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Solving the recognition problem of Lorenz braids via matrices of inversions for permutations

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Abstract.

In this work, we present some needed results about matrices of inversions for permutations. Then we apply it for solving the recognition problem of Lorenz braids. Each Lorenz braid is uniquely determined by a unique simple binary matrix. Then, we got a quick algorithm for counting the trip number (minimal braid index) hence, crossing number and minimal braid representative of the Lorenz knots.

Keywords: Lorenz Knots and links; Braid groups; Matrices of inversions for permutations.

Mathematics Subject Classification 2000: 37D45, 20F36, 05A05.

1 Introduction

A knot is a simple closed curve embedded in $\mathbb{R}^3$, while a link is a finite collection of knots [1]. Lorenz knots and links arise from non-linear dynamical systems. They include all torus knots, algebraic knots and modular knots. They are closed positive braids and fibered. See [2], [3], [4] for details. They are prime links [5], twists links [6], and modular links [7]. Also, Lorenz knots can not be a satellite of non-Lorenz knots, only parallel cables with possible twists can occur [8].

A lorenz knot is a periodic orbit of a flow in $\mathbb{R}^3$, determined by the Lorenz system of ordinary differential equations $\dot{x} = 10(y-x), \dot{y} = 28x - y - xz, \dot{z} = xy - 8/3z$ [3]. A template is a branched 2−dimensional manifold in $\mathbb{R}^3$ with boundary. Figure 1a shows the Lorenz template. A Lorenz link is a finite collection of disjoint simple closed curves embedded in the Lorenz template. The crossings in the regular projection of a Lorenz link to a plane is all positive, as in figure 1b.

Fig.1: a) Lorenz Template b) Positive crossing c) Lorenz braid Template

A braid groups B_n has a presentation of $n-1$ generators $\sigma_i, i = 1, 2, ..., n-1$ subject to the relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, i = 1, 2, ..., n-2$, $\sigma_i\sigma_j = \sigma_j\sigma_i$, $|i-j| \geq 2$. A positive braid is called a positive permutation braid if each pair of its strands cross in a positive sense at most once. The braid where each two strands cross each other exactly once in a positive sense is called the fundamental braid, denoted Δ_n. Where,

$$\Delta_1 = I, \Delta_2 = \sigma_1, \Delta_3 = \sigma_1.\sigma_2\sigma_1, \Delta_n = \Delta_{n-1}\sigma_{n-1}\sigma_{n-2}...\sigma_2\sigma_1, \Delta_n^2 = (\sigma_1\sigma_2...\sigma_{n-1})^n$$

Any oriented knot or link K can be viewed as a closed braid $\widehat{\beta}$, for a braid word β in some B_n. The braid index for a knot or link is the smallest integer n, such that it can be represented as a closed $n-$braid.

Definition 1 *A lorenz braid $L(l, r)$ is a finite set of strands that embeds on the the Lorenz braid template, figure 1c. It is a two groups of strands, a left group of l strands and a right group of r strands, $l + r = n$. It is a positive permutation braid with the restrictions,*

- *Strands in the same group never cross one another.*
- *Strands in the left group always pass over those in the right group.*
- *Each strand in any group should cross some strands in the other group.*
- *The permutation π at the end of strands and as a product of disjoint cycles $\pi = \mu_1\mu_2...\mu_k, k \geq 2$, no two cycles $\mu_i = \mu_j$ of the same length s, and $\mu_i(x) = \mu_j(x) + t$, $x = 1, 2, ..., s$, for some integer s. The permutation at the ends of the strands of a Lorenz braid is called a Lorenz permutation.*
- *A Lorenz knot or link is a closed Lorenz braid $\widehat{L}(l, r)$.*

