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Fractional Power Series Method For Solving The time-fractional biological population model equation

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Abstract

In this paper, we present approximate analytical solution of the time-fractional biological population equation using the fractional power series method (FPSM). The fractional derivatives are described in the Caputo sense. Some examples are given and the results are compared with the exact solutions.The results reveal that FPSM is very effective simple and efficient technique to handle fractional differential equations.

Keyword: Biological population equation; fractional power series; fractional differential equation; Caputo fractional derivative.

1 Introduction

Fractional differential equations (FDEs) have gained importance and popularity during the past three decades or so, mainly due to its demonstrated applications in numerous seemingly diverse fields of science and engineering. FDEs are also used in modeling of many chemical processes, mathematical biology and many other problems in physics and engineering [8]. Unfortunately, most of FDEs do not have exact analytical solutions;therefore considerable heed has been focused on the approximate and numerical solutions of these equations. In recent years, many methods have been developed for constructing approximate analytical solutions such as, Adomian decomposition method [10] homotopy analysis method, homotopy perturbation method and others ([10]-[12]). Recently, published a very interesting work whereby the approximate analytical solution of some FDEs was given using a new method called fractional power series method([5],[13]). It was shown that this new method is very efficient, Fractional power series method is an important method to solve mathematical problems.

In this paper, we consider the nonlinear fractional biological population model in the form [9]

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^2}{\partial x^2}(u^2)+\frac{\partial^2}{\partial y^2}(u^2)+f(u),0<\alpha\le 1,t>0,x,y\in\Re \tag{1}$$

with the initial condition $u(x,y,0)=\mathrm{g}(x,y)$, where $u(x,y,t)$ denotes the population density and f represents the population supply due to birth and death, α is a parameter describing the order of the fractional derivative. Eq.(1) describes the nonlinear of biologic equation ([2], [4], [5]).

The paper is organized as follows: In section 2, we provide the Basic definitions fractional calculus In addition property. which will be used throughout the paper. Section 3, Application models of fractional biological equation. Sections 4,Conclusion.

2 Basic definitions

2.1 *Definition* ([13])

The fractional derivative of $f(x)$ in caputo sense is defined as

$$D^{\alpha}f(x)=\frac{1}{\Gamma(m-\alpha)}\int_{0}^{x}(x-s)^{m-\alpha-1}f^{(m)}(s)ds,$$

2.2 *Definition* ([5])

A power series representation of the form

$$\sum_{n=0}^{\infty}c_n(t-t_0)^{n\alpha}=c_0+c_1(t-t_0)^{\alpha}+c_2(t-t_0)^{2\alpha}+\dots, \quad (2)$$

where $0\leq m-1<\alpha\leq m, m\in \mathrm{N}^{+}$ and $t>t_0$ is called a fractional power series (FPS) about t_0 ,where t is a varible and C_n are the coeffients of the series.

In addtion, we also need the following property:

2.3 *Theorem 1*([13])

Suppose that the FPS $\sum_{n=0}^{\infty}c_n t^{n\alpha}$ has radius of convergence $R>0.f(t)$ is a function defined by $f(t)$ = $\sum_{n=0}^{\infty}c_n t^{n\alpha}$ on $0\leq t<R$, then for $m-1<\alpha\leq m,$ and $0\leq t<R$, we have:

$$D^{\alpha}f(t)=\sum_{n=1}^{\infty}c_n\frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)}t^{(n-1)\alpha} \quad (3)$$

3 Application models of fractional biological equation

In this section, the applicability of FPSM shall be demonstrated by test examples

3.1 Example :[9]

We consider the time- fractional biological equation in the form

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}=\frac{\partial^{2}u^{2}}{\partial x^{2}}+\frac{\partial^{2}u^{2}}{\partial y^{2}}+ku \qquad 0<\alpha\leq 1 \quad (4)$$

subject to the initial condition $u(x,y,o)=\sqrt{xy}.$

If we put $\alpha=1,$ we obtain the exact solution $u(x,y,t)=e^{nt}\sqrt{xy}.$

To apply FPSM , we suppose that the solution of (4) takes the form

$$u(x,y,t)=\sum_{n=0}^{\infty}u_n(x,y)t^{\alpha n} \qquad (5)$$
$$=u_o(x,y)+u_1(x,y)t^{\alpha}+u_2(x,y)t^{2\alpha}+\ldots$$

by theorem

$$D_t^{\alpha}u=\sum_{n=0}^{\infty}u_n(x,y)\frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)}t^{(n-1)\alpha} \qquad (6)$$

