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Compactness in asymmetric quasi normed spaces

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Abstract

In this paper it is represented a study of precompact and compact subsets on asymmetric quasi normed spaces.

Keywords: Asymmetric quasi norm; topology; compactness.

1. Introduction

Definition 1.1. A function $p : X \to R^+$ is an *asymmetric quasi norm* on X if for every $x, y, z \in X$, $\lambda \in R^+$ and $k \geq 1$:

1) $p(x) = p(-x) = 0 \Leftrightarrow x = 0$
2) $p(\lambda x) = \lambda p(x)$
3) $p(x+y) \leq k\left(p(x) + p(y)\right)$.

For $k = 1$ the function is called the *asymmetric norm function*, and the pair (X, p) is called the asymmetric normed space. More information on this structure can be found in [1] and [2].

The function $p^{-1} : X \to R^+$ defined by $p^{-1}(x) = p(-x)$ is also an asymmetric quasi norm.

While the formula $p^s(x) = \max\left\{p(x), p^{-1}(x)\right\}$ gives a quasi norm on X.

Definition 1.2. A *quasi metric* on a set X is a function $d : X \times X \to R^+$ that satisfies:

1) $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$
2) $d(x+y) \leq d(x, z) + d(z, y)$, for every $x, y, z \in X$.

Each quasi metric d on X generates a topology $T(d)$ on X, that in general is a T_0 topology. The basic open sets can be defined as the d-balls:

$$B_d(x, r) = \left\{y \in X : d(x, y) < r\right\},\ x \in X,\ r > 0.$$

An asymmetric quasi norm p on a linear space X induces the quasi metric d_p by means of the formula:

$$d_p(x, y) = p(y - x),\ x, y \in X.$$

Thus the sets: $B^p_\varepsilon(0) = \left\{x \in X : p(x) < \varepsilon\right\}$, $\varepsilon > 0$, define a fundamental system of neighborhoods of zero for the topology $T(d_p)$, and for all $y \in X$, the sets $B^p_\varepsilon(y) = y + B^p_\varepsilon(0)$ define a fundamental system of neighborhoods of y (these sets are convex). Than we say that the pair (X, p) is an asymmetric quasi normed linear space.

Now let us deal that with the topologies induced by p, p^{-1}, p^s, we will write these symbols before the property we are referring if necessary; for instance, we will write p- compact set, or p^s- compact set to refer to compactness of a set with respect to the topology induced by p (resp. by p^s).

If the space (X, p^s) is complete, we say that (X, p) is a bi-Banach space (see [3]).

Denote by $B^p_{\leq,\varepsilon}$ the set:

$$B^p_{\leq,\varepsilon}(0) = \{x \in X : p(x) \leq \varepsilon\},\ \varepsilon > 0.$$

Let (X, p) an asymmetric quasi normed space and $x \in X$, denote by Ψ_x the set defined by:

$$\Psi_x = \{y \in X : d_p(x, y) = p(y - x) = 0\}.$$

In particular:

$$\Psi_0 = \{y \in X : d_p(0, y) = p(y) = 0\}.$$

Observe that Ψ_x is the closure of $\{x\}$ in (X, p^{-1}).

Given a set $A \subset X$ of an asymmetric quasi normed space (X, p)(analogue Lemma 2 in [4]), we have that:

$$\bigcup_{x \in A} \Psi_x = A + \Psi_0$$

where:

$$A + \Psi_0 = \{z \in X : z = x + y, x \in A, y \in \Psi_0\}.$$

We have also that $B^p_\varepsilon(x) = B^p_\varepsilon(x) + \Psi_0, x \in X$ (see [4]) and if $A \subset X$ is an open set, then $A = A + \Psi_0$.

Definition 1.3. (Its analogue is in [4]) Let (X, p) be an asymmetric quasi normed space. We say it is *right-bounded* if there exists a real constant $r > 0$, such that:

$$rB^p_1(0) \subset B^{p^s}_1(0) + \Psi_0.$$

Note that the inclusion $B^{p^s}_\varepsilon(x) + \Psi_0 \subset B^p_\varepsilon(x)$ holds in any asymmetric quasi norm space (X, p), for every $\varepsilon > 0$ and $x \in X$. In fact if $y \in B^{p^s}_\varepsilon(x) + \Psi_0$, then there are some $x_0 \in B^{p^s}_\varepsilon(x)$ and $z_0 \in \Psi_0$ such that $y = x_0 + z_0$. By the triangle inequality we have that $p(y - x) < \varepsilon$. Then $y \in B^p_\varepsilon(x)$.

2. Main results

Let (X, p) be an asymmetric quasi normed space.

