(t, x)$, then we have

$$\int_{-\infty}^{\infty} \left(\left\langle \partial_\tau u_h, u(t+h,x) - u(t,x) \right\rangle + \lambda \left\langle u_h, u(t+h,x) - u(t,x) \right\rangle \right) dt +$$
$$+ \int_{-\infty}^{\infty} \left\langle \left(\sum_{i,j=1,\dots,l} a_{ij} \frac{\partial}{\partial x_j} u \right)_h, \frac{\partial}{\partial x_i} \left(u(t+h,x) - u(t,x) \right) \right\rangle dt + \int_{-\infty}^{\infty} \left\langle b_h, u(t+h,x) - u(t,x) \right\rangle dt =$$
$$= \int_{-\infty}^{\infty} \left\langle f_h, u(t+h,x) - u(t,x) \right\rangle dt,$$

we change averaging in the first term to the second "factor" and the difference at first, so

$$\int_{-\infty}^{\infty} \frac{\left\| \Delta_h u \right\|^2}{h} dt + \lambda \int_{-\infty}^{\infty} \left\langle u_h, \Delta_h u \right\rangle dt$$
$$+ \int_{-\infty}^{\infty} \left\langle \Delta_h \left(\sum_{i,j=1,\dots,l} a_{ij} \frac{\partial}{\partial x_j} u \right), \frac{\partial}{\partial x_i} u_h \right\rangle dt + \int_{-\infty}^{\infty} \left\langle \Delta_h b, u_h \right\rangle dt =$$
$$= \int_{-\infty}^{\infty} \left\langle \Delta_h f, u_h \right\rangle dt,$$

We use Hölder inequality, we obtain

$$\int_{-\infty}^{\infty} \frac{\left\| \Delta_h u \right\|^2_{L^2(R^l)}}{h} dt \le \varepsilon(h) \xrightarrow{h \to 0} 0 \ ,$$

so, the lemma 1 is proved.

Estimation of Hölder constant. Let be the condition parabolicity (ellipticity) conditions (4), (5) and each generalized solution $u(t,x)$ from $V_{1,0}^2$ is bounded, show that under these conditions $u \in H^{\alpha,\frac{\alpha}{2}}$ for some $\alpha > 0$ and we will obtain estimation of norm $|u|^{(\alpha)}$. Thus, element $u \in V_{1,0}^2$, for any element $\varphi \in W_{1,0}^2$, we obtain inequality

$$\langle u(\tau),\varphi(\tau)\rangle|_{t_1}^{t_2} + \int_{t_1}^{t_2}\left(-\langle u(\tau),\partial_\tau\varphi(\tau)\rangle + \lambda\langle u(\tau),\varphi(\tau)\rangle\right)d\tau + \int_{t_1}^{t_2}\left\langle \sum_{i,j=1,\dots,l} a_{ij}\frac{\partial}{\partial x_j}u,\frac{\partial}{\partial x_i}\varphi\right\rangle d\tau \le \left|\int_{t_1}^{t_2}\langle b,\varphi\rangle d\tau\right| + \int_{t_1}^{t_2}\left|\langle f,\varphi\rangle\right|d\tau,$$

and since for any element $\varphi \in W_{1,0}^2$, the following condition on function b

$$\left|\int_{t_1}^{t_2}\langle b,\varphi\rangle d\tau\right| \le \left|\int_{t_1}^{t_2}\langle \mu_1|\nabla u| + \mu_2|u| + \mu_3,|\varphi|\rangle d\tau\right|,$$

then, we estimate

$$\langle u(\tau),\varphi(\tau)\rangle|_{t_1}^{t_2} + \int_{t_1}^{t_2}\left(-\langle u(\tau),\partial_\tau\varphi(\tau)\rangle + \lambda\langle u(\tau),\varphi(\tau)\rangle\right)d\tau + \int_{t_1}^{t_2}\left\langle \sum_{i,j=1,\dots,l} a_{ij}\frac{\partial}{\partial x_j}u,\frac{\partial}{\partial x_i}\varphi\right\rangle d\tau \le$$
$$\le \int_{t_1}^{t_2}\left(\langle \mu_1|\nabla u|,|\varphi|\rangle + \langle \mu_2|u|,|\varphi|\rangle + \langle \mu_3,|\varphi|\rangle\right)d\tau + \int_{t_1}^{t_2}\left|\langle f,\varphi\rangle\right|d\tau,$$