$$u^2=u_0^2+2u_0u_1t^{\alpha}+(2u_0u_2+u_1^2)t^{2\alpha}+\ldots$$

$$(u^2)_x=2u_0u_{0x}+2(u_{0x}u_1+u_0u_{1x})t^{\alpha}+2(u_0u_{2x}+u_{0x}u_2+2u_1u_{1x})t^{2\alpha}+\ldots$$

$$(u^2)_{xx}=2(u_0u_{0xx}+(u_{0x})^2)+2(u_0u_{1xx}+2u_{0x}u_{1x}+u_{0xx}u_1)t^{\alpha} \qquad (7)$$
$$+2(u_0u_{2xx}+2u_{0x}u_{2x}+u_{0xx}u_2+2(u_1u_{1xx}+(u_{1x})^2)t^{2\alpha}+\ldots$$

$$(u^2)_y=2u_0u_{0y}+2(u_{0y}u_1+u_0u_{1y})t^{\alpha}+2(u_0u_{2y}+u_{0y}u_2+2u_1u_y)t^{2\alpha}+\ldots$$

$$(u^2)_{yy}=2(u_0u_{0yy}+(u_{0y})^2)+2(u_0u_{1yy}+2u_{0y}u_{1y}+u_{0yy}u_1)t^{\alpha}a8 \qquad (8)$$
$$+2(u_0u_{2yy}+2u_{0y}u_{2y}+u_{0yy}u_2+2(u_1u_{1yy}+(u_{1y})^2)t^{2\alpha}+..$$

substituting (5), (6) ,(7), (8) into (4) and comparing the cofficients of t^{α}

$$\sum_{n=1}^{\infty}u_n(x,y)\frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)}t^{(n-1)\alpha}=$$

$$[2(u_0u_{0xx}+(u_{0x})^2)+2(u_0u_{1xx}+2u_{0x}u_{1x}+u_{0xx}u_1)t^{\alpha}$$
$$+2(u_0u_{2xx}+2u_{0x}u_{2x}+u_{0xx}u_2+2(u_1u_{1xx}+(u_{1x})^2)t^{2\alpha}+\ldots]$$
$$+[2(u_0u_{0yy}+(u_{0y})^2)+2(u_0u_{1yy}+2u_{0y}u_{1y}+u_{0yy}u_1)t^{\alpha}$$
$$+2(u_0u_{1yy}+2u_{0y}u_{1y}+u_{0yy}u_1)t^{2\alpha}+\ldots]$$
$$+k[u_o+u_1t^{\alpha}+u_1t^{2\alpha}+..]$$

using initial condition $u(x,y,o)=\sqrt{xy}$

we have $u_0(x,y)=\sqrt{xy}$

Next we determine the $u_n(n=1,2,\ldots)$.

$$u_1\Gamma(\alpha+1)=2(u_0u_{0xx}+(u_{0x})^2)+2(u_0u_{0yy}+(u_{0y})^2)+ku_o \qquad (9)$$

and

$$u_2\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}=2(u_0u_{1xx}+2u_{0x}u_{1x}+u_{0xx}u_1)+2(u_0u_{1yy}+2u_{0y}u_{1y}+u_{0yy}u_1)+ku_1 \qquad (10)$$

therefore we obtain the approximate solution of equation (4)

$$u(x,y,t) = u_o(x,y) + u_1(x,y)t^{\alpha} + u_2(x,y)t^{2\alpha} + ..$$

For example , if $u_0(x,y) = \sqrt{xy}$ then form (9) and (10) we get

$$u_1(x,y) = \frac{k\sqrt{xy}}{\Gamma(\alpha+1)},$$

$$u_2(x,y) = \frac{k^2\sqrt{xy}}{\Gamma(2\alpha+1)}$$

then $u_n(x,y) = \dfrac{k^n\sqrt{xy}}{\Gamma(n\alpha+1)}$

$$u(x,y,t) = \sum_{n=0}^{\infty} u_n(x,y)t^{\alpha n} = \sum_{n=0}^{\infty} \frac{k^n\sqrt{xy}}{\Gamma(n\alpha+1)}t^{\alpha n} = \sqrt{xy}\sum_{n=0}^{\infty}\frac{(kt^{\alpha})^n}{\Gamma(n\alpha+1)} = \sqrt{xy}\,E_{\alpha}(kt^{\alpha}),$$

where $E_{\alpha}(kt^{\alpha})$ is Mittag-Leffler function, which is an exact solution to the standard form biological population equation and which is in full agreement with the results obtained by ([1],[4],[5],[9]) .

