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Comparison of Sumudu and Laplace decomposition method for solving fractional Lane-Emden type differential equations

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Abstract

In this work we discuss the solution of delay differential equations of the fractional order of Lane-Emden type equation using Sumudu and Laplace transforms with decomposition method, definitions and some useful theorems are stated.

Keywords: Sumudu transform; Lane-Emden equation of fraction; Caputo fractional derivative; inverse of Sumudu transform.

1. Introduction

The Adomian decomposition method first was introduced by George Adomian $[1]$, the method is powerful and effective in solving a many different equations in mathematical and engineering field. The Lane-Emden type equations published by Janathan Homer Lane in $[6]$, and Emden who extended details of Lane-Emden equation used the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical lows of thermodynamics. The Laplace transform of $f(t)$ is defined as a function $F(s)$ by the integral:

$$L[f(t)]=\int_0^{\infty} e^{-st} f(t)\,dt \quad , \qquad 0<t<\infty$$

Beside the Sumudu transform is defined over the set of a functions :

$$A=\left\{ f(t) \backslash \exists\, M, \tau_1, \tau_2 > 0, |f(t)| < M e^{\frac{|t|}{\tau_j}}, if\ t \in (-1)^j \times [0,\infty) \right\}$$

by: $F(u)=S[f(t)]=\frac{1}{u}\int_0^{\infty} e^{\frac{-t}{u}} f(t)\,dt \quad , u\in(-\tau_1,\tau_2)$

The solution of fractional Lane-Emden type equation by Sumudu decomposition method

In this section we use the technique of Sumudu decomposition method to solve Lane-Emden equation of fraction order.

Theorem 1

Let $f \in A$, then the Sumudu transform integer order derivative of f^{n} is given by :

$$S\left[f^{n}(t)\right]=\frac{S(u)}{u^{n}}-\sum_{k=0}^{n-1}\frac{f^{k}(0)}{u^{n-k}} \qquad , n\geq 0$$

For more see $[3]$.

Theorem 2

The Sumudu transform of fraction function $t\,D^{\alpha}y$ where $n-1<\alpha\leq n$, $n=1,2$.

$$S\left[t\,D^{\alpha}y\right]=u\frac{d}{du}\left[u\,S(D^{\alpha}y)\right] \qquad , \quad n-1<\alpha\leq n \quad , \quad n=1,2$$

Proof :

$$S\left[D^{\alpha}y\right]=\frac{1}{u}\int_{0}^{\infty}e^{\frac{-t}{u}}D^{\alpha}y\,dt$$

$$u\,S\left[D^{\alpha}y\right]=\int_{0}^{\infty}e^{\frac{-t}{u}}D^{\alpha}y\,dt$$

$$\frac{d}{du}\left[u\,S\left[D^{\alpha}y\right]\right]=\int_{0}^{\infty}\frac{d}{du}e^{\frac{-t}{u}}D^{\alpha}y\,dt$$

$$=\int_{0}^{\infty}\frac{1}{u^{2}}e^{\frac{-t}{u}}t\,D^{\alpha}ydt$$

$$=\frac{1}{u}\int_{0}^{\infty}\frac{1}{u}e^{\frac{-t}{u}}t\,D^{\alpha}ydt$$

$$=\frac{1}{u}S\left[t\,D^{\alpha}y\right]$$

$$u\frac{d}{du}\left[u\,S\left[D^{\alpha}y\right]\right]=S\left[t\,D^{\alpha}y\right].\square$$

Definition 1

Sumudu transform of the Caputo fractional derivative is defined as follows:

$$S\left[D^{\alpha}f(t)\right]=\frac{S\left[f(t)\right]}{u^{\alpha}}-\sum_{k=0}^{n-1}\frac{f^{k}(0)}{u^{\alpha-k}} \qquad , \qquad n-1<\alpha\leq n$$

Analysis of the method

Here we consider the Lane-Emden equation of fraction of the form:

$$D^{\alpha}y(t)+\tfrac{2n}{t}y'(t)+\tfrac{n(n-1)}{t^{2}}y(t)+f(t,y)=g(t) \quad , n=1,2 \qquad (1)$$

With initial condition:

$$y(0)=A \qquad , \qquad y'(0)=B$$

Where f is a real function, g is a known function, A and B are constant, if $n=1$ we obtain:

$$D^{\alpha}y(t)+\frac{2}{t}y'(t)+f(t,y)=g(t) \tag{2}$$

Multiplying equation (2) with t and applying the Sumudu transform we get:

$$S\left[t\,D^{\alpha}y(t)+2y'(t)+t\,f(t,y)\right]=S\left[t\,g(t)\right] \tag{3}$$

By using the theorem (2) we have:

$$u\frac{d}{du}\left[uS(D^{\alpha}y)\right]+2\left[\frac{Y(u)-y(0)}{u}\right]+S\left[t\,f(t,y)\right]=S\left[t\,g(t)\right]$$

$$u\frac{d}{du}\left[u\left(\frac{Y(u)-y(0)}{u^{\alpha}}-\frac{y'(0)}{u^{\alpha-1}}\right)\right]+2\left[\frac{Y(u)-y(0)}{u}\right]+S\left[t\,f(t,y)\right]=S\left[t\,g(t)\right] \tag{4}$$

Simplifying equation (4) we obtain:

$$\frac{d}{du}\left[\frac{Y(u)-y(0)}{u^{\alpha-1}}\right]+\frac{2Y(u)}{u^{2}}-\frac{2y(0)}{u^{2}}+\frac{1}{u}S\left[t\,f(t,y)\right]=\frac{1}{u}S\left[t\,g(t)\right] \tag{5}$$