Definition 2.1. A subset A of X is p*-bounded* if there is a positive constant M such that $p(x) \leq M$ for all $x \in A$. It is obvious that if a set A is p-bounded and p^{-1}-bounded, then A is p^s-bounded.

Definition 2.2. We say that a subset A of X is p*-precompact* if for all $\varepsilon > 0$ we can find a finite set of points $\{a_1, \ldots, a_n\}$ in A such that: $A \subset \bigcup_{i=1}^n B^p_\varepsilon(a_i)$.

We say that a subset A of an asymmetric quasi normed space (X, p) is *outside* p*-precompact* if for each $\varepsilon > 0$ there is a finite set $\{x_1, \ldots, x_n\}$in X such that $A \subset \bigcup_{i=1}^n B^p_\varepsilon(x_i)$. Obviously if a set A is p-precompact, then it is outside p-precompact; the converse is not true in general, a p*-convergent* sequence is outside p-precompact but not necessarily p-precompact.

The relationship between p-precompactness and outside p-precompactness is given by the following proposition.

Proposition 2.3. Let (X, p) be an asymmetric quasi normed space. A subset A of X is p-precompact if and only if for all $\varepsilon > 0$ there is a finite set $\{x_1,...,x_n\}$ in X such that $A \subset \bigcup_{i=1}^{n} B_\varepsilon^p(x_i)$ and $B_\varepsilon^{p^{-1}}(x_i) \cap A \neq \Phi$ for all $i \in \{1,...,n\}$.

Proof. The direct implication is obvious by the definition of p-precompact set. To prove the converse fix a positive ε and choose a finite set $\{x_1,...,x_n\}$ in X such that $A \subset \bigcup_{i=1}^{n} B_{\frac{\varepsilon}{2k}}^p(x_i)$ and $B_{\frac{\varepsilon}{2k}}^{p^{-1}}(x_i) \cap A \neq \Phi$ for some $k \geq 1$. Take $a_i \in B_{\frac{\varepsilon}{2k}}^{p^{-1}}(x_i) \cap A$; we must prove that $B_{\frac{\varepsilon}{2k}}^p(x_i) \subset B_\varepsilon^p(a_i)$. If $x \in B_{\frac{\varepsilon}{2k}}^p(x_i)$, then:

$$p(x-a_i) \leq k\left(p(x-x_i)+p(x_i-a_i)\right) < k\left(\frac{\varepsilon}{2k}+p^{-1}(a_i-x_i)\right) = k\left(\frac{\varepsilon}{2k}+\frac{\varepsilon}{2k}\right) = \varepsilon .$$

Then the set A is p-precompact.■

Proposition 2.4. Let (X, p) be an asymmetric quasi normed space then:

(1) The finite sum and the finite union of p-precompact sets is p-precompact.

(2) The convex hull of a p-precompact set is p-precompact.

Proof. (1) This result is an immediate consequence of the definition. We give the proof for the case of the sum of two p-precompact sets A_1 and A_2. Let $\varepsilon > 0$. Take $\frac{\varepsilon}{2k}$ for some $k \geq 1$, and consider the sets $\{x_1^1,...,x_n^1\} \subset A_1$ and $\{x_1^2,...,x_m^2\} \subset A_2$ such that:

$$A_1 \subset \bigcup_{i=1}^{n} B_{\frac{\varepsilon}{2k}}^p(x_i^1) \text{ and } A_2 \subset \bigcup_{i=1}^{m} B_{\frac{\varepsilon}{2k}}^p(x_i^2).$$

If $z \in A_1 + A_2$ then $z = z_1 + z_2$, with $z_1 \in A_1$ and $z_2 \in A_2$. There are elements x_i^1 and x_j^2 such that $p(z_1 - x_i^1) < \frac{\varepsilon}{2k}$ and $p(z_2 - x_j^2) < \frac{\varepsilon}{2k}$. Then:

$$p(z-(x_i^1+x_j^2)) \leq k\left(p(z_1-x_i^1)+p(z_2-x_j^2)\right) < k\left(\frac{\varepsilon}{2k}+\frac{\varepsilon}{2k}\right) = \varepsilon .$$

Thus the set $\{x_i^1 + x_j^2; i=1,...,n; j=1,...,m\}$ define an adequate set of centers of p-balls of radius ε to cover the set $A_1 + A_2$.