An inversion of a permutation $\pi = (\pi_1\pi_2 ... \pi_n)$ in S_n is a pair (i, j) with $i < j$ and $\pi_i > \pi_j$. The matrix of inversions for a permutation $\pi = (\pi_1\pi_2 ... \pi_n)$ in S_n is a matrix $M_\pi = (m_{ij})_{n\times n}$, where $m_{ij} = 1$ if $i < j$ and $\pi_i > \pi_j$, otherwise $m_{ij} = 0$. The set $M_n(F) = \{M_\pi : \pi \in S_n,\ m_{ij} \in F = \{0, 1\}\}$ of all possible matrices of inversions for permutations over S_n is a group with a specific binary operation. The set $M_n(F)$ is isomorphic to S_n. Each permutation π in S_n induces a unique positive permutation braid with a canonical form, in such a way. Every element of S_n associates a unique canonical word in the Hecke algebra $H_{n-1}(z)$. That provides an effective and simple algorithm for counting a linear basis of Hecke algebra H_n, as binary matrices. Also, provides an algorithm for recognizing such these matrices from all binary strictly upper triangle matrices. See [9], [10] for more details about the properties of such these matrices.

In the next section we give an algorithm to recover Lorenz braids hence Lorenz knots and links from a specific matrices of inversions for permutations. Then find the trip number of a Lorenz knot or link and extract a minimal braid representative.

2 Recognition of Lorenz braids

Definition 2 *Let $L(l, r)$ be a Lorenz braid with associated Lorenz permutation $\pi_{L(l,r)}$. The corresponding matrix of inversions $M_{\pi_{L(l,r)}}$ for $\pi_{L(l,r)}$ is called a Lorenz matrix.*

The recognition problem in mathematics is to decide whether an element belongs to a specific category. Here our recognition problem is to decide whether a matrix of inversion for a permutation is a Lorenz matrix. In another words, can we recover Lorenz braids hence Lorenz knots and links from their matrices of inversion for permutations.

Theorem 3 *A matrix $M_\pi = (m_{ij})$ of inversions for a permutation $\pi = (\pi_1\pi_2 ... \pi_n)$ in S_n is a Lorenz matrix if and only if there exist two integers $l, r, 1 < l, r < n, l + r = n$, such that*

$$m_{ij} = \left\{ \begin{array}{ll} 1 & \forall\ 1 \le i \le l, l + 1 \le j \le l + \pi_i - i \\ 0 & \textit{otherwise} \end{array} \right\}$$

Proof. Let π be a Lorenz permutation, then we have two groups of strands with no crossing in each group. But $\pi_i < \pi_j\ \ \forall\ 1 \le i < j \le l$ and $\forall\ l + 1 \le i < j \le n$, then in both cases there are no inversions, which means that $m_{ij} = 0\ \forall\ 1 \le i < j \le l$ (the $l \times l$ submatrix at the first l rows and l columns of the matrix $M_{\pi_{L(l,r)}}$). Also $m_{ij} = 0\ \forall\ l + 1 \le i < j \le n$ (the $r \times r$ submatrix at the last r rows and r columns of the matrix $M_{\pi_{L(l,r)}}$). Also $\pi_1 < \pi_k\ \ \forall\ k > 1, k = l + 1, l + 2, ..., l + \pi_1 - 1$, then $m_{1k} = 1\ \forall\ k = l+1, l+2, ..., l+\pi_1-1$. In general $\pi_i < \pi_k\ \forall\ k > i, k = l+1, l+2, ..., l+\pi_i-i$, then $m_{ik} = 1\ \forall\ k = l + 1, l + 2, ..., l + \pi_i - i$. $i = 1, 2, ..., l$. The converse is obvious. The matrix M_π has the pattern as in figure 2 ■