By integrating equation (5) with respect to u we get:

$$\frac{Y(u)-y(0)}{u^{\alpha-1}}=\int_0^u\frac{1}{u}S\left[t\,g(t)\right]du+\int_0^u\frac{2y(0)}{u^{2}}du-\int_0^u\frac{2Y(u)}{u^{2}}\,du-\int_0^u\frac{1}{u}S\left[t\,f(t,y)\right]du\,,$$

we arrange the above equation, we have

$$Y(u)=y(0)+u^{\alpha-1}\int_0^u\frac{1}{u}S\left[t\,g(t)\right]du+u^{\alpha-1}\int_0^u\frac{2y(0)}{u^{2}}du-u^{\alpha-1}\int_0^u\frac{2Y(u)}{u^{2}}\,du$$
$$-u^{\alpha-1}\int_0^u\frac{1}{u}S\left[t\,f(t,y)\right]du \tag{6}$$

Where $f(t,y)$ is defined as follow:

$$f(t,y)=K(y(t))+M(y(t)) \tag{7}$$

We define the solution of $y(t)$ and non-linear function $f(t,y)$ by infinite series as

$$y(t)=\sum_{n=0}^{\infty}y_n(t)\qquad,\qquad f(t,y)=\sum_{n=0}^{\infty}A_n(t) \tag{8}$$

Respectively, where A_n is defined by the following formula:

$$A_n=\frac{1}{n!}\left[\frac{d^n}{d\lambda^n}M\left(\sum_{n=0}^{\infty}\lambda^n u_n\right)_{\lambda=0}\right]\quad,\quad n=0,1,2,\ldots \tag{9}$$

Equation (9) is given by:

$$A_0=M\left[u_0\right]\qquad,\qquad A_1=u_1M'\left[u_0\right]$$
$$A_2=u_2M'\left[u_0\right]+\frac{1}{2!}u_1^2M''\left[u_0\right] \tag{10}$$
$$A_3=u_3M'\left[u_0\right]+u_1u_2M''\left[u_0\right]+\frac{1}{3!}u_1^3M'''\left[u_0\right]$$

Substituting equation (8) in to equation (6) yield to :

$$S\left[\sum_{n=0}^{\infty} y_n(t)\right] = y(0) + u^{\alpha-1}\int_0^u \frac{1}{u} S\left[t\,g(t)\right]du + u^{\alpha-1}\int_0^u \frac{2y(0)}{u^2}du$$
$$-u^{\alpha-1}\int_0^u \frac{2}{u^2} S\left[\sum_{n=0}^{\infty} y_n(t)\right]du - u^{\alpha-1}\int_0^u \frac{1}{u}\left(S\left[t\,k\,y_n(t) + t\,A_n(t)\right]\right)du \tag{11}$$

$$S\left[y_0(t)\right] = y(0) + u^{\alpha-1}\int_0^u \frac{1}{u} S\left[t\,g(t)\right]du \tag{12}$$

By taking inverse of Sumudu transform of (6) we have :

$$y_0(t) = S^{-1}\left[\, y(0) + u^{\alpha-1}\int_0^u \frac{1}{u} S\left[t\,g(t)\right]du \,\right] \tag{13}$$

$$y_{n+1}(t) = S^{-1}\left[u^{\alpha-1}\int_0^u \frac{2y(0)}{u^2}du - u^{\alpha-1}\int_0^u \frac{2}{u^2} S\left[y_n(t)\right] - u^{\alpha-1}\int_0^u \frac{1}{u}\left(S\left[t\,k\,y_n(t) + t\,A_n(t)\right]\right)du\right] \tag{14}$$

We assume the inverse Sumudu transform exist for each terms in the right hand side of equation (14).

Numerical examples:

Example 1

Consider the singular fractional Lane-Emden equation:

$$D^{\alpha}y + \tfrac{2}{t}y' = -y \qquad , \qquad 1 < \alpha \le 2 \tag{15}$$

With the initial:

$$y(0) = 1 \quad , \quad y'(0) = 0 \tag{16}$$

Multiplying both side of equation (15) by t we have:

$$t\,D^{\alpha}y + 2y' = -ty \tag{17}$$

On using Sumudu transform to equation (17) :

$$S\left[t\,D^{\alpha}y\right] + S\left[2y'\right] = S\left[-ty\right]$$

Applying definition 1 and theorem 2 weget:

$$u\frac{d}{du}\left[uS(D^{\alpha}y)\right] + 2\left[\frac{Y(u) - y(0)}{u}\right] = S\left[-ty\right]$$

$$u\frac{d}{du}\left[u\left(\frac{Y(u) - y(0)}{u^{\alpha}} - \frac{y'(0)}{u^{\alpha-1}}\right)\right] + 2\left[\frac{Y(u) - y(0)}{u}\right] = S\left[-t\,y\right]$$

Apply the initial condition:

$$u\frac{d}{du}\left[\left(\frac{Y(u) - 1}{u^{\alpha-1}}\right)\right] + 2\left[\frac{Y(u) - 1}{u}\right] = S\left[-t\,y\right]$$

$$\frac{d}{du}\left[\left(\frac{Y(u)-1}{u^{\alpha-1}}\right)\right]+2\left[\frac{Y(u)-1}{u^{2}}\right]=\frac{1}{u}S[-t\,y],$$

$$\frac{d}{du}\left[\frac{Y(u)-1}{u^{\alpha-1}}\right]=\frac{1}{u}S[-ty]+\frac{2}{u^{2}}-\frac{2Y(u)}{u^{2}}, \qquad (18)$$