For the proof of (2), let see A a p-precompact subset of X. For $\varepsilon > 0$, we can find a set of points of A, $\{x_1,...,x_n\}$ such that $A \subset \{x_1,...,x_n\} + B_{\frac{\varepsilon}{2}}^p(0)$. Denote by $convex(A)$ the convex hull of A; we have:

$$convex(A) \subset convex(\{x_1,...,x_n\}) + B_{\frac{\varepsilon}{2}}^p(0).$$

Note that since $convex(\{x_1,...,x_n\})$ is p^s-compact, then it is p-precompact. Thus we can define a set $\{y_1,...,y_n\}$ in $convex(\{x_1,...,x_n\})$ such that:

$$convex(\{x_1,...,x_n\}) \subset \{y_1,...,y_n\} + B_{\frac{\varepsilon}{2}}^p(0).$$

Then we conclude that:

$$convex(A) \subset \{y_1,...,y_n\} + B^p_{\frac{\varepsilon}{2}}(0) + B^p_{\frac{\varepsilon}{2}}(0) \subset \{y_1,...,y_n\} + B^p_{\varepsilon}(0)$$

and confirm that $convex(A)$ is p-precompact. ■

Proposition 2.5. A subset A of (X,p) is p-precompact if and only if the p^{-1}-closure of A is p-precompact.

Proof. If A is p-precompact and $\varepsilon > 0$, $k \geq 1$, there is a finite set in A, $\{x_1,...,x_n\}$ such that:

$$A \subset \bigcup_{i=1}^{n} B^p_{\frac{\varepsilon}{2k}}(x_i) \subset \bigcup_{i=1}^{n} B^p_{\leq,\frac{\varepsilon}{2k}}(x_i).$$

Note that the sets $B^p_{\leq,\frac{\varepsilon}{2k}}(a_i)$ are p^{-1}-closed. Then:

$$\overline{A}^{p^{-1}} \subset \overline{\bigcup_{i=1}^{n} B^p_{\leq,\frac{\varepsilon}{2k}}(x_i)}^{p^{-1}} \subset \bigcup_{i=1}^{n} \overline{B^p_{\leq,\frac{\varepsilon}{2k}}(x_i)}^{p^{-1}} = \bigcup_{i=1}^{n} B^p_{\leq,\frac{\varepsilon}{2k}}(x_i) \subset \bigcup_{i=1}^{n} B^p_{\leq,\varepsilon}(x_i).$$

Conversely: If $\overline{A}^{p^{-1}}$ is p-precompact, for $\varepsilon > 0$ and some $k \geq 1$, there is a finite subset of $\overline{A}^{p^{-1}}$, $\{x_1,...,x_n\}$, such that:

$$A \subset \overline{A}^{p^{-1}} \subset \bigcup_{i=1}^{n} \left(x_i + B^p_{\frac{\varepsilon}{2k}}(0) \right).$$

We have that for every $i \in \{1,...,n\}, x_i \in \overline{A}^{p^{-1}}$. Then for a fixed index i there is some $a_i \in A$ such that:

$$p^{-1}(a_i - x_i) = p(x_i - a_i) < \frac{\varepsilon}{2k}.$$

Now let us prove that: $x_i + B^p_{\frac{\varepsilon}{2k}}(0) \subset a_i + B^p_{\varepsilon}(0)$.

Let $y \in x_i + B^p_{\frac{\varepsilon}{2k}}(0)$. Then $p(y - x_i) < \frac{\varepsilon}{2}$ and:

$$p(y-a_i) \leq k\left(p(y-x_i) + p(x_i - a_i)\right) < k\left(\frac{\varepsilon}{2k} + \frac{\varepsilon}{2k}\right) = \varepsilon.$$

So we have A is p-precompact.■

Let (X,p) be an asymmetric quasi normed space and $K \subset X$.

Then K is compact respect to the topology $T(d_p)$ induced by the asymmetric quasi norm p if and only if $K + \Psi_0$ is compact for the same topology (see [4]). An analogue proposition and its proof of the following proposition, for the case of the asymmetric norm is given in [5]

Proposition 2.6. Let (X,p) be an asymmetric quasi normed space. If K is a subset of X such that $K_0 \subset K \subset K_0 + \Psi_0$ where K_0 is p^s-compact, then K is p-compact.

Theorem 2.7. Let (X,p) be an asymmetric quasi normed space. Let K be a subset of X, then:

(1) If (X,p) is a bi-Banach right-bounded space with constant $r=1$ and K is p-precompact set then there is a p^s-compact subset K_0 of X such that $K\subset K_0+\Psi_0$.

(2) If there is a p^s-precompact subset K_0 of X such that $K\subset K_0+\Psi_0$ then K is outside p-precompact.