then, if necessary, can be re-used by averaging because the calculations are the same as were made above, we put in the last inequality $\varphi(t,x) = (\xi(t,x))^2 u(t,x) \equiv \xi^2 u,$ and after integration by parts first term on time variables, we obtain inequality

$$\frac{1}{2}\|u(\tau)\xi(\tau)\|^2_{K(\delta)}|_{t_1}^{t_2} + \int_{t_1}^{t_2}\left(-\langle u^2(\tau)\xi(\tau),\partial_\tau\xi(\tau)\rangle_{K(\delta)} + \lambda\langle u^2(\tau),\xi^2(\tau)\rangle_{K(\delta)}\right)d\tau +$$
$$+\int_{t_1}^{t_2}\left\langle \sum_{i,j=1,\dots,l} a_{ij}\frac{\partial}{\partial x_j}u,\frac{\partial}{\partial x_i}\left(u\xi^2(\tau)\right)\right\rangle_{K(\delta)} d\tau \le$$
$$\le \int_{t_1}^{t_2}\left(\langle \mu_1|\nabla u|,|u\xi^2(\tau)|\rangle_{K(\delta)} + \langle \mu_2|u|,|u\xi^2(\tau)|\rangle_{K(\delta)} + \langle \mu_3,|u\xi^2(\tau)|\rangle_{K(\delta)}\right)d\tau + \int_{t_1}^{t_2}\left|\langle f,u\xi^2(\tau)\rangle_{K(\delta)}\right|d\tau,$$

where $K(\delta)$ is cube in R^l with sides of length δ.

The first part of this inequality look classic and can be investigated by using standard methods ellipticity, on the second part of it is necessary to use the conditions of form-bounded functions μ_i, $i = 1,2,3$.

Then, we estimate

$$\left|\langle \mu_1|\nabla u|,|u|\xi^2(\tau)\rangle\right| \le \|\mu_1\xi^2(\tau)\|\,\||\nabla u|u\| \le \frac{1}{2}\left(\frac{1}{\varepsilon^2}\|\mu_1\xi^2(\tau)\|^2 + \varepsilon^2\||\nabla u|u\|^2\right),$$

$$\|\mu_1\xi^2(\tau)\|^2 = \left\langle\left(\mu_1\xi^2(\tau)\right)^2\right\rangle \le \beta\langle\nabla\xi^2\circ a\circ\nabla\xi^2\rangle + c(\beta)\|\xi^2\|^2,$$

similarly

$$\left\langle \mu_3\xi^2(\tau),u\right\rangle \le \left\|\mu_3\xi^2(\tau)\right\|\|u\| \le \left(\beta\left\langle\nabla\xi\circ a\circ\nabla\xi\right\rangle + c(\beta)\|\xi\|^2\right)^{\frac{1}{2}}\|u\|.$$

Next, we estimate

$$\left\||\nabla u|u\right\|^2 \le \frac{1}{2}\left(\frac{1}{\varepsilon_1^{\,2}}\left\||\nabla u|^2\right\|^2 + \varepsilon_1^{\,2}\left\|u^2\right\|^2\right).$$

So, we obtain inequality

$$\left\|u(t_2)\xi(t_2)\right\|^2_{K(\delta)} + K(\beta,\varepsilon,\varepsilon_1,...,l)\int_{t_1}^{t_2}\left\langle \sum_{i,j=1,....,l} a_{ij}\frac{\partial}{\partial x_j}u, \frac{\partial}{\partial x_i}\left(u\xi^2(\tau)\right)\right\rangle_{K(\delta)} d\tau \le$$

$$\le \left\|u(t_1)\xi(t_1)\right\|^2_{K(\delta)} + \int_{t_1}^{t_2}\left(K_1\|\nabla\xi\| + K_2\|\xi\| + K_3\left\langle\xi|\xi_\tau|\right\rangle\right)_{K(\delta)} d\tau + K_4\int_{t_1}^{t_2}\left(F(f,\xi^2)\|u\|\right)_{K(\delta)} d\tau,$$