Integrating both side of equation (18) with respect to u we get :

$$\frac{Y(u)-1}{u^{\alpha-1}}=\int_0^u\frac{2}{u^{2}}du+\int_0^u\frac{1}{u}S[-t\,y]du-\int_0^u\frac{2Y(u)}{u^{2}}\,du$$

$$Y(u)-1=u^{\alpha-1}\int_0^u\frac{2}{u^{2}}du+u^{\alpha-1}\int_0^u\frac{1}{u}S[-t\,y]du-u^{\alpha-1}\int_0^u\frac{2Y(u)}{u^{2}}$$

$$Y(u)=1+u^{\alpha-1}\int_0^u\frac{2}{u^{2}}du+u^{\alpha-1}\int_0^u\frac{1}{u}S[-t\,y]du-u^{\alpha-1}\int_0^u\frac{2Y(u)}{u^{2}}\,du\,, \qquad (19)$$

by taking inverse of Sumudu transform of (19) we have:

$$y(t)=S^{-1}[1]+S^{-1}\left[u^{\alpha-1}\int_0^u\frac{2}{u^{2}}du\right]du+S^{-1}\left[u^{\alpha-1}\int_0^u\frac{1}{u}S[-t\,y]du\right]-S^{-1}\left[u^{\alpha-1}\int_0^u\frac{2}{u^{2}}Y(u)du\right]$$

$$y_0=S^{-1}[1]=1$$

$$y_{n+1}(u)=S^{-1}\left[u^{\alpha-1}\int_0^u\frac{2}{u^{2}}du\right]du+S^{-1}\left[u^{\alpha-1}\int_0^u\frac{1}{u}S[-t\,y_n]du\right]-S^{-1}\left[u^{\alpha-1}\int_0^u\frac{2}{u^{2}}S(y_n)du\right] \qquad (20)$$

By substituting $n=0$ in equation (20) we have:

$$y_1(t)=S^{-1}\left[u^{\alpha-1}\int_0^u\frac{2}{u^{2}}du\right]du+S^{-1}\left[u^{\alpha-1}\int_0^u\frac{1}{u}S[-t\,y_0]du\right]-S^{-1}\left[u^{\alpha-1}\int_0^u\frac{2}{u^{2}}S(y_0)du\right]$$

$$y_1(t)=S^{-1}\left[u^{\alpha-1}\int_0^u\frac{1}{u}(-u)du\right]=S^{-1}\left[-u^{\alpha}\right]=\frac{-t^{\alpha}}{\Gamma(\alpha+1)}$$

By substituting $n=1$ in equation (20) we have:

$$y_2(t)=S^{-1}\left[u^{\alpha-1}\int_0^u\frac{1}{u}S[-t\,y_1]du\right]-S^{-1}\left[u^{\alpha-1}\int_0^u\frac{2}{u^{2}}S(y_1)du\right]$$

$$y_2=S^{-1}\left[u^{\alpha-1}\left(\frac{u^{\alpha+1}\Gamma(\alpha+2)}{(\alpha+1)\Gamma(\alpha+1)}\right)\right]+S^{-1}\left[\frac{2u^{2\alpha-2}}{(\alpha-1)}\right]=\frac{t^{2\alpha}\Gamma(\alpha+2)}{(\alpha+1)\Gamma(\alpha+1)\Gamma(2\alpha+1)}+\frac{2t^{2\alpha-2}}{(\alpha-1)\Gamma(2\alpha-1)}$$

Again substituting $n=2$ in equation (20) we have:

$$y_3(u)=S^{-1}\left[u^{\alpha-1}\int_0^u\frac{1}{u}S[-t\,y_2]du\right]-S^{-1}\left[u^{\alpha-1}\int_0^u\frac{2}{u^{2}}S(y_2)du\right]$$

$$y_3(u)=S^{-1}\left[u^{\alpha-1}\int_0^u \frac{1}{u}S\left[-t\left(\frac{t^{2\alpha}\Gamma(\alpha+2)}{(\alpha+1)\Gamma(\alpha+1)\Gamma(2\alpha+1)}+\frac{2t^{2\alpha-2}}{(\alpha-1)\Gamma(2\alpha-1)}\right)\right]du\right]$$
$$-S^{-1}\left[u^{\alpha-1}\int_0^u \frac{2}{u^2}S\left(\frac{t^{2\alpha}\Gamma(\alpha+2)}{(\alpha+1)\Gamma(\alpha+1)\Gamma(2\alpha+1)}+\frac{2t^{2\alpha-2}}{(\alpha-1)\Gamma(2\alpha-1)}\right)du\right]$$

$$y_3=\frac{-t^{3\alpha}\Gamma(\alpha+2)\Gamma(2\alpha+2)}{(\alpha+1)\Gamma(\alpha+1)\Gamma(2\alpha+1)(2\alpha+1)\Gamma(3\alpha+1)}+\frac{-2t^{3\alpha-2}\Gamma(2\alpha)}{(\alpha-1)\Gamma(2\alpha-1)(2\alpha-1)\Gamma(3\alpha-1)}$$
$$-\frac{2t^{3\alpha-2}\Gamma(\alpha+2)}{(\alpha+1)\Gamma(\alpha+1)(2\alpha-1)\Gamma(3\alpha-1)}-\frac{4t^{3\alpha-4}}{(\alpha-1)(2\alpha-3)\Gamma(3\alpha-3)}$$