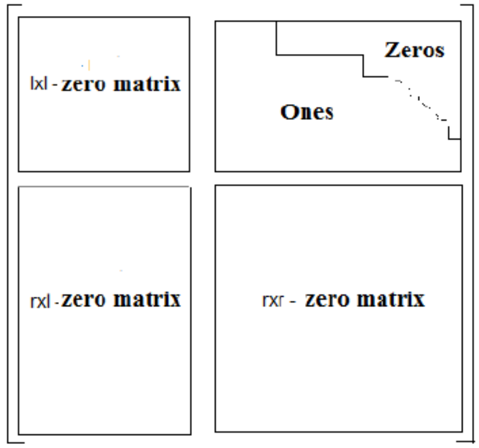


Fig. 2: A patern of a Lorenz matrix

Remark 4 *In a Lorenz matrix* $M_{\pi_{L(l,r)}}$,

- *It is remarkable that* $m_{il+1} = 1 \forall i = l, 2, ..., l$. *Therefore the* $i\underline{th}$ *row has* $\pi_i - i$ *ones,* $1 \leq i \leq l$. *Also if* $1 \leq i < i+1 \leq l+1$, *then*

$$\pi_i < \pi_{i+1} \Rightarrow \pi_i \leq \pi_{i+1} - 1 \Rightarrow \pi_i - i \leq \pi_{i+1} - (i+1)$$

this means that the number of ones in $i\underline{th}$ *row does not exceed the number of ones in* $(i+1)\,\underline{st}$ *row. But ones in all rows starts from the* $(l+1)\,\underline{st}$ *column. We find that the ones accumulate next to each other and do not allow for zeros that enters them. In fact the ones form a shap looks like a stairs, as in figure 2. Each step is the rows of ones with the same length. The lengths of rows (steps) increase from top to botton, if two successive rows* i *and* $i+1$ *have equal lengths, then* $\pi_i - i = \pi_{i+1} - (i+1)$, *so* $\pi_{i+1} = \pi_i + 1$. *But* $\pi_l = n = l + r \Rightarrow \pi_l - l = r$. *Then the bais of the stairs is the* $l\underline{th}$ *row, with length* r. *While the height of the stairs is the number of ones in the* $(l+1)\,\underline{st}$ *column, which equals* l.

- *The total crossing number* c_L *of the associated Lorenz braid* $L(l, r)$ *is the sum of all ones in the associated Lorenz matrix, in other words, it is the sum of the lengths of rows,* $c_L = \sum_{i=1}^{l} (\pi_i - i)$.

Corollary 5 *The ordered set* $S_\pi = \{(l_i, h_i), i = 1, 2, ...k\}$ *is a complete invariant for the associated Lorenz braid. Where* l_i *and* h_i *are the length and the height of* $i\underline{th}$ *step, respectively.*

Proof. The two integers l and r are determined by the height and the basis length of the stairs, take $n = l + r$. For $i = 1, 2, ..., l$, take $l_i = (\pi_i - i)$, then $\pi_i = l_i + i$.

Now we know the first l values of the permutation π. Then complete the remaining values from the set $\{1,2,...,n\} \setminus \{\pi_i, i=1,2,...,l,\}$ in an increasing order. Then the Lorenz braid L is determined entirely by a lorenz matrix ■

The trip number of a Lorenz braid is the maximal number of strands from the left band that cross over the same number of strands from the right band. This concept was first introduced in the study of Lorenz knots from the point of view of symbolic dynamics. The braid index of a Lorenz knot or link is its trip number [3]. The comming lemma provides a quick algorithm for computing the trip number.

Lemma 6 *The trip number of a Lorenz braid is the length of the widest square submatrix of the associated Lorenz matrix.*

Proof. For a Lorenz braid $L(l,r)$, $\pi_i < \pi_{i+1} \forall i = 1,2,...,l$. Also $\pi_i < \pi_{i+1} \forall i = l+1, l+2, ..., l+r$. The trip number of a Lorenz braid is a number t, such that $t = \#\{i : l < \pi_i, i \in \{1,2,...,l\}\} = \#\{j : l > \pi_j, j \in \{l+1, l+2, ..., l+r\}\}$. So it will be strands numbered $l-t+1, l-t+2, ..., l, l+1, l+2, ..., l+t$, at the top of the braid. In fact $m_{ij} = 1$ for $1 \le i \le l, l < \pi_i$ and $l+1 \le j \le l+r = n, l+1 > \pi_j$. $\{i :, i \in \{1,2,...,l\}\}$ ■

The following lemma provides another proof of a result due to J. Birman and R. Williams [3].