Proof. (1) *Step 1*. First we construct an special family of balls covering the set K in order to find an adequate p^s-compact set. By the definition of p-precompactness we have that for $\varepsilon=\frac{1}{4k}$ and $\varepsilon=\frac{1}{2k}$, $k\geq 1$:

$$K\subset\bigcup_{i=1}^{n_1}B^p_{\frac{1}{2k}}(x_i^1),\ \left\{x_1^1,\dots,x_{n_1}^1\right\}\subset K \text{ and } K\subset\bigcup_{i=1}^{n_2}B^p_{\frac{1}{4k}}(x_i^2),\ \left\{x_1^2,\dots,x_{n_2}^2\right\}\subset K.$$

It follows that for all $i=1,\dots,n_2$, there is an index $j_i\in\{1,\dots,n_1\}$ such that $x_i^2\in B^p_{\frac{1}{2k}}(x_{j_i}^1)$.

We also have that $B^p_{\frac{1}{2k}}(x_{j_i}^1)\subset B^{p^s}_{\frac{1}{2k}}(x_{j_i}^1)+\Psi_0$. Thus $x_i^2=\overline{x}_i^2+z$ with $\overline{x}_i^2\in B^{p^s}_{\frac{1}{2k}}(x_{j_i}^1)$ and $z\in\Psi_0$.

If $y\in B^p_{\frac{1}{4k}}(x_i^2)$ then:

$$p(y-\overline{x}_i^2)\leq k\left(p(y-x_i^2)+p(z)\right)<k\left(\frac{1}{4k}\right)=\frac{1}{4}.$$

Thus $B^p_{\frac{1}{4k}}(x_i^2)\subset B^p_{\frac{1}{4k}}(\overline{x}_i^2)$, and $\left\{B^p_{\frac{1}{4k}}(\overline{x}_i^2):i=1,\dots,n_2\right\}$ defines a p-cover of K.

Following this construction for each $N\in\mathbb{N}$ we obtain a family $\left\{\overline{x}_1^N,\dots,\overline{x}_{n_N}^N\right\}$, such that for each $\overline{x}_i^N, i=1,\dots,n_N$ there is $\overline{x}_{j_i}^{N-1}$ such that:

$$p^s\left(\overline{x}_i^N-\overline{x}_{j_i}^{N-1}\right)<\frac{1}{2^{N-1}}$$

and:

$$K\subset\bigcup_{i=1}^{n_N}B^p_{\frac{1}{2^N}}\left(\overline{x}_i^N\right).$$

Let $L=\left\{\overline{x}_1^1,\overline{x}_2^1,\dots,\overline{x}_{n_1}^1,\overline{x}_1^2,\dots,\overline{x}_{n_2}^2,\dots\right\}$, with $\overline{x}_i^1=x_i^1$ for $i=1,\dots,n_1$. We will prove first that L is p^s-precompact.

Let $\varepsilon>0$ and consider some $N\in\mathbb{N}$ such that $\frac{1}{2^{N-2}}<\frac{\varepsilon}{k}$ for some $k\geq 1$. We omit the subindexes for the aim of clarity. Take some $\overline{x}^m\in L$; then we have two cases:

Case 1: $m\leq N$. Then we have:

$$\overline{x}^m\in B^{p^s}_{\varepsilon}\left(\overline{x}^m\right)\subset\bigcup_{i=1}^{n_m}B^{p^s}_{\varepsilon}\left(\overline{x}_i^m\right)\subset\bigcup_{l=1}^{N}\bigcup_{i=1}^{n_l}B^{p^s}_{\varepsilon}\left(\overline{x}_i^l\right).$$

Case 2: $m > N$, fixed $\overline{x}^{m}$ there is an element $\overline{x}^{m-1}$ such that $p^s\left(\overline{x}^{m} - \overline{x}^{m-1}\right) < \frac{1}{2^{m-1}}$.

In the same way there is $\overline{x}^{m-2}$ such that $p^s\left(\overline{x}^{m-1} - \overline{x}^{m-2}\right) < \frac{1}{2^{m-2}}$ and if we continue like this we obtain $\overline{x}^{N}$ such that $p^s\left(\overline{x}^{N} - \overline{x}^{N-1}\right) < \frac{1}{2^{N-1}}$. Then we have:

$$p^s\left(\overline{x}^{l} - \overline{x}^{l-1}\right) < \frac{1}{2^{l-1}} \text{ for } l = N, \dots, m.$$