The series solution is given by:

$$y(t)=y_0(t)+y_1(t)+y_2(t)+\dots$$

$$y(t)=1+\frac{-t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}\Gamma(\alpha+2)}{(\alpha+1)\Gamma(\alpha+1)\Gamma(2\alpha+1)}+\frac{2t^{2\alpha-2}}{(\alpha-1)\Gamma(2\alpha-1)}$$
$$+\frac{-t^{3\alpha}\Gamma(\alpha+2)\Gamma(2\alpha+2)}{(\alpha+1)\Gamma(\alpha+1)\Gamma(2\alpha+1)(2\alpha+1)\Gamma(3\alpha+1)}+\frac{-2t^{3\alpha-2}\Gamma(2\alpha)}{(\alpha-1)\Gamma(2\alpha-1)(2\alpha-1)\Gamma(3\alpha-1)}$$
$$-\frac{2t^{3\alpha-2}\Gamma(\alpha+2)}{(\alpha+1)\Gamma(\alpha+1)(2\alpha-1)\Gamma(3\alpha-1)}-\frac{4t^{3\alpha-4}}{(\alpha-1)(2\alpha-3)\Gamma(3\alpha-3)}$$

In particular case $\alpha=2$ then we have:

$$y(t)=1-\frac{3t^2}{2}-\frac{5t^4}{3\times 4!}-\frac{t^6}{6!}-\dots$$

Example 2

Consider the fractional nonlinear Lane-Emden equation:

$$D^{\alpha}y+\tfrac{2}{t}y'=-e^{y} \quad , \quad 1<\alpha\le 2 \tag{21}$$

With the initial:

$$y(0)=0 \quad , \quad y'(0)=0 \tag{22}$$

By using the same method in example 1 we have:

$$u\frac{d}{du}\left[u\left(\frac{Y(u)-y(0)}{u^{\alpha}}-\frac{y'(0)}{u^{\alpha-1}}\right)\right]+2\left[\frac{Y(u)-y(0)}{u}\right]=S\left[-t\,e^{y}\right]$$

Apply the initial condition:

$$u\frac{d}{du}\left[\left(\frac{Y(u)}{u^{\alpha-1}}\right)\right]+2\left[\frac{Y(u)}{u}\right]=S\left[-t\,e^{y}\right]$$

$$\frac{d}{du}\left[\left(\frac{Y(u)}{u^{\alpha-1}}\right)\right]+2\left[\frac{Y(u)}{u^2}\right]=\frac{1}{u}S\left[-t\,e^{y}\right]$$

$$\frac{d}{du}\left[\frac{Y(u)}{u^{\alpha-1}}\right]=\frac{1}{u}S\left[-t\,e^{y}\right]-\frac{2Y(u)}{u^2} \tag{23}$$

Integrating both side of equation (23) with respect to u we get:

$$\frac{Y(u)}{u^{\alpha-1}} = \int_0^u \frac{1}{u} S\left[-t\, e^{y}\right] du - \int_0^u \frac{2Y(u)}{u^2}\, du$$

$$Y(u) = u^{\alpha-1} \int_0^u \frac{1}{u} S\left[-t\, e^{y}\right] du - u^{\alpha-1} \int_0^u \frac{2Y(u)}{u^2}\, du \qquad (24)$$

Applying inverse of Sumudu transform of equation (24) we obtain:

$$y(u) = S^{-1}\left[u^{\alpha-1}\int_0^u \frac{1}{u} S\left[-t\, e^{y}\right] du\right] - S^{-1}\left[u^{\alpha-1}\int_0^u \frac{2}{u^2} Y(u) du\right]$$

$$y_0 = 0$$

$$y_{n+1}(u) = S^{-1}\left[u^{\alpha-1}\int_0^u \frac{1}{u} S\left[-t\, A_n\right] du\right] - S^{-1}\left[u^{\alpha-1}\int_0^u \frac{2}{u^2} S(y_n) du\right] \qquad (25)$$

Substituting $n = 0$ in equation (25) we have:

$$y_1(u) = S^{-1}\left[u^{\alpha-1}\int_0^u \frac{1}{u} S\left[-t\, A_0\right] du\right] - S^{-1}\left[u^{\alpha-1}\int_0^u \frac{2}{u^2} S(y_0) du\right] \qquad (26)$$

By using equation (10), we have :

$$A_0 = M\left[y_0\right] = y_0^5$$
$$A_1 = y_1 M'\left[y_0\right] = y_1 y_0'$$
$$A_2 = y_1 y_2 M'\left[y_0\right] + \frac{1}{2!} y_1^2 M''\left[y_0\right] = y_1 y_0' + \frac{1}{2} y_1^2 y_0''$$
$$A_3 = y_3 M'\left[y_0\right] + y_1 y_2 M''\left[y_0\right] + \frac{1}{3!} y_1^3 M'''\left[y_0\right] = y_3 y_0' + y_1 y_2 y_0'' + \frac{1}{3!} y_1^3 y_0'''$$

Apply this to equation (26) we get :

$$y_1(u) = S^{-1}\left[u^{\alpha-1}\int_0^u \frac{1}{u} S\left[-t\,(e^{y_0})\right] du\right] - S^{-1}\left[u^{\alpha-1}\int_0^u \frac{2}{u^2} (0) du\right]$$

$$y_1(u) = S^{-1}\left[u^{\alpha-1}\int_0^u \frac{1}{u} (-u) du\right] = S^{-1}\left[-u^{\alpha}\right] = \frac{-t^{\alpha}}{\Gamma(\alpha+1)}$$

Substituting $n = 1$ in equation (25) we have :

$$y_2(u) = S^{-1}\left[u^{\alpha-1}\int_0^u \frac{1}{u} S\left[-t\, A_1\right] du\right] - S^{-1}\left[u^{\alpha-1}\int_0^u \frac{2}{u^2} S(y_1) du\right]$$

$$y_2(u) = S^{-1}\left[u^{\alpha-1}\int_0^u \frac{1}{u} S\left[-t\left(y_1 e^{y_0}\right)\right] du\right] - S^{-1}\left[u^{\alpha-1}\int_0^u \frac{2}{u^2} S\left(\frac{-t^{\alpha}}{\Gamma(\alpha+1)}\right) du\right]$$

$$y_2(u) = \frac{t^{2\alpha}\Gamma(\alpha+2)}{\Gamma(\alpha+1)(\alpha+1)\Gamma(2\alpha+1)} + \frac{2t^{2\alpha-2}}{(\alpha-1)\Gamma(2\alpha-1)}$$

