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Certain Orthogonal Polynomials for Solving Boundary Value Problems Using the Variational Iteration Method

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Abstract.

This paper illustrates the use of certain orthogonal polynomials (Chebyshev polynomials) as trial functions in a variational iterative approach for solving 6th order boundary value problems (BVPs). The proposed method is validated through some numerical examples without any form of discretization or perturbation. The resulting numerical evidences show that the method is effective and accurate. All calculations are analyzed using Maple18 software.

Keywords: Boundary value problems, Chebyshev polynomials, orthogonal polynomials, Variational iteration method.

1. Introduction

Consider a general sixth order boundary value problem of the form:

$$a_6 \frac{d^6}{dx^6} y + a_0 y = f(x), a < x < b \qquad (1)$$

with boundary conditions

$$y(a) = k_1 , y'(a) = k_2 , y''(a) = k_3 , y(b) = l_1, y'(b) = l_2, y''(b) = l_3$$

where a_6 *and* a_0 are constants, $f(x)$continuous on [*a*,*b*]. These types of problems are prevalent in most fields of science and engineering such as viscoelastic flow, heat transfer, and in other fields of science and technology. However, solving this kind of problems has prompted many authors to propose some numerical techniques. [1] investigated the numerical solution for fifth order boundary value problem using sixth degree B-spline. [2] applied a modified variational iteration method (VIM) with canonical polynomials to seek the numerical solution to eight order boundary value problems. [3] equally obtained the numerical solution of fifth order boundary value problem using Mamadu-Njoseh polynomials as trial functions. [4] proposed the power series approximation method for a generalized boundary value problem (BVP). Also, the method of tau and tau-collocation approximation method was excessively used by [5] to seek the solution of first and second ordinary differential equations. [6] applied the

Adomian decomposition method with Laguerre polynomials for solving ordinary differential equation. [7-11] adopted the variational iteration decomposition method and the variational iteration homotopy perturbation method for the numerical solution of higher order boundary value problems. In like manner, [12] coupled the homotopy perturbation method with the standard variational iteration method to seek the numerical solution of ninth and tenth-order boundary value problems. [13] used the homotopy perturbation method and the variational iteration method to obtain the numerical solution of seventh order boundary value problems. [14] used the Collocation method to obtain the numerical solution of sixth-order boundary value problems

The objective of this is to seek the numerical solution of the 6^{th} order BVP using VIM with certain orthogonal polynomials as trial solutions. The VIM involves the construction of a correction functional for the required problem to enable the computation of the general Lagrange multiplier optimally using the variational theory. Then, the class of orthogonal polynomials – Chebyshev are introduced as trial functions in the approximation of the analytic solution of the given BVP.

2. The Standard Variational Iteration Method

Consider the following general differential equation [2-3]

$$Lu + Nu - g(x) = 0, \tag{2}$$

Where L is a linear operator, N a nonlinear operator and $g(x)$ is the inhomogeneous term
Constructing the correction functional by variational iteration method, the problem (2.1) is given by

$$u_{n+1} = u_n(x) + \int_0^x \lambda(t)\left[Lu_n(t) + N\widetilde{u_n(t)} - g(t)\right]dt \tag{3}$$

Where $\lambda(t)$ is a Lagrange multiplier (which can be identified optimally via variational theory). The subscripts n denote the nth approximation, $\widetilde{u_n(t)}$is considered as a restricted variation. i.e.$\widetilde{u_n}$= 0. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. In this method, we need to determine the Lagrange multiplier $\lambda(t)$ optimally and hence the successive approximation of the solution u will be readily obtained upon using the Lagrange multiplier and our $u_0(x)$ .and the solution is given by

$$\lim_{n-\infty} u_n = u$$

The Lagrange Multiplier plays a major role in the determination of the solution of the problem. Hence we summarize the formula for the Lagrange Multiplier given as [3]