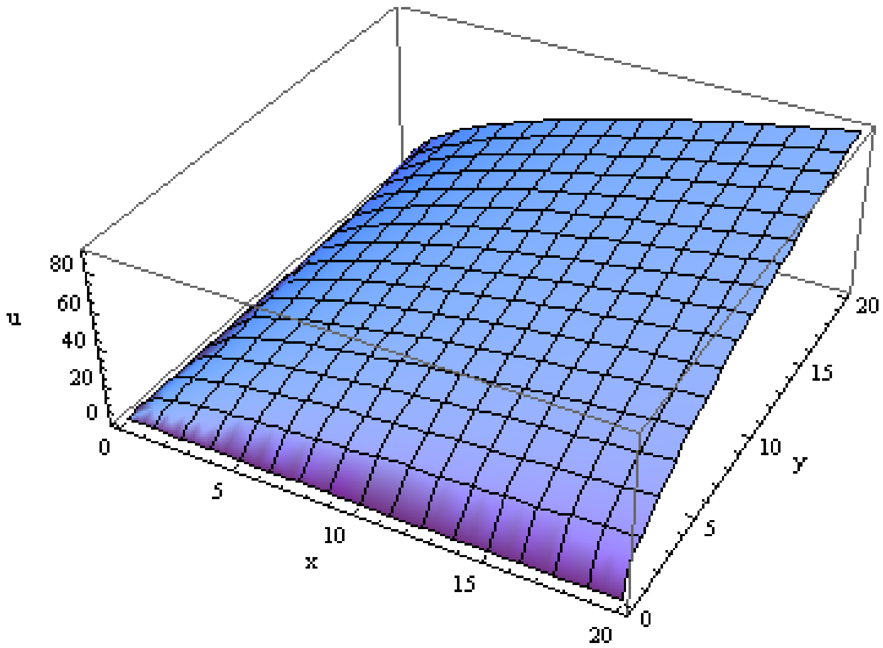

Figure 1. The behavior of the approximate solution at $k = 1, t = 1.5, \alpha = 1$.

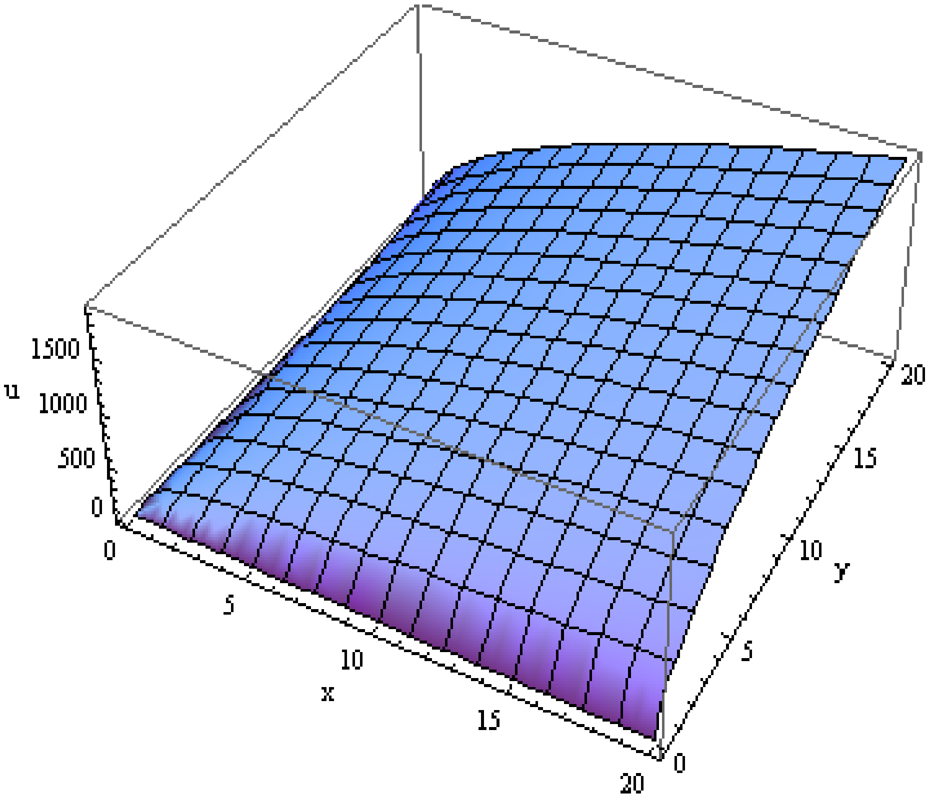

Figure 2. The behavior of the approximate solution at $k = 2.5, t = 1.5, \alpha = 0.85$.