By substituting $n=2$ in equation (25) we get :

$$y_3(u)=S^{-1}\left[u^{\alpha-1}\int_0^u \frac{1}{u}S\left[-t\,A_2\right]du\right]-S^{-1}\left[u^{\alpha-1}\int_0^u \frac{2}{u^2}S(y_2)du\right]$$

$$y_3(u)=S^{-1}\left[u^{\alpha-1}\int_0^u \frac{1}{u}S\left[-t\left(y_2+\frac{1}{2}y_1^2\right)e^{y_0}\right]du\right]$$

$$-S^{-1}\left[u^{\alpha-1}\int_0^u \frac{2}{u^2}S\left(\frac{t^{2\alpha}\Gamma(\alpha+2)}{(\alpha+1)\Gamma(2\alpha+1)\Gamma(\alpha+1)}+\frac{2t^{2\alpha-2}}{(\alpha-1)\Gamma(2\alpha-1)}\right)du\right]$$

$$y_3(u)=\frac{-t^{3\alpha}\Gamma(2\alpha+2)\Gamma(\alpha+2)}{(\alpha+1)\Gamma(2\alpha+1)\Gamma(\alpha+1)(2\alpha+1)\Gamma(3\alpha+1)}-\frac{2t^{3\alpha-2}\Gamma(2\alpha)}{(\alpha-1)\Gamma(2\alpha-1)(2\alpha-1)\Gamma(3\alpha-1)}$$

$$-\frac{1}{2}\frac{t^{3\alpha}\Gamma(2\alpha+2)}{\left[\Gamma(\alpha+1)\right]^2(2\alpha+1)\Gamma(3\alpha+1)}-\frac{2t^{3\alpha-2}\Gamma(\alpha+2)}{(\alpha+1)(2\alpha-1)\Gamma(\alpha+1)\Gamma(3\alpha-1)}$$

$$-\frac{4t^{3\alpha-4}}{(\alpha-1)(2\alpha-3)\Gamma(3\alpha-3)}$$

The series solution is given by :

$$y(t)=y_0(t)+y_1(t)+y_2(t)+\ldots$$

$$y(t)=1+\frac{-t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}\Gamma(\alpha+2)}{\Gamma(\alpha+1)(\alpha+1)\Gamma(2\alpha+1)}+\frac{2t^{2\alpha-2}}{(\alpha-1)\Gamma(2\alpha-1)}$$

$$\frac{-t^{3\alpha}\Gamma(2\alpha+2)\Gamma(\alpha+2)}{(\alpha+1)\Gamma(2\alpha+1)\Gamma(\alpha+1)(2\alpha+1)\Gamma(3\alpha+1)}-\frac{2t^{3\alpha-2}\Gamma(2\alpha)}{(\alpha-1)\Gamma(2\alpha-1)(2\alpha-1)\Gamma(3\alpha-1)}$$

$$-\frac{1}{2}\frac{t^{3\alpha}\Gamma(2\alpha+2)}{\left[\Gamma(\alpha+1)\right]^2(2\alpha+1)\Gamma(3\alpha+1)}-\frac{2t^{3\alpha-2}\Gamma(\alpha+2)}{(\alpha+1)(2\alpha-1)\Gamma(\alpha+1)\Gamma(3\alpha-1)}$$

$$-\frac{4t^{3\alpha-4}}{(\alpha-1)(2\alpha-3)\Gamma(3\alpha-3)}$$

In particular case $\alpha=2$ then we have :

$$y(t)=\frac{-3t^2}{2}+\frac{-5t^4}{3\times 4!}+\frac{-4t^6}{6!}+\ldots$$

Solution Of fractional Lane-Emden type equation by Laplace decomposition method

In this section we discuss the solution of fractional Lane-Emden differential equation type by Laplace decomposition method and compare the solution with that by Sumudu transform.

In the following theorem we discuss the Laplace transform of non-constant fractional derivative.

Theorem 3

$$L\left[t\,D^{\alpha}y\right]=-\frac{d}{ds}L\left[D^{\alpha}y\right] \quad , \quad n-1<\alpha\le n \quad , \quad n=1,2,3,\ldots$$

Proof:

$$L\left[D^{\alpha}y\right]=\frac{1}{u}\int_{0}^{\infty}e^{-st}D^{\alpha}y\,dt$$

$$\frac{d}{ds}L\left[D^{\alpha}y\right]=\frac{d}{ds}\int_{0}^{\infty}e^{-st}D^{\alpha}y\,dt$$

$$\frac{d}{ds}L\left[D^{\alpha}y\right]=\int_{0}^{\infty}\frac{d}{ds}e^{-st}D^{\alpha}y\,dt$$

$$=\int_{0}^{\infty}-t\,e^{-st}D^{\alpha}y\,dt$$

$$\frac{d}{ds}L\left[D^{\alpha}y\right]=-\int_{0}^{\infty}e^{-st}t\,D^{\alpha}ydt$$

$$=-L\left[t\,D^{\alpha}y\right]$$

$$L\left[t\,D^{\alpha}y\right]=-\frac{d}{ds}L\left[D^{\alpha}y\right].$$

Definition 2

The Laplace transform of Caputo fractional derivative for $m-1<\alpha\leq m,\;\; m\in\mathbb{N}$ can be obtained in the form of :

$$L\left[D^{\alpha}f\,(t)\right]=\frac{s^{m}F(s)-s^{m-1}f\,(0)-s^{m-2}f\,'(0)-...-f^{\,(m-1)}(0)}{s^{m-\alpha}}\qquad ,$$