Thus:

$$p^s\left(\overline{x}^{m} - \overline{x}^{N}\right) \le k\left(p^s\left(\overline{x}^{m} - \overline{x}^{m-1}\right) + p^s\left(\overline{x}^{m-1} - \overline{x}^{m-2}\right) + \dots + p^s\left(\overline{x}^{N+1} - \overline{x}^{N}\right)\right)$$

$$\le \frac{1}{2^{m-1}} + \frac{1}{2^{m-2}} + \dots + \frac{1}{2^{N}}$$

$$\le \frac{1}{2^{N-1}}\left(\sum_{j=1}^{\infty} \frac{1}{2^{j}}\right) = \frac{1}{2^{N-1}}.$$

Then we conclude $p^s\left(\overline{x}^{m} - \overline{x}^{N}\right) \le k\frac{1}{2^{N-1}} < k\frac{\varepsilon}{k} = \varepsilon$, that is: $\overline{x}^{m} \in B_{\varepsilon}^{p^s}\left(\overline{x}^{N}\right) \subset \bigcup_{i=1}^{n_N} B_{\varepsilon}^{p^s}\left(\overline{x}_i^{N}\right)$.

Therefore $L \subset \bigcup_{l=1}^{N} \bigcup_{i=1}^{n_l} B_{\varepsilon}^{p^s}\left(\overline{x}_i^{l}\right)$ which proves that L is p^s-precompact, thus $\overline{L}^{p^s}$ is p^s-compact.

Step 2. Let $K_0 = \overline{L}^{p^s}$. We must prove that $K \subset K_0 + \Psi_0$.

If $x \in K$, for each $n \in \mathbb{N}$, there is some $\overline{x}^{n}$ of the corresponding family obtained in the previous step (we omit the indexes since there in no confusion) such that:

$$x \in B_{\frac{1}{2^n}}^{p}\left(\overline{x}^{n}\right) = B_{\frac{1}{2^n}}^{p^s}\left(\overline{x}^{n}\right) + \Psi_0.$$

Then for every $n \in \mathbb{N}$, there are $\overline{y}^{n} \in B_{\frac{1}{2^n}}^{p^s}\left(\overline{x}^{n}\right)$ and $z^n \in \Psi_0$ such that $x = \overline{y}^{n} + z^n$.

Consider the sequence $\left\{\overline{x}^{n}\right\}_{n\in\mathbb{N}} \subset \overline{L}^{p^s}$; since $\overline{L}^{p^s}$ is p^s-compact there is a subsequence $\left\{\overline{x}^{n_l}\right\}_l$, p^s-convergent to $x_0 \in \overline{L}^{p^s}$.

Let us now prove that $p(x - x_0) = 0$. For a positive ε there is some index l_0 such that for all $l \ge l_0$, we have: $p^s(\overline{x}^{n_l} - x_0) < \frac{\varepsilon}{2}$, note that: $p^s(\overline{x}^{n_l} - \overline{y}^{n_l}) < \frac{1}{2^{n_l}}$.

If we choose l_1 such that $\frac{1}{2^{n_l}} < \frac{\varepsilon}{2k}$ for $l \ge l_1$ and some $k \ge 1$, and consider $l_2 = \max\{l_0, l_1\}$, then for all $l \ge l_2$ we have:

$$p(x - x_0) \le k\left(p(x - \overline{y}^{n_l}) + p(\overline{y}^{n_l} - \overline{x}^{n_l}) + p(\overline{x}^{n_l} - x_0)\right)$$

$$\leq k\left(0+p^{s}(\overline{y}^{n_l}-\overline{x}^{n_l})+p^{s}(\overline{x}^{n_l}-x_0)\right)<k\left(\frac{1}{2^{n_l}}+\frac{\varepsilon}{2k}\right)<k\left(\frac{\varepsilon}{2k}+\frac{\varepsilon}{2k}\right)=\varepsilon .$$

Since this can be done for every $\varepsilon > 0$, we obtain that $p(x-x_0)=0$ so $x-x_0 \in \Psi_0$. We conclude that:

$$x \in \overline{L}^{p^s} + \Psi_0 = K_0 + \Psi_0 .$$

End of the proof for (1).

For the proof of (2), choose some $\varepsilon > 0$. Since K_0 is p^s-precompact, there is a finite set $\{x_1,...,x_n\}$ in K_0 such that $K_0 \subset \bigcup_{i=1}^{n} B_{\varepsilon}^{p^s}(x_i)$. Then (see [4]):

$$K_0 \subset \bigcup_{i=1}^{n} B_{\varepsilon}^{p^s}(x_i) + \Psi_0 = \bigcup_{i=1}^{n}\left(B_{\varepsilon}^{p^s}(x_i)+\Psi_0\right) \subset \bigcup_{i=1}^{n} B_{\varepsilon}^{p}(x_i) .$$

That is clear that K is outside p-precompact. ■

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