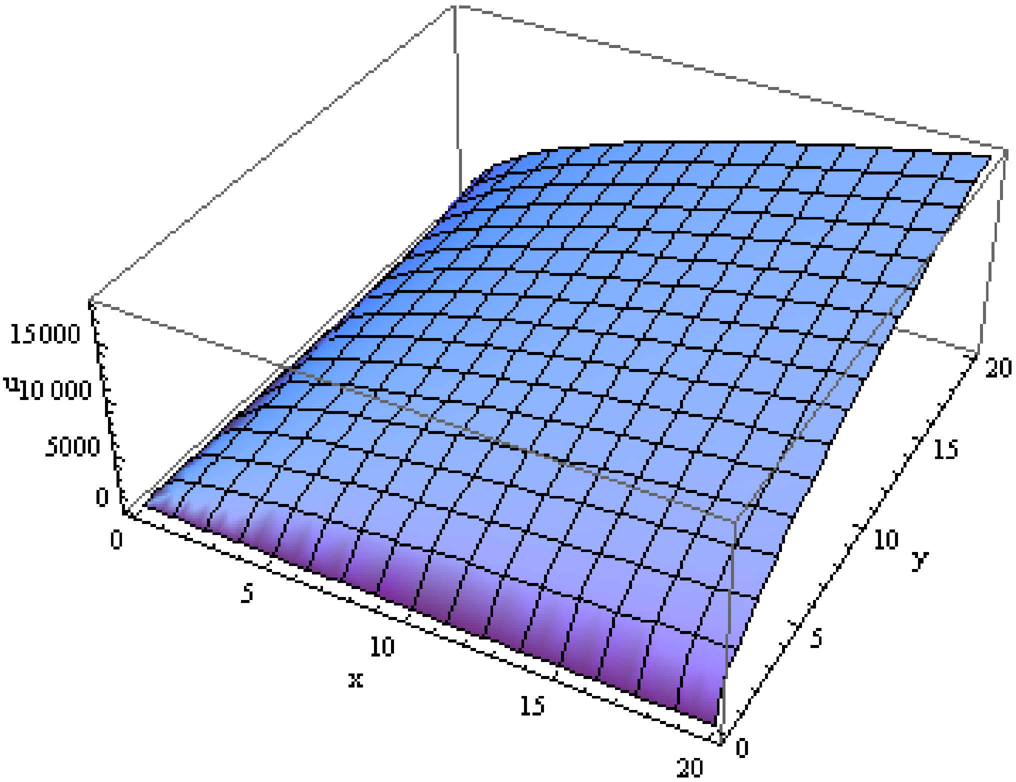

Figure 3. The behavior of the approximate solution at $k = 3.5, t = 0.5, \alpha = 0.5$.

where in Figure 1-3, we presented the behavior of the approximate solution with different values of α ($\alpha = 1, 0.85$ and 0.5, respectively) and different values of r . from these fpgures we can see that the obtained solutions are in full agreement with the results obtained by ([1],[4],[5],[9]).

3.2 Example :[9]

we consider the time- fractional biological equation in the form

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^2 u^2}{\partial x^2}+\frac{\partial^2 u^2}{\partial y^2}+ku^a(1-ru^b) \qquad a=b=k=1 \qquad 0<\alpha\le 1 \tag{11}$$

subject to the initial condition $u(x,y,o)=e^{\sqrt{\frac{r}{8}}(x+y)}$

To apply FPSM, we suppose that the solution of (11) takes the form

$$u(x,y,t)=\sum_{n=0}^{\infty}u_n(x,y)t^{\alpha n} \tag{12}$$

$$=u_o(x,y)+u_1(x,y)t^\alpha+u_2(x,y)t^{2\alpha}+\ldots$$

by theorem

$$D_t^\alpha u=\sum_{n=0}^{\infty}u_n(x,y)\frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)}t^{(n-1)\alpha} \tag{13}$$

$$u^2=u_0^2+2u_0u_1t^\alpha+(2u_0u_2+u_1^2)t^{2\alpha}+\ldots \tag{14}$$

$$(u^2)_x=2u_0u_{0x}+2(u_{0x}u_1+u_0u_{1x})t^\alpha+2(u_0u_{2x}+u_{0x}u_2+2u_1u_{1x})t^{2\alpha}+\ldots$$

$$(u^2)_{xx}=2(u_0u_{0xx}+(u_{0x})^2)+2(u_0u_{1xx}+2u_{0x}u_{1x}+u_{0xx}u_1)t^\alpha \tag{15}$$