that is, obtained a qualitative estimation, where K, K_1, K_2, K_3 are some positive nuvbers, depending on the initial data of the problem, constants $\varepsilon,\varepsilon_1,..,\varepsilon_4$ are arbitrary constants are selected so as to neutralize term that contains a gradient, for example

$$\frac{1}{2}\left(\frac{1}{\varepsilon^2}\left\|\mu_1\xi^2(\tau)\right\|^2 + \varepsilon^2\left\||\nabla u|u\right\|^2\right) \le$$

$$\le \frac{1}{2}\left(\frac{1}{\varepsilon^2}\beta\left\langle\nabla\xi^2\circ a\circ\nabla\xi^2\right\rangle + c(\beta)\left\|\xi^2\right\|^2 + \varepsilon^2\frac{1}{2}\left(\frac{1}{\varepsilon_1^{\,2}}\left\||\nabla u|^2\right\|^2 + \varepsilon_1^{\,2}\left\|u^2\right\|^2\right)\right),$$

usually it is rationally to choose $\varepsilon^2 = c\beta$. Thus, we obtained a priori estimation of the solution of equation (1).

We assume that an element $u \in V^2_{1,0}$ is solution of equation (1), then for any element $v \in W^2_{1,0}(R^l, d^l x)$ such that $vrai\max|v(t,x)| < \infty$, $t \in [0,T]$ make integrated identity

$$\left\langle u(\tau),v(\tau)\right\rangle|_0^t + \int_0^t\left(-\left\langle u(\tau),\partial_t v(\tau)\right\rangle + \lambda\left\langle u(\tau),v(\tau)\right\rangle\right)d\tau + \int_0^t\left\langle\frac{\partial}{\partial x_i}\sum_{i,j=1,....,l} a_{ij}\frac{\partial}{\partial x_j}u,v\right\rangle d\tau + \int_0^t\left\langle b,v\right\rangle d\tau = \int_0^t\left\langle f,v\right\rangle d\tau,$$

by differentiating the third term on spatial variables, we obtain equality

$$\left\langle u(\tau),v(\tau)\right\rangle|_0^t + \int_0^t\left(-\left\langle u(\tau),\partial_t v(\tau)\right\rangle + \lambda\left\langle u(\tau),v(\tau)\right\rangle\right)d\tau +$$

$$+\int_0^t\left\langle\left(\frac{\partial}{\partial x_i}\sum_{i,j=1,...N} a_{ij}\right)\frac{\partial}{\partial x_j}u,v\right\rangle d\tau + \int_0^t\left\langle\sum_{i,j=1,....,l} a_{ij}\left(\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}u\right),v\right\rangle d\tau +$$

$$+\int_0^t\left\langle b,v\right\rangle d\tau = \int_0^t\left\langle f,v\right\rangle d\tau.$$

Next, we put $v = u$, then

$$\frac{1}{2}\|u(\tau)\|^2|_0^t + \lambda\int_0^t\|u(\tau)\|^2 d\tau + \int_0^t\left\langle\left(\frac{\partial}{\partial x_i}\sum_{i,j=1,...N} a_{ij}\right)\frac{\partial}{\partial x_j}u,u\right\rangle d\tau +$$

$$+\int_0^t\left\langle\sum_{i,j=1,....,l} a_{ij}\left(\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}u\right),u\right\rangle d\tau = \int_0^t\left\langle f,u\right\rangle d\tau - \int_0^t\left\langle b,u\right\rangle d\tau.$$

The right part estimate similar to how it did before, that is, use the form of form-bounded condition and Hölder estimation. To estimate the third summand we need additional conditions on derivatives of elliptical matrix, so we assume that the space variables derived matrix is bounded i.e. and symbol $\frac{\partial}{\partial x_i}\sum_{i,j=1,....,l} a_{ij}$ can be taken out of spatial integrals.