If $1<\alpha\leq 2$ then:

$$L\left[D^{\alpha}f\,(t)\right]=\frac{s^{2}F(s)-s\,f\,(0)-f\,'(0)}{s^{2-\alpha}}$$

Example 3

Consider the fractional Lane-Emden equation:

$$D^{\alpha}y+\tfrac{2}{t}y'=-y\qquad,\qquad n-1<\alpha\leq n \tag{27}$$

With the initial condition:

$$y(0)=1\,,\quad y'(0)=0 \tag{28}$$

Multiplying both side of equation (27) with t , we get :

$$t\,D^{\alpha}y+2y'=-ty \tag{29}$$

Applying Laplace transform to equation (29) we get:

$$L\left[t\,D^{\alpha}y\right]+L\left[2y'\right]=L\left[-ty\right] \tag{30}$$

On using theorem 3 and definition 2 we have :

$$-\frac{d}{ds}\left[L(D^{\alpha}y)\right]+2\left[sY\,(s)-y\,(0)\right]=L\left[-ty\right]$$

$$-\frac{d}{ds}\left[\frac{sY\,(s)-1}{s^{1-s}}\right]+2\left[sY\,(s)-1\right]=L\left[-t\,y\right]\;-\frac{d}{ds}\left[L(D^{\alpha}y)\right]+2\left[sY\,(s)-y\,(0)\right]=L\left[-ty\right]$$

$$-\frac{d}{ds}\left[\frac{sY(s)-1}{s^{1-s}}\right]=2sY(s)-2-L[-ty] \tag{31}$$

Integrating both side of equation (31) with respect to u, then we have :

$$\frac{sY(s)-1}{s^{1-s}}=\int_0^s 2sY(s)ds-\int_0^s 2ds-\int_0^s L[-ty]ds$$

$$sY(s)-1=s^{1-\alpha}\int_0^s 2sY(s)ds-s^{1-\alpha}\int_0^s 2ds-s^{1-\alpha}\int_0^s L[-ty]ds$$

$$sY(s)=1+s^{1-\alpha}\int_0^s 2sY(s)ds-s^{1-\alpha}\int_0^s 2ds-s^{1-\alpha}\int_0^s L[-ty]ds$$

$$Y(s)=\frac{1}{s}+\frac{1}{s}.s^{1-\alpha}\int_0^s 2sY(s)ds-\frac{1}{s}.s^{1-\alpha}\int_0^s 2ds-\frac{1}{s}.s^{1-\alpha}\int_0^s L[-ty]ds \tag{32}$$

By taking inverse of Laplace transform of (32) we

get:

$$y(s)=L^{-1}\left[\frac{1}{s}\right]+L^{-1}\left[s^{-\alpha}\int_0^s 2sY(s)ds\right]-L^{-1}\left[s^{-\alpha}\int_0^s 2ds\right]-L^{-1}\left[s^{-\alpha}\int_0^s L[-ty]ds\right]$$

$$y_0=L^{-1}\left[\frac{1}{s}\right]=1$$

$$y_{n+1}=L^{-1}\left[s^{-\alpha}\int_0^s 2s\,L[y_n]ds\right]-L^{-1}\left[s^{-\alpha}\int_0^s 2ds\right]-L^{-1}\left[s^{-\alpha}\int_0^s L[-t\,y_n]ds\right] \tag{33}$$

Substituting $n=0$ in equation (33) we get :

$$y_1=L^{-1}\left[s^{-\alpha}\int_0^s 2s\,L[y_0]ds\right]-L^{-1}\left[s^{-\alpha}\int_0^s 2ds\right]-L^{-1}\left[s^{-\alpha}\int_0^s L[-t\,y_0]ds\right]$$

$$y_1=-L^{-1}\left[s^{-\alpha}(s^{-\alpha})\right]=-L^{-1}\left[\frac{1}{s^{\alpha+1}}\right]=-\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

Substituting $n=1$ in equation (33) we get :

$$y_2=L^{-1}\left[s^{-\alpha}\int_0^s 2s\,L[y_1]ds\right]-L^{-1}\left[s^{-\alpha}\int_0^s L[-t\,y_1]ds\right]$$

$$y_2=L^{-1}\left[s^{-\alpha}\int_0^s -2s\frac{1}{s^{\alpha+1}}ds\right]-L^{-1}\left[s^{-\alpha}\int_0^s \frac{\Gamma(\alpha+2)}{\Gamma(\alpha+1)}\frac{1}{s^{\alpha+2}}ds\right]$$

$$y_2=\frac{2}{\alpha-1}.\frac{t^{2\alpha-2}}{\Gamma(2\alpha-1)}+\frac{\Gamma(\alpha+2)}{\Gamma(\alpha+1)(\alpha+1)}.\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