$$+2(u_0u_{2xx}+2u_{0x}u_{2x}+u_{0xx}u_2+2(u_1u_{1xx}+(u_{1x})^2)t^{2\alpha}+\ldots$$

$$(u^2)_y=2u_0u_{0y}+2(u_{0y}u_1+u_0u_{1y})t^\alpha+2(u_0u_{2y}+u_{0y}u_2+2u_1u_y)t^{2\alpha}+\ldots$$

$$(u^2)_{yy}=2(u_0u_{0yy}+(u_{0y})^2)+2(u_0u_{1yy}+2u_{0y}u_{1y}+u_{0yy}u_1)t^\alpha \tag{16}$$

$$+2(u_0u_{2yy}+2u_{0y}u_{2y}+u_{0yy}u_2+2(u_1u_{1yy}+(u_{1y})^2)t^{2\alpha}+\ldots$$

substituting (12), (13), (14) and (15), (16) into (11) and comparing the cofficients of t^α

$$\sum_{n=1}^{\infty}u_n(x,y)\frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)}t^{(n-1)\alpha}=[2(u_0u_{0xx}+(u_{0x})^2)+2(u_0u_{1xx}+2u_{0x}u_{1x}+u_{0xx}u_1)t^\alpha$$

$$+2(u_0u_{2xx}+2u_{0x}u_{2x}+u_{0xx}u_2+2(u_1u_{1xx}+(u_{1x})^2)t^{2\alpha}+\ldots]$$

$$+[2(u_0u_{0yy}+(u_{0y})^2)$$

$$+2(u_0u_{1yy}+2u_{0y}u_{1y}+u_{0yy}u_1)t^\alpha+2(u_0u_{1yy}+2u_{0y}u_{1y}+u_{0yy}u_1)t^{2\alpha}+\ldots]$$

$$+[u_o+u_1t^\alpha+u_1t^{2\alpha}+\ldots]$$

$$-r(u_0^2+2u_0u_1t^\alpha+(2u_0u_2+u_1^2)t^{2\alpha}+\ldots)$$

using initial condition $u(x,y,o)=e^{\sqrt{\frac{r}{8}}(x+y)}$.

we have $u_0(x,y)=e^{\sqrt{\frac{r}{8}}(x+y)}$

Next we determine the $u_n \quad (n=1,2,...)$.

$$u_1\Gamma(\alpha+1)=2(u_0u_{0xx}+(u_{0x})^2)+2(u_0u_{0yy}+(u_{0y})^2)+u_o-ru_0^2 \tag{17}$$

and

$$u_2\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}=2(u_0u_{1xx}+2u_{0x}u_{1x}+u_{0xx}u_1)+2(u_0u_{1yy}+2u_{0y}u_{1y}+u_{0yy}u_1)+u_1-r2u_0u_1 \tag{18}$$

therefore we obtain the approximate solution of equation (11)

$$u(x,y,t)=u_o(x,y)+u_1(x,y)t^{\alpha}+u_2(x,y)t^{2\alpha}+..$$

For example, if $u_0(x,y)=e^{\sqrt{\frac{r}{8}}(x+y)}$ then form (17) and (18) we get

$$u_1(x,y)=\frac{e^{\sqrt{\frac{r}{8}}(x+y)}}{\Gamma(\alpha+1)},$$

$$u_2(x,y)=\frac{e^{\sqrt{\frac{r}{8}}(x+y)}}{\Gamma(2\alpha+1)}$$

then $u_n(x,y)=\dfrac{e^{\sqrt{\frac{r}{8}}(x+y)}}{\Gamma(n\alpha+1)}$

$$u(x,y,t)=\sum_{n=0}^{\infty}u_n(x,y)t^{\alpha n}=e^{\sqrt{\frac{r}{8}}(x+y)}\sum_{n=0}^{\infty}\frac{t^{\alpha n}}{\Gamma(n\alpha+1)}=e^{\sqrt{\frac{r}{8}}(x+y)}E_\alpha(t^\alpha).$$

where $E_\alpha(kt^\alpha)$ is Mittag-Leffler function, which is an exact solution to the standard form biological population equation and which is in full agreement with the results obtained by([1],[4],[5],[9]).