$$\lambda(t) = (-1)^n \frac{1}{(n-1)!}(t-x)^{n-1} \tag{4}$$

Where n is the highest order of the given BVP

3 Chebyshev Polynomials

The Chebyshev polynomial of the first kind is defined as

$$T_r(x) = cos(rcos^{-1}x) = \sum_{i=0}^{n} C_r^{(n)} x^i,\ -1 \le x \le 1,$$

with

$$T_{r+1}(x) = 2xT_r(x) - T_{r-1}(x),$$

satisfying the conditions

$$T_0(x) = 1 \text{ and } T_1(x) = x.$$

Now, if $x \in [-1,1]$ is mapped objectively to $a \leq x \leq b$, then equation (3) becomes

$$T_{r+1}^{*}(x) = 2xT_{r}^{*}(x) - T_{r-1}^{*}(x),$$

satisfying the conditions

$$T_{0}^{*}(x) = 1 \text{ and } T_{1}^{*}(x) = \frac{2x-a-b}{b-a}.$$

Equation (5) is called the nth degree shifted Chebyshev polynomials. Thus, the first seven (7) Chebyshev polynomials are presented below:

$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_2(x) = 2x^2 - 1$$
$$T_3(x) = 4x^3 - 3x$$
$$T_4(x) = 8x^4 - 8x^2 + 1$$
$$T_5(x) = 16x^5 - 20x^3 + 5x$$
$$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$$
$$T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$$

5. Variational Iteration Method Using Chebyshev Polynomials

Using (2) and (3), we now assume an approximate solution of the form

$$u_{n,N}(x) = \sum_{i=0}^{N} a_{i,N} T_{i,N}^{*}(x)$$

Where $T_{i,N}(x)$ are Shifted Chebyshev Polynomials, $a_{i,N}$are constants to be determined, and N the degree of approximant. Hence we obtain the following iterative method

$$u_{n+1,N} = \sum_{i=0}^{N} a_{i,N} T_{i,N}^{*}(x) + \int_{0}^{x} \lambda(t) \left[L \sum_{i=0}^{N} a_{i,N} T_{i,N}^{*}(x) + N \sum_{i=0}^{N} a_{i,N} T_{i,N}^{*}(x) - g(t) \right] dt$$

6. Numerical Applications

In this section we applied the proposed technique to solve examples of which are linear and non-linear BVPs. The main objective here is to solve these two examples using the MVIM given in section 5 and compare our results with the presented result in **[9].**

Example 6.1 [9]

Consider the following sixth order linear boundary value problem

$$u^{(vi)}(x) = u(x) - 6e^{x}, \ 0 \leq x \leq 1 \quad (6)$$

subject to boundary conditions

$$u(0) = 1, u'(0) = 0, u''(0) = -1, u(1) = 0, u'(1) = -e, u''(1) = -2e \quad (7)$$

The exact solution is

$$u(x) = (1 - x)e^x \tag{8}$$

The correction functional for the boundary value problem is

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(t)\left[\frac{d^6 u_n}{dt^6} - u_n(t) + 6e^t\right]dt$$

Making the correction functional stationary, $\lambda(t) = \frac{(t-x)^5}{5!}$ as the Lagrange multiplier, we have the following

$$u_{n+1}(x) = u_n(x) + \int_0^x \frac{(t-x)^5}{5!}\left[\frac{d^6 u_n}{dt^6} - u_n(t) + 6e^t\right]dt$$

Applying the proposed method, we assume an approximate solution of the form

$$u_{n,6}(x) = \sum_{i=0}^{6} a_{i,6} T_{i,6}^*(x)$$

$$u_{n+1,N}(x) = \sum_{i=0}^{6} a_{i,6} T_{i,6}^*(x) + \int_0^x \frac{(t-x)^5}{5!}\left[\frac{d^6}{dt^6}\left(\sum_{i=0}^{6} a_{i,6} T_{i,6}^*(t)\right) - \sum_{i=0}^{6} a_{i,6} T_{i,6}^*(t) + 6e^t\right]dt$$