Substituting $n=2$ in equation (33) we get :

$$y_3 = L^{-1}\left[s^{-\alpha}\int_0^s 2s\, L\left[y_2\right]ds \right] - L^{-1}\left[s^{-\alpha}\int_0^s L\left[-t\, y_2\right]ds \right]$$

$$y_3 = \frac{4}{(\alpha-1)}\cdot\frac{-1}{(2\alpha-3)}L^{-1}\left[\frac{1}{s^{3\alpha-3}}\right] + \frac{2\Gamma(\alpha+2)}{\Gamma(\alpha+1)(\alpha+1)}\cdot\frac{-1}{(2\alpha-1)}L^{-1}\left[\frac{1}{s^{3\alpha-1}}\right]$$
$$+\frac{2\Gamma(2\alpha)}{(\alpha-1)\Gamma(2\alpha-1)}\cdot\frac{1}{(2\alpha-1)}L^{-1}\left[\frac{1}{s^{3\alpha-1}}\right] + \frac{\Gamma(\alpha+2)\Gamma(2\alpha+2)}{\Gamma(\alpha+1)(\alpha+1)\Gamma(2\alpha+1)}\cdot\frac{1}{(2\alpha+1)}L^{-1}\left[\frac{1}{s^{3\alpha+1}}\right]$$

$$y_3 = -\frac{4}{(\alpha-1)}\cdot\frac{1}{(2\alpha-3)}\frac{t^{3\alpha-4}}{\Gamma(3\alpha-3)} - \frac{2\Gamma(\alpha+2)}{\Gamma(\alpha+1)(\alpha+1)}\cdot\frac{1}{(2\alpha-1)}\frac{t^{3\alpha-2}}{\Gamma(3\alpha-1)}$$
$$-\frac{2\Gamma(2\alpha)}{(\alpha-1)\Gamma(2\alpha-1)}\cdot\frac{1}{(2\alpha-1)}\frac{t^{3\alpha-2}}{\Gamma(3\alpha-1)} - \frac{\Gamma(\alpha+2)\Gamma(2\alpha+2)}{\Gamma(\alpha+1)(\alpha+1)\Gamma(2\alpha+1)}\cdot\frac{1}{(2\alpha+1)}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

The series solution is given by :

$$y(t) = y_0(t) + y_1(t) + y_2(t) + \ldots$$

$$y(t) = 1 + \frac{-t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}\Gamma(\alpha+2)}{(\alpha+1)\Gamma(\alpha+1)\Gamma(2\alpha+1)} + \frac{2t^{2\alpha-2}}{(\alpha-1)\Gamma(2\alpha-1)}$$
$$+\frac{-t^{3\alpha}\Gamma(\alpha+2)\Gamma(2\alpha+2)}{(\alpha+1)\Gamma(\alpha+1)\Gamma(2\alpha+1)(2\alpha+1)\Gamma(3\alpha+1)} + \frac{-2t^{3\alpha-2}\Gamma(2\alpha)}{(\alpha-1)\Gamma(2\alpha-1)(2\alpha-1)\Gamma(3\alpha-1)}$$
$$-\frac{2t^{3\alpha-2}\Gamma(\alpha+2)}{(\alpha+1)\Gamma(\alpha+1)(2\alpha-1)\Gamma(3\alpha-1)} - \frac{4t^{3\alpha-4}}{(\alpha-1)(2\alpha-3)\Gamma(3\alpha-3)}$$

In particular case $\alpha = 2$ then we have :

$$y(t) = 1 - \frac{3t^2}{2} - \frac{5t^4}{3\times 4!} - \frac{t^6}{6!} - \ldots$$

Example 4

Consider the singular fractional Lane-Emden equation :

$$D^{\alpha}y + \tfrac{2}{t}y' = -e^{y}\ , \qquad n-1<\alpha\le n \tag{34}$$

With the initial condition :

$$y(0)=0\ , \quad y'(0)=0 \tag{35}$$

Multiplying both side of equation (34) with t , we get :

$$t\,D^{\alpha}y + 2y' = -t\, e^{y} \tag{36}$$

Applying Laplace transform to equation (36) :

$$L\left[t\,D^{\alpha}y\right] + L\left[2y'\right] = L\left[-t\, e^{y}\right] \tag{37}$$

Using theorem 3 and definition 2 we have :

$$-\frac{d}{ds}\left[L(D^{\alpha}y)\right] + 2\left[sY(s) - y(0)\right] = L\left[-t\, e^{y}\right]$$

$$-\frac{d}{ds}\left[\frac{s^2Y(s) - sy(0) - y'(0)}{s^{2-s}}\right] + 2\left[sY(s) - y(0)\right] = L\left[-t\, e^{y}\right]$$

$$-\frac{d}{ds}\left[\frac{sY(s)}{s^{1-s}}\right]+2\left[sY(s)\right]=L\left[-t\,e^{y}\right]$$

$$-\frac{d}{ds}\left[\frac{sY(s)}{s^{1-s}}\right]=2sY(s)-L\left[-t\,e^{y}\right] \quad (38)$$

Integrating both side equation (38) with respect to u ,then we have :

$$\frac{sY(s)}{s^{1-s}}=\int_0^s 2sY(s)ds-\int_0^s L\left[-t\,e^{y}\right]ds$$

$$sY(s)=s^{1-\alpha}\int_0^s 2sY(s)ds-s^{1-\alpha}\int_0^s L\left[-t\,e^{y}\right]ds$$

$$sY(s)=s^{1-\alpha}\int_0^s 2sY(s)ds-s^{1-\alpha}\int_0^s L\left[-t\,e^{y}\right]ds$$

$$Y(s)=\frac{1}{s}.s^{1-\alpha}\int_0^s 2sY(s)ds-\frac{1}{s}.s^{1-\alpha}\int_0^s L\left[-t\,e^{y}\right]ds \quad (39)$$

by taking inverse of Laplace transform of equation (39) we get:

$$y(s)=L^{-1}\left[s^{-\alpha}\int_0^s 2sY(s)ds\right]-L^{-1}\left[s^{-\alpha}\int_0^s L\left[-t\,e^{y}\right]ds\right]$$

$$y_0=0$$

$$y_{n+1}=L^{-1}\left[s^{-\alpha}\int_0^s 2s\,L\left[y_n\right]ds\right]-L^{-1}\left[s^{-\alpha}\int_0^s L\left[-t\,A_n\right]ds\right] \quad (40)$$