The ellipticity conditions can be written as

$$\nu\|\xi\|^2 \le \sum_{ij=1,...,l} a_{ij}\xi_i\xi_j \le \mu\|\xi\|^2 \quad \forall\xi\in R^l .$$

Consequently, there is a positive real constant number C_1 that estimation is correct

$$\left\langle \sum_{i,j=1,....,l} a_{ij}\left(\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}u\right),u\right\rangle \le C_1\|\Delta u\|\|u\|,$$

then, we use inequalities of Hölder and Young, we get that spatial derivatives of elliptical matrix are bounded and solution u belong to space W_2^2, i.e. we have improvement of the smoothness of the solution equation.

Theorem 4. *If the Cauchy problem*

$$\frac{\partial}{\partial t}u+\lambda u-\frac{\partial}{\partial x_i}\left(a_{ij}(t,x,u)\frac{\partial}{\partial x_j}u\right)+b(t,x,u,\nabla u)=f(t,x), \quad ,$$

$$u(0,x)=u_0 ,$$

where is the unknown function $u(t,x)$ *and* $\lambda>0$ *is real number, and* $f(t,x)=f$ *is given function has a solution* $u\in W_{1,1}^2$ *, the conditions form-function bounded* b *and if* $vrai\max\left|\frac{\partial a_{jk}}{\partial x_i}\right|<\infty$*;*

then the solution $u(t,x)$ *will be belong to space* $W_{2,1}^2$ *, that will improve the properties of the solution, narrowing the class of functions in which we sought solution.*

4. The existence of the solution of the system (1)

Theorem 6. *If the conditions (4), (5) is in space* $W_1^2([0,T]\times R^l)$*, then there is a solution of equation (1).*

Proof. We construct a sequence of approximate solutions $\{u_m(t,x)\}$, $m=1,2,....$ equation

$$\frac{\partial}{\partial t}u+\lambda u-\sum_{i,j=1,...,l}\frac{\partial}{\partial x_i}\left(a_{ij}(t,x,u)\frac{\partial}{\partial x_j}u\right)+b(t,x,u,\nabla u)=f,$$

which will be sought in the form $\{u_m(t,x)\}=\left\{\sum_{i=1}^{m}c_i^m(t)\varphi_i(x)\right\}$; where elements $\{\varphi_n(x)\}$ $n=1,2,....$ form a basis of functional vector space $W_1^2(R^l)$ whith the properties: $(\varphi_i,\varphi_j)=\delta_{ij}$ and $\max_{R^l}|\varphi_i,\varphi_{ix}|\le c_i<\infty$. Functional coefficients $c_n^m(t)$ are determined by equation of ordinary differential equations using substitution $\{u_m(t,x)\}=\left\{\sum_{i=1}^{m}c_i^m(t)\varphi_i(x)\right\}$

$$\langle \partial_t u_m, \varphi_n \rangle + \lambda \langle u_m, \varphi_n \rangle +$$
$$+\left\langle \sum_{i,j=1,\dots,l} a_{ij} \frac{\partial}{\partial x_j} u_m, \frac{\partial}{\partial x_i} \varphi_n \right\rangle + \langle b, \varphi_n \rangle = \langle f, \varphi_n \rangle, \; n = 1, 2, \dots, m$$

and initial conditions

$$c_n^m(0) = (u_0, \varphi_n(x)), \; n = 1, 2, \dots, m \; .$$

The conditions of the problem implies that $|c_n^m| \le const, n = 1, 2, \dots, m$ for $t \in [0, T]$. We will show that solutions uniformly bounded on $t \in [0, T]$ this follows from the limitations of the bottom elliptical matrix and conditions for nonlinear perturbation. We multiply n-th equation at c_n^m and on summands at n from 1 to m, then we obtain inequality

$$\frac{1}{2}\|u_m(t)\|^2 + \int_0^t \langle \nabla u_m \circ a \circ \nabla u_m \rangle d\tau + \lambda \int_0^t \|u_m\|^2 d\tau \le$$
$$\le \left(\frac{1}{\sqrt{\beta}} + \frac{c(\beta)}{2\sqrt{\beta}} + c(\beta) \right) \int_0^t \|u_m\|^2 d\tau +$$
$$+\sqrt{\beta}\left(1 + \sqrt{\beta}\right) \int_0^t \langle \nabla u_m \circ a \circ \nabla u_m \rangle d\tau +$$
$$+\frac{\sqrt{\beta}}{2} \int_0^t \|f\|^2 d\tau + \frac{\sqrt{\beta}}{2} \int_0^t \|\mu_3\|^2 d\tau.$$

Next, we use a known lemma.