$$\begin{aligned} u_{n+1,N}(x) = a_{0,6} &+ a_{1,6}(2x - 1) + a_{2,6}(8x^2 - 8x + 1) + a_{3,6}(32x^3 - 48x^2 + 18x - 1) \\ &+ a_{4,6}(128x^4 - 48x^3 + 160x^2 - 32x + 1) \\ &+ a_{5,6}(512x^5 - 1280x^4 + 1120x^3 - 400x^2 + 50x - 1) \\ &+ a_{6,6}(2048x^6 - 6144x^5 + 6912x^4 - 3584x^3 + 840x^2 - 70x + 1) \\ &+ \int_0^x \frac{(t-x)^5}{5!}\left[\frac{d^6}{dt^6}\left(\sum_{i=0}^{6} a_{i,6} T_{i,6}^*(t)\right) - \sum_{i=0}^{6} a_{i,6} T_{i,6}^*(t) + 6e^t\right]dt \end{aligned}$$

Iterating and applying the boundary in *Equation (7)* the values of the unknown constants can be determined as follows

$$a_{0,6} = 0.6624924734, \qquad a_{1,6} = -0.4805374555, \qquad a_{2,6} = -0.1638612995$$

$$a_{3,6} = -0.02280554316, \qquad a_{4,6} = -0.002121611362, \qquad a_{5,6} = -0.00015492375$$

$$a_{6,6} = -0.00007484949877$$

Consequently, the series solution is given as and the corresponding results are shown in table 6.1

$$\begin{aligned} u(x) = 0.9999999996 &- 0.499999999x^2 - 0.3333334120x^3 - 0.1249998279x^4 - 0.03333342796x^5 \\ &- 0.006944434047x^6 - 0.001190583117x^7 - 0.0001732992382x^8 \\ &- 0.00002248940926x^9 - 0.000002137988454x^{10} - 0.0000003887715248x^{11} + O(x)^{12} \end{aligned}$$

Table 6.1: The result of the proposed method compared with Variational decomposition method

x	Exact solution	Approximate Solution	Present Method Error	VDM Error
0.0	1.0000000	1.0000000	0.0000e+00	0.0000e+00
0.1	0.9946538	0.9946538	1.0000e-10	4.0933e-04
0.2	0.9771222	0.9771222	1.0000e-10	7.7820e-04
0.3	0.9449012	0.9449012	1.1000e-09	1.0704e-03
0.4	0.8950948	0.8950948	1.6000e-09	1.2578e-03
0.5	0.8243606	0.8243606	1.9000e-09	1.3223e-03
0.6	0.7288475	0.7288475	1.6000e-09	1.2578e-03
0.7	0.6041258	0.6041258	1.2000e-09	1.0740e-03
0.8	0.4451082	0.4451082	4.0000e-10	7.7820e-04
0.9	0.2459603	0.2459603	1.0000e-10	4.0933e-04
1.0	0.0000000	0.0000000	0.0000e+00	0.0000e+00

-

Example 6.2 [14]

Consider the following sixth order linear boundary value problem of the form

$$u^{(vi)}(x) = u(x) - 6cosx \quad ,0 \leq x \leq 1, \tag{9}$$

subject to boundary conditions

$$u(0) = 0, u^{'}(0) = -1, u^{''}(0) = 2, u(1) = 0, u^{'}(1) = \sin(1), u^{''}(1) = 2\cos(1) \tag{10}$$

The exact solution is

$$u(x) = (x-1)\sin(x) \tag{11}$$

The correction functional for the boundary value problem is

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(t)\left[\frac{d^6 u_n}{dt^6} - u_n(t) + 6\cos(t)\right]dt$$

Making the correction functional stationary, $\lambda(t) = \frac{(t-x)^5}{5!}$ as the Lagrange multiplier, we have the following

$$u_{n+1}(x) = u_n(x) + \int_0^x \frac{(t-x)^5}{5!}\left[\frac{d^6u_n}{dt^6} - u_n(t) + 6\cos(t)\right]dt$$