Substituting $n=0$ in equation (40) we get :

$$y_1=L^{-1}\left[s^{-\alpha}\int_0^s 2s\,L\left[y_0\right]ds\right]-L^{-1}\left[s^{-\alpha}\int_0^s L\left[-t\,A_0\right]ds\right]$$

$$y_1=-L^{-1}\left[s^{-\alpha}(s^{-1})\right]=-L^{-1}\left[\frac{1}{s^{\alpha+1}}\right]=-\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

Substituting $n=1$ in equation (40) we get:

$$y_2=L^{-1}\left[s^{-\alpha}\int_0^s 2s\,L\left[y_1\right]ds\right]-L^{-1}\left[s^{-\alpha}\int_0^s L\left[-t\,A_1\right]ds\right]$$

$$y_2=L^{-1}\left[s^{-\alpha}\left(\frac{-2}{s^{\alpha-1}}.\frac{-1}{(\alpha-1)}\right)\right]-L^{-1}\left[s^{-\alpha}\int_0^s \frac{\Gamma(\alpha+2)}{\Gamma(\alpha+1)}\frac{1}{s^{\alpha+2}}ds\right]$$

$$y_2=\frac{2}{(\alpha-1)}\frac{t^{2\alpha-2}}{\Gamma(2\alpha-1)}+\frac{\Gamma(\alpha+2)}{\Gamma(\alpha+1)(\alpha+1)}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

Substituting $n=2$ in equation (40) we get :

$$y_3 = -L^{-1}\left[s^{-\alpha}\int_0^s L\left[-t\left(y_2 + \frac{1}{2}y_1^2 \right)\right] ds \right]$$
$$+ L^{-1}\left[s^{-\alpha}\int_0^s 2s\, L\left[\frac{2}{(\alpha-1)}\frac{t^{2\alpha-2}}{\Gamma(2\alpha-1)} + \frac{\Gamma(\alpha+2)}{\Gamma(\alpha+1)(\alpha+1)}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right] ds \right]$$

$$y_3 = -L^{-1}\left[s^{-\alpha}\int_0^s L\left[-t\left(\left(\frac{2}{(\alpha-1)}\frac{t^{2\alpha-2}}{\Gamma(2\alpha-1)} + \frac{\Gamma(\alpha+2)}{\Gamma(\alpha+1)(\alpha+1)}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \right) + \frac{1}{2}\left[\frac{-t^{\alpha}}{\Gamma(\alpha+1)} \right]^2 \right)\right] ds \right]$$
$$+ L^{-1}\left[s^{-\alpha}\int_0^s 2s\left[\frac{2}{(\alpha-1)}\cdot\frac{1}{s^{2\alpha-1}} + \frac{\Gamma(\alpha+2)}{\Gamma(\alpha+1)(\alpha+1)}\cdot\frac{1}{s^{2\alpha+1}}\right] ds \right]$$
$$y_3 = \frac{-2\Gamma(2\alpha)}{(\alpha-1)\Gamma(2\alpha-1)(2\alpha-1)}\frac{t^{3\alpha-2}}{\Gamma(3\alpha-1)} - \frac{\Gamma(\alpha+2)\Gamma(2\alpha+2)}{\Gamma(\alpha+1)(\alpha+1)\Gamma(2\alpha+1)(2\alpha+1)}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$
$$-\frac{1}{2}\frac{\Gamma(2\alpha+2)}{\left[\Gamma(\alpha+1)\right]^2(2\alpha+1)}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{4}{(\alpha-1)(2\alpha-3)}\cdot\frac{t^{3\alpha-4}}{\Gamma(3\alpha-3)} - \frac{2\Gamma(\alpha+2)}{\Gamma(\alpha+1)(\alpha+1)(2\alpha-1)}\cdot\frac{t^{3\alpha-2}}{\Gamma(3\alpha-1)}$$

The series solution is given by:

$$y(t) = y_0(t) + y_1(t) + y_2(t) + \dots$$

$$y(t) = 1 + \frac{-t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}\Gamma(\alpha+2)}{\Gamma(\alpha+1)(\alpha+1)\Gamma(2\alpha+1)} + \frac{2t^{2\alpha-2}}{(\alpha-1)\Gamma(2\alpha-1)}$$
$$\frac{-t^{3\alpha}\Gamma(2\alpha+2)\Gamma(\alpha+2)}{(\alpha+1)\Gamma(2\alpha+1)\Gamma(\alpha+1)(2\alpha+1)\Gamma(3\alpha+1)} - \frac{2t^{3\alpha-2}\Gamma(2\alpha)}{(\alpha-1)\Gamma(2\alpha-1)(2\alpha-1)\Gamma(3\alpha-1)}$$
$$-\frac{1}{2}\frac{t^{3\alpha}\Gamma(2\alpha+2)}{\left[\Gamma(\alpha+1)\right]^2(2\alpha+1)\Gamma(3\alpha+1)} - \frac{2t^{3\alpha-2}\Gamma(\alpha+2)}{(\alpha+1)(2\alpha-1)\Gamma(\alpha+1)\Gamma(3\alpha-1)}$$
$$-\frac{4t^{3\alpha-4}}{(\alpha-1)(2\alpha-3)\Gamma(3\alpha-3)}$$

In particular case $\alpha = 2$ then we have:

$$y(t) = \frac{-3t^2}{2} + \frac{-5t^4}{3\times 4!} + \frac{-4t^6}{6!} + \dots$$

Conclusion:

In this paper we studied tow method for solving fractional of Lane-Emden type equation the two method are very powerful and efficient for solving different kinds of linear and non-linear fractional differential equations in different fields of science and engineering.

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