Lemma 2. *Let absolutely continuous at $t \in [0, T]$ positive function $\psi(t)$ such that $\psi(0) = 0$ and almost all $t \in [0, T]$ satisfies the following inequality*

$$\frac{d}{d}\psi(t) \le c(t)\psi(t) + F(t)$$

where functions $c(t)$ and $F(t)$ are positive and integrated at $t \in [0, T]$. Then we have

$$\psi(t) \le \exp\left(\int_0^t c(\tau) d\tau \right) \int_0^t F(\tau) d\tau,$$

and

$$\frac{d}{d}\psi(t) \le c(t) \exp\left(\int_0^t c(\tau) d\tau \right) \int_0^t F(\tau) d\tau + F(t) .$$

Thus, provided that $u_0 \in L^2(R^l)$ is true we have a priori estimate

$$\max_{t \in [0, T]} \sum_{n=1}^m \left(c_n^{mk}(t) \right)^2 = \max_{t \in [0, T]} \|u_m\|^2 \le const \; .$$

Functions $c_n^m(t)=\left(u^m(t,x),\varphi_n(x)\right), m,n=1,2,....$ is continuous on $t\in[0,T]$. To study functions $c_n^m(t)=\left(u^m(t,x),\varphi_n(x)\right), m,n=1,2,....$ in the absolute continuity at $t\in[0,T]$, we look at integrals at t to $t+\Delta t$, we use estimation that was obtained above, so we have

$$\left\langle\left|u_m(t+\Delta t,x)-u_m(t,x)\right|,\varphi_n\right\rangle\le$$
$$\le\int_t^{t+\Delta t}\left(\left\langle\left|\sum_{i,j=1,...,ml}a_{ij}\frac{\partial}{\partial x_j}u_m,\frac{\partial}{\partial x_i}\varphi_n\right|\right\rangle\right)d\tau+\int_t^{t+\Delta t}\left\langle f,\varphi_n\right\rangle d\tau+\lambda\int_t^{t+\Delta t}\left\langle u_m,\varphi_n\right\rangle d\tau+$$
$$+\int_t^{t+\Delta t}\left\langle\mu_1(t,x)\left|\nabla u_m\right|+\mu_2(t,x)\left|u_m\right|+\mu_3(t,x),\varphi_n\right\rangle d\tau\le$$

$$\le c_n\int_t^{t+\Delta t}\left(\left\langle\sum_{i,j=1,....,l}\left|a_{ij}\frac{\partial}{\partial x_j}u_m\right|\right\rangle\right)d\tau-\lambda c_n\int_t^{t+\Delta t}\left\|u_m\right\|d\tau+$$
$$+c_n const(\beta)\int_t^{t+\Delta t}\left\|u_m\right\|^2d\tau+c_n const(\beta)\int_t^{t+\Delta t}\left\langle\nabla u_m\circ a\circ\nabla u_m\right\rangle d\tau+$$
$$+c_n const(\beta)\left(\int_t^{t+\Delta t}\left\|f\right\|^2d\tau+\int_t^{t+\Delta t}\left\|\mu_3\right\|^2d\tau\right)\le Const(n,\varphi,l)\Delta t.$$

Consequently, constants $Const(n,\varphi,l)$ dependent on n,φ,l but do not depend on m provided $m\ge n$ i.e. inequality

$$\left|c_n^m(t+\Delta t)-c_n^m(t)\right|\le\varepsilon\left(\Delta t\right)\left\|\varphi_n\right\|\underset{t\to 0}{\to}0.$$