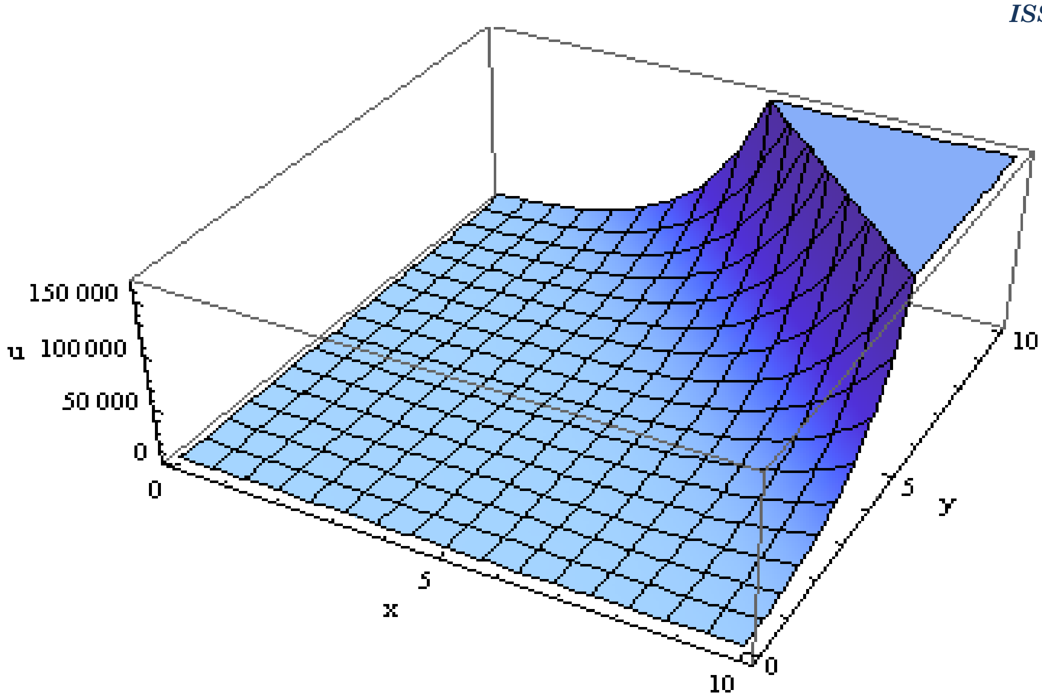

Figure 4. The behavior of the approximate solution at $r = 1, t = 2, \alpha = 1$.

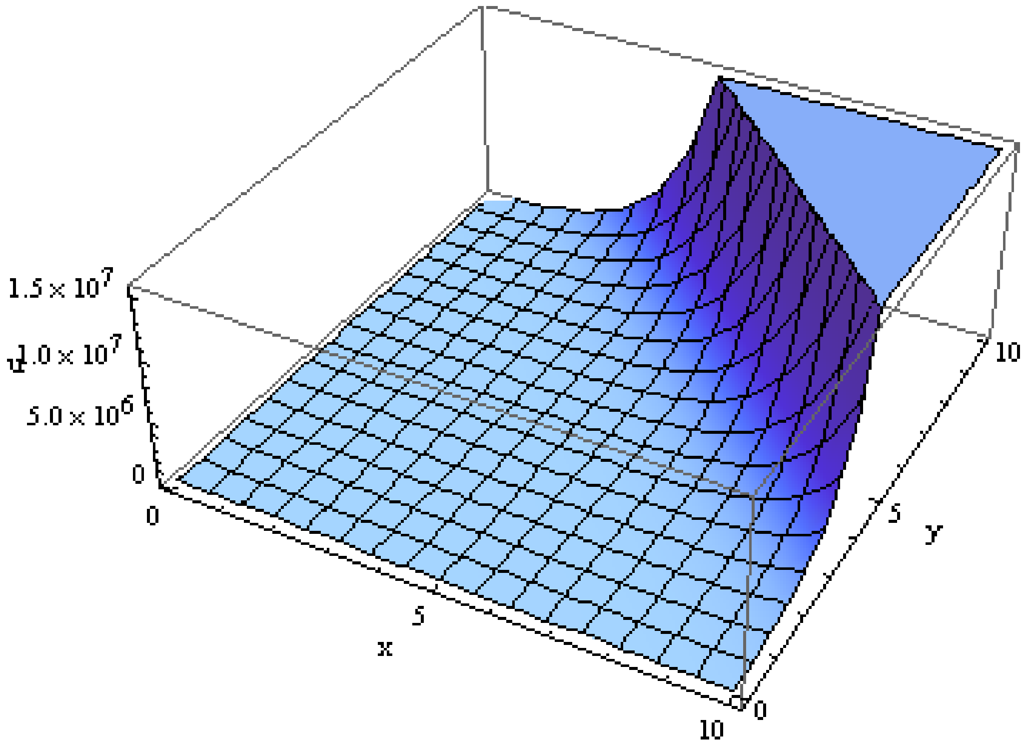

Figure 5. The behavior of the approximate solution at $r = 3, t = 2, \alpha = 0.85$.