Applying the proposed method, we assume an approximate solution of the form

$$u_{n,6}(x) = \sum_{i=0}^{6} a_{i,6}T^*_{i,6}(x)$$

$$u_{n+1,N}(x) = \sum_{i=0}^{6} a_{i,6}T^*_{i,6}(x) + \int_0^x \frac{(t-x)^5}{5!}\left[\frac{d^6}{dt^6}\left(\sum_{i=0}^{6} a_{i,6}T^*_{i,6}(t)\right) - \sum_{i=0}^{6} a_{i,6}T^*_{i,6}(t) + 6\cos(t)\right]dt$$

$$\begin{aligned} u_{n+1,N}(x) = a_{0,6} &+ a_{1,6}(2x-1) + a_{2,6}(8x^2-8x+1) + a_{3,6}(32x^3-48x^2+18x-1) \\ &+ a_{4,6}(128x^4-48x^3+160x^2-32x+1) \\ &+ a_{5,6}(512x^5-1280x^4+1120x^3-400x^2+50x-1) \\ &+ a_{6,6}(2048x^6-6144x^5+6912x^4-3584x^3+840x^2-70x+1) \\ &+ \int_0^x \frac{(t-x)^5}{5!}\left[\frac{d^6}{dt^6}\left(\sum_{i=0}^{6} a_{i,6}T^*_{i,6}(t)\right) - \sum_{i=0}^{6} a_{i,6}T^*_{i,6}(t) + 6\cos(t)\right]dt \end{aligned}$$

Iterating and applying the boundary condition in *Equation (10)* the values of the unknown constants can be determined as follows

$$a_{0,6} = -0.1189983652, \qquad a_{1,6} = 0.004433792806, \qquad a_{2,6} = 0.1194906850$$

$$a_{3,6} = -0.005206469651, \qquad a_{4,6} = -0.01244558500, \qquad a_{5,6} = 0.000025775748$$

$$a_{6,6} = 0.000003337662563$$

Finally the series solution is

$$\begin{aligned} u(x) = -x + x^2 &+ 0.166666024x^3 - 0.166665218x^4 - 0.0083341261x^5 + 0.008333341961x^6 \\ &+ 0.00019836501x^7 - 0.0001982736292x^8 - 0.000002953518270x^9 \\ &- 0.0000029083098x^{10} - 0.00000003659566792x^{11} + O(x)^{12} \end{aligned}$$

Table 6.2: The result of the proposed method compared with Variational Iteration Decomposition method

x	Exact solution	Approximate Solution	Present Method Error	VIDM Error
0.0	0.0000000	0.0000000	0.0000e+00	0.0000e+00
0.1	-0.0898501	-0.0898501	5.0000e-11	4.0933e-06
0.2	-0.1589355	-0.1589355	4.0000e-10	7.7820e-06
0.3	-0.2068641	-0.2068641	8.0000e-10	1.0704e-05
0.4	-0.2336510	-0.2336510	1.3000e-09	1.2578e-05
0.5	-0.2397128	-0.2397128	1.8000e-09	1.3223e-05
0.6	-0.2258570	-0.2258570	1.6000e-09	1.2578e-05
0.7	-0.1932653	-0.1932653	1.3000e-09	1.0740e-05
0.8	-0.1434712	-0.1434712	7.0000e-10	7.7820e-04
0.9	-0.0783327	-0.0783327	3.5000e-10	4.0933e-04
1.0	0.0000000	-0.0000000	9.4000e-11	0.0000e+00

7. Discussion of Results

The numerical evidences show that VIM is an effective and accurate solver for 6th order BVPs as shown in the Tables 6.1 and 6.2. In Table 6.1, a maximum error of order 10^{-11} was obtained with the present method reflecting its superiority over VDM at every interpolation point. Likewise, the table 6.2, incurred a maximum error of order 10^{-11} against that of VIDM with a maximum error of order 10^{-6}.

8. Conclusion

In this paper, the modified variational iteration method has been found to be a good and efficient method for linear sixth order boundary value problems. The modification involves the use of Chebyshev polynomials as trial function coupled with standard variational iterative scheme. From table **6.1** and **6.2** the maximum error of the proposed method was observed to be $\mathbf{10}^{-10}$ and $\mathbf{10}^{-11}$ respectively. This makes it an advantage over methods as available in literature.

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