By diagonal way, we construct a subsequence $c_n^{m(i)}, i=1,2,....$ coinciding evenly on $[0,T]$ to some continuous function $c_n(t), n=1,2,....$ for everyone n. The sequence of functions $c_n(t), n=1,2,....$ determines the function $u(t,x)$ by rule $u(t,x)=\sum_{i=1}^{\infty}c_i(t)\varphi_i(x)$. The sequence of functions $\left\{u_m(t,x)\right\}=\left\{\sum_{i=1}^{m}c_i^m(t)\varphi_i(x)\right\}$ limits to $u(t,x)=\sum_{i=1}^{\infty}c_i(t)\varphi_i(x)$ weak in $L^2(R^l)$ and evenly at $t\in[0,T]$. We have

$$\left(u_{m(i)}-u,v\right)=\sum_{n=1}^{s}(v,\varphi_n)\left(u_{m(i)}-u,\varphi_n\right)+\left(u_{m(i)}-u,\sum_{n=s+1}^{\infty}(v,\varphi_n)\varphi_n\right),$$

we use estimation

$$\left|\left(u_{m(i)}-u,\sum_{n=s+1}^{\infty}(v,\varphi_n)\varphi_n\right)\right|\le const\left(\sum_{n=s+1}^{\infty}(v,\varphi_n)^2\right)^{\frac{1}{2}}.$$

Let the number s sufficiently large, then for any positive number ε given in advance we have inequality $const\left(\sum_{n=s+1}^{\infty}(v,\varphi_n)^2\right)^{\frac{1}{2}}\le\frac{\varepsilon}{2}$, then for all $t\in[0,T]$ for sufficiently large $m(i)$ the first amount is also less $\frac{\varepsilon}{2}$ to all $t\in[0,T]$, that proved uniformly for all $t\in[0,T]$, then sequence $\left\{u_{m(i)}\right\}$ limits to u weakly in $L^2(R^l)$ relatively $t\in[0,T]$. There is a subsequence of the sequence $\left\{u_{m(i)}\right\}$ which converges to u weak in $L^2(R^l)$ with its derivatives $\left\{\partial_j u_{m(i)}\right\}$, again we denote it as $\left\{u_m\right\}$.

We show that the limit function u is the solution of the Cauchy problem for the equation (1). We can write identity

$$\langle u_m(\tau), v(\tau)\rangle|_0^t + \int_0^t \left(-\langle u_m(\tau), \partial_t v(\tau)\rangle + \lambda\langle u_m(\tau), v(\tau)\rangle\right)d\tau +$$
$$+\int_0^t \left\langle \sum_{i,j=1,\ldots,l} a_{ij}\frac{\partial}{\partial x_j}u_m, \frac{\partial}{\partial x_i}v\right\rangle d\tau + \int_0^t \langle b, v\rangle d\tau = \int_0^t \langle f, v\rangle d\tau,$$

which is true for any function $v = \sum_{i=1}^{m} d_i^m(t)\varphi_i(x)$ where $d_i^m(t)$ - continuous functions of argument $t \in [0, T]$, which have bounded generalized derivatives. The set of such functions is denoted $\wp_m$ and function u_m belongs to $\wp_m$. Let set $\wp$, is formed by the union at m sets $\wp_m$ is dense in W_1^2.

We have for any function $v = \sum_{i=1}^{m} d_i^m(t)\varphi_i(x)$, (due to weak in $L^2(R^l)$) the sequence $\{u_m\}$ convergence to function u evenly over $t \in [0, T]$, sequence $\{u_m\}$ convergence to function u strong in $L^2([0, T] \times R^l)$, henc follows convergence in $L^2(R^l)$ almost all $t \in [0, T]$ and almost everywhere. Using the estimates that were obtained above go to the limit and we obtain identity

$$\int_0^t \left(-\langle u(\tau), \partial_t v(\tau)\rangle + \lambda\langle u(\tau), v(\tau)\rangle\right)d\tau + \int_0^t \left\langle \sum_{i,j=1,\ldots,l} a_{ij}\frac{\partial}{\partial x_j}u, \frac{\partial}{\partial x_i}v\right\rangle d\tau +$$
$$+\int_0^t \langle b, v\rangle d\tau + \langle u(\tau), v(\tau)\rangle|_0^t = \int_0^t \langle f, v\rangle d\tau,$$

it is true for any v element of the set $\wp$.