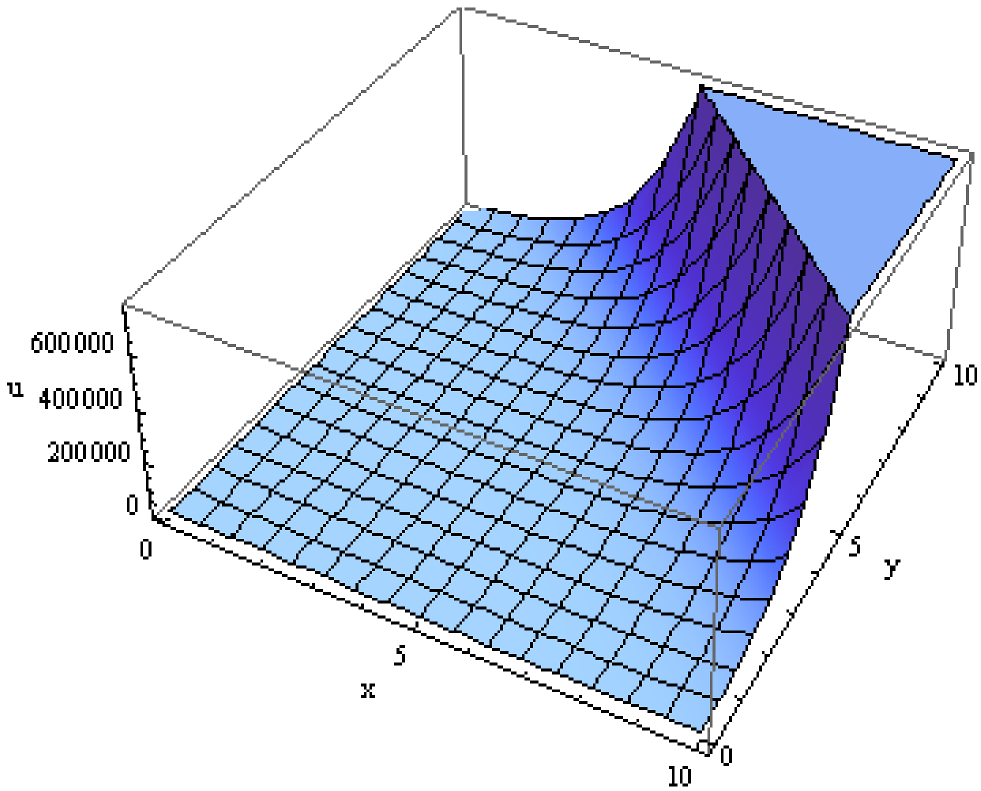

Figure 6. The behavior of the approximate solution at $r = 5, t = 1, \alpha = 0.5$.

where in Figure 4-6, we presented the behavior of the approximate solution with different values of α ($\alpha = 1, 0.85$ and 0.5, respectively) and different values of r . from these fpgures we can see that the obtained solutions are in full agreement with the results obtained by ([1],[4],[5],[9]).

3.3 Example :[14]

We consider the time- fractional biological equation in the form

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + u \qquad 0 < \alpha \le 1 \tag{19}$$

subject to the initial condition $u(x, y, o) = \sqrt{\sin x \sinh y}$.

To apply FPSM ,we suppose that the solution of (19) takes the form

$$u(x, y, t) = \sum_{n=0}^{\infty} u_n(x, y) t^{\alpha n} \tag{20}$$

$$= u_o(x, y) + u_1(x, y) t^\alpha + u_2(x, y) t^{2\alpha} + \dots$$

by theorem

$$D_t^\alpha u = \sum_{n=0}^{\infty} u_n(x, y) \frac{\Gamma(n\alpha + 1)}{\Gamma((n-1)\alpha + 1)} t^{(n-1)\alpha} \tag{21}$$

$$u^2 = u_0^2 + 2u_0 u_1 t^\alpha + (2u_0 u_2 + u_1^2) t^{2\alpha} + \dots \tag{22}$$

$$(u^2)_x = 2u_0u_{0x} + 2(u_{0x}u_1 + u_0u_{1x})t^{\alpha} + 2(u_0u_{2x} + u_{0x}u_2 + 2u_1u_{1x})t^{2\alpha} + \ldots$$