Then we use the inequality monotony of type, that is, for any φ element of the set $\wp_m$, we must prove the following inequality

$$\int_0^t \left\langle \sum_{i,j=1,\ldots,l} a_{ij}(\tau, x, u_m)\frac{\partial}{\partial x_j}u_m - a_{ij}(\tau, x, \varphi)\frac{\partial}{\partial x_j}\varphi, \frac{\partial}{\partial x_i}\left(u_m(\tau) - \varphi(\tau)\right)\right\rangle d\tau +$$
$$+ function\left(\|u_m - \varphi\|\right) \geq 0.$$

Indeed, let $v = u_m - \varphi$ then we have

$$\langle u_m(\tau), u_m - \varphi\rangle|_0^t + \int_0^t \left(-\langle u_m(\tau), \partial_t(u_m - \varphi)(\tau)\rangle + \lambda\langle u_m(\tau), (u_m - \varphi)(\tau)\rangle\right)d\tau +$$
$$+\int_0^t \left\langle \sum_{i,j=1,\ldots,l} a_{ij}\frac{\partial}{\partial x_j}u_m, \frac{\partial}{\partial x_i}(u_m - \varphi)\right\rangle d\tau + \int_0^t \langle b, (u_m - \varphi)\rangle d\tau = \int_0^t \langle f, (u_m - \varphi)\rangle d\tau,$$

and therefore

$$\int_0^t \left\langle \sum_{i,j=1,\dots,l} a_{ij} \frac{\partial}{\partial x_j} u_m, \frac{\partial}{\partial x_i}(u_m - \varphi) \right\rangle d\tau =$$

$$= -\left(\langle u_m(\tau), u_m - \varphi \rangle |_0^t + \int_0^t \left(-\langle u_m(\tau), \partial_t (u_m - \varphi)(\tau) \rangle + \lambda \langle u_m(\tau), (u_m - \varphi)(\tau) \rangle \right) d\tau + \int_0^t \langle b, (u_m - \varphi) \rangle d\tau \right) +$$

$$+ \int_0^t \langle f, (u_m - \varphi) \rangle d\tau,$$

and then

$$\int_0^t \left(-\langle u_m(\tau), \partial_t (u_m - \varphi)(\tau) \rangle + \lambda \langle u_m(\tau), (u_m - \varphi)(\tau) \rangle \right) d\tau +$$

$$+ \int_0^t \left\langle \sum_{i,j=1,\dots,l} a_{ij} \frac{\partial}{\partial x_j} u_m, \frac{\partial}{\partial x_i}(u_m - \varphi) \right\rangle d\tau + \int_0^t \langle b, (u_m - \varphi) \rangle d\tau -$$

$$- \int_0^t \langle f, (u_m - \varphi) \rangle d\tau - \frac{1}{2} \|u_m\|^2 |_{t=0}^{t=t} + (u_m, \varphi) |_{t=0}^{t=t} + function(\|u_m - \varphi\|) \geq 0.$$

The last inequality at a fixed function φ almost all $t \in [0, T]$ you can go to the limit $m \to \infty$, then we get the following inequality

$$\int_0^t \left(-\langle u(\tau), \partial_t (u - \varphi)(\tau) \rangle + \lambda \langle u(\tau), (u - \varphi)(\tau) \rangle \right) d\tau +$$

$$+ \int_0^t \left\langle \sum_{i,j=1,\dots,l} a_{ij} \frac{\partial}{\partial x_j} u, \frac{\partial}{\partial x_i}(u - \varphi) \right\rangle d\tau + \int_0^t \langle b, (u - \varphi) \rangle d\tau -$$

$$- \int_0^t \langle f, (u - \varphi) \rangle d\tau - \frac{1}{2} \|u\|^2 |_{t=0}^{t=t} + (u, \varphi) |_{t=0}^{t=t} + function(\|u - \varphi\|) \geq 0.$$

If we take $v = u$ then we have (in order to make this replacement we must use estimates that were obtained earlier, since the function u, generally speaking are not differentiated by $t \in [0, T]$)

$$\frac{1}{2} \|u\|^2 |_0^t + \lambda \int_0^t \|u(\tau)\|^2 d\tau +$$

$$+ \int_0^t \left\langle \sum_{i,j=1,\dots,l} a_{ij} \frac{\partial}{\partial x_j} u, \frac{\partial}{\partial x_i} u \right\rangle d\tau + \int_0^t \langle b, u \rangle d\tau = \int_0^t \langle f, u \rangle d\tau.$$

We use arbitrary function $v \in \wp_m$ and any m , so for any element $v \in \wp = \bigcup_{m=1}^{\infty} \wp_m$, we get

$$\int_0^t \left(-\langle u(\tau), \partial_t (u - v)(\tau) \rangle + \lambda \langle u(\tau), (u - v)(\tau) \rangle \right) d\tau +$$

$$+ \int_0^t \left\langle \sum_{i,j=1,\dots,l} a_{ij} \frac{\partial}{\partial x_j} u, \frac{\partial}{\partial x_i}(u - v) \right\rangle d\tau + \int_0^t \langle b, (u - v) \rangle d\tau -$$

$$- \int_0^t \langle f, u - v \rangle d\tau + function(\|u - v\|) \geq 0.$$

Since the set $\wp$ which is formed by the union m sets $\wp_m$ which is dense in W_1^2, then for any $\varepsilon > 0$ and for any function $\varphi \in \wp$ we can put $v = u - \varepsilon\varphi$, then

$$\varepsilon\int_0^t\left(-\langle u(\tau),\partial_t\varphi(\tau)\rangle+\lambda\langle u(\tau),\varphi(\tau)\rangle\right)d\tau+$$
$$+\varepsilon\int_0^t\left\langle\sum_{i,j=1,\dots,l}a_{ij}\frac{\partial}{\partial x_j}u,\frac{\partial}{\partial x_i}\varphi\right\rangle d\tau+\varepsilon\int_0^t\langle b,\varphi\rangle d\tau-$$
$$-\varepsilon\int_0^t\langle f,\varphi\rangle d\tau+function\left(\varepsilon\|\varphi\|\right)\geq 0.$$

we go to the limit $\varepsilon\to 0$, we obtain

$$\int_0^t\left(-\langle u(\tau),\partial_t\varphi(\tau)\rangle+\lambda\langle u(\tau),\varphi(\tau)\rangle\right)d\tau+$$
$$+\int_0^t\left\langle\sum_{i,j=1,\dots,l}a_{ij}\frac{\partial}{\partial x_j}u,\frac{\partial}{\partial x_i}\varphi\right\rangle d\tau+\int_0^t\langle b,\varphi\rangle d\tau-\int_0^t\langle f,\varphi\rangle d\tau\geq 0.$$

As set $\wp$ is dense in W_1^2, then the last inequality implies that for every $\varphi\in W_1^2$ is true equality

$$\int_0^t\left(-\langle u(\tau),\partial_t\varphi(\tau)\rangle+\lambda\langle u(\tau),\varphi(\tau)\rangle\right)d\tau+$$
$$+\int_0^t\left\langle\sum_{i,j=1,\dots,l}a_{ij}\frac{\partial}{\partial x_j}u,\frac{\partial}{\partial x_i}\varphi\right\rangle d\tau+\int_0^t\langle b,\varphi\rangle d\tau-\int_0^t\langle f,\varphi\rangle d\tau=0,$$

which means that the element $u\in W_1^2$ is the solution of a given equation (1).

Conclusions

In this article we developed a new method for studying existence of solution of quasi-linear evolution system. We build new differential form $h_\lambda^p:\left(\underset{1}{\overset{N}{\times}}W_1^p(R^l,d^lx)\right)\times\left(\underset{1}{\overset{N}{\times}}W_1^q(R^l,d^lx)\right)\to R$ and operator associated with this form $A^p:W_1^p\left(R^l,d^lx\right)\to W_{-1}^p\left(R^l,d^lx\right)$. We proved a priori estimates for quasi-linear parabolic system. It is shown that quasi-linear parabolic systems possess unique solutions for sufficiently smooth initial values.

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