$$\begin{aligned}(u^2)_{xx} &= 2(u_0u_{0xx} + (u_{0x})^2) + 2(u_0u_{1xx} + 2u_{0x}u_{1x} + u_{0xx}u_1)t^{\alpha} \\ &+ 2(u_0u_{2xx} + 2u_{0x}u_{2x} + u_{0xx}u_2 + 2(u_1u_{1xx} + (u_{1x})^2)t^{2\alpha} + \ldots\end{aligned} \quad (23)$$

$$(u^2)_y = 2u_0u_{0y} + 2(u_{0y}u_1 + u_0u_{1y})t^{\alpha} + 2(u_0u_{2y} + u_{0y}u_2 + 2u_1u_y)t^{2\alpha} + \ldots$$

$$\begin{aligned}(u^2)_{yy} &= 2(u_0u_{0yy} + (u_{0y})^2) + 2(u_0u_{1yy} + 2u_{0y}u_{1y} + u_{0yy}u_1)t^{\alpha} \\ &+ 2(u_0u_{2yy} + 2u_{0y}u_{2y} + u_{0yy}u_2 + 2(u_1u_{1yy} + (u_{1y})^2)t^{2\alpha} + \ldots\end{aligned} \quad (24)$$

substituting (20), (21), (22) and (23), (24) into (19) and comparing the cofficients of t^{α}

$$\sum_{n=1}^{\infty} u_n(x,y)\frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)}t^{(n-1)\alpha} =$$

$$\begin{aligned}&[2(u_0u_{0xx} + (u_{0x})^2) + 2(u_0u_{1xx} + 2u_{0x}u_{1x} + u_{0xx}u_1)t^{\alpha} \\ &+ 2(u_0u_{2xx} + 2u_{0x}u_{2x} + u_{0xx}u_2 + 2(u_1u_{1xx} + (u_{1x})^2)t^{2\alpha} + \ldots] \\ &+ [2(u_0u_{0yy} + (u_{0y})^2) \\ &+ 2(u_0u_{1yy} + 2u_{0y}u_{1y} + u_{0yy}u_1)t^{\alpha} + 2(u_0u_{1yy} + 2u_{0y}u_{1y} + u_{0yy}u_1)t^{2\alpha} + \ldots] \\ &+ [u_o + u_1t^{\alpha} + u_1t^{2\alpha} + \ldots]\end{aligned}$$

using initial condition $u(x,y,o) = \sqrt{\sin x \sinh y}$.

we have $u_0(x,y) = \sqrt{\sin x \sinh y}$

Next we determine the u_n $(n = 1, 2, \ldots)$.

$$u_1\Gamma(\alpha+1) = 2(u_0u_{0xx} + (u_{0x})^2) + 2(u_0u_{0yy} + (u_{0y})^2) + u_o \quad (25)$$

and

$$u_2\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} = 2(u_0u_{1xx} + 2u_{0x}u_{1x} + u_{0xx}u_1) + 2(u_0u_{1yy} + 2u_{0y}u_{1y} + u_{0yy}u_1) + u_1 \quad (26)$$

therefore we obtain the approximate solution of equation(19)

$$u(x,y,t) = u_o(x,y) + u_1(x,y)t^{\alpha} + u_2(x,y)t^{2\alpha} + ..$$

For example , if $u_0(x,y) = \sqrt{\sin x \sinh y}$. then form (25) and (26) we get

$$u_1(x,y) = \frac{\sqrt{\sin x \sinh y}}{\Gamma(\alpha+1)},$$

$$u_2(x,y) = \frac{\sqrt{\sin x \sinh y}}{\Gamma(2\alpha+1)}$$

then $u_n(x,y) = \dfrac{\sqrt{\sin x \sinh y}}{\Gamma(n\alpha+1)}$

$$u(x,y,t) = \sum_{n=0}^{\infty} u_n(x,y) t^{\alpha n} = \sqrt{\sin x \sinh y} \sum_{n=0}^{\infty} \frac{t^{\alpha n}}{\Gamma(n\alpha+1)} = \sqrt{\sin x \sinh y}\, E_\alpha(t^\alpha).$$

4 Conclusion.

In this paper, the fractional power series method has been successfully applied to study the time-fractional biological equation.The results show that FPSM is an efficient and easy- to- use technique for finding exact and approximate solutions for nonlinear fractional partial differential equations.The obtained approximate solutions using the suggested method is in excellent agreement with the exact solution and show that these approaches can be solved the problem effectively and illustrates the validity and the great potential of the proposed technique.

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