Lemma 7 *A Lorenz link $L(l,r)$ with trip number t has a minimal $t-$braid representative*

$$\begin{aligned} & \left(\sigma_1\sigma_2...\sigma_{t-1}\right)^t \ \Pi_{i=1}^{l-t}\left(\sigma_1\sigma_2...\sigma_{n_i}\right) \ \Pi_{j=n}^{r-t}\left(\sigma_{t-1}\sigma_{t-2}...\sigma_{m_j}\right), \\ n_i &= (\pi_i - i) - 1 = \textit{number of ones in the ith row } -1, \\ m_j &= (j - \pi_j) - 1 = \textit{number of ones in the jth column } -1 \end{aligned}$$

Proof. The ones in a Lorenz matrix lies in three blocks. The widest square submatrix of length t, a top block with $l-t$ rows and a right block with $r-t$ columns, as the matrix in example 8. Connecting opposite ends of the strands of the 1st positions at top and bottom of the braid, we get a curl which starts at the position $l+1$ at the top and ends at the position π_1 at the bottom. After removing the trivial curl, we get an arc starts at the position $l+1$ at the top and ending at the position π_1 of bottom, crossing $n_1 = (\pi_1 - 1) - 1 = \#$ {ones at the first row in the matrix} -1 strands at the left of the braid. This contributes a braid word $(\sigma_1\sigma_2...\sigma_{n_1})$. In general, the arc starts at the position i at the top and ending at the position π_i at contributes a braid word $(\sigma_1\sigma_2...\sigma_{n_i}), i = 1,2,...,l-t$. The similar sing will hold at the right of the braid, which completes the proof ■

Example 8 *Let us take the early example of Birman and Williams [3]. Consider the permutation* $\pi = (5\ 7\ 11\ 12\ 13\ 14\ 15\ 18\ 21\ 22\ 1\ 2\ 3\ 4\ 6\ 8\ 9\ 10\ 16\ 17\ 19\ 20)$ *in* S_{22}, *with Lorenz braid* $L(10,12)$*. Then its associated matrix of inversions is*

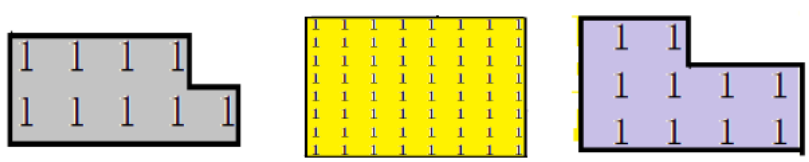
$$M_\pi = \left[\begin{array}{cccccccccccccccccccccc}
0&0&0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right]$$

The matrix has a stairs of ones with five steps, with invariant set

$$S_\pi = \{(4,1),(5,1),(8,5),(10,1),(12,2)\}$$

The stairs consists of three blocks, as in figure 3. The top block is the first two rows, with 4 *and* 5 *ones, From top to bottom. That contribute* $(\sigma_1\sigma_2\sigma_3)$ *and* $(\sigma_1\sigma_2\sigma_3\sigma_4)$*, respectively, as in figure 3a. The widest square submatrix contributes* Δ_8^2*, as in figure 3b. The square of length equals* 8*, then the trip number (braid index) is* 8*. The right block is the last four columns, with* $2,2,3$ *and* 3 *ones, from right to left. That contribute* $\sigma_7, \sigma_7, , \sigma_7\sigma_6, \sigma_7\sigma_6$ *respectively, as in figure 3c. So the given Lorenz link has the twisted minimal positive braid,*

$$\begin{aligned} b_\pi &= (\sigma_1\sigma_2\sigma_3)(\sigma_1\sigma_2\sigma_3\sigma_4).\Delta_8^2.\sigma_7.\sigma_7(\sigma_7\sigma_6)(\sigma_7\sigma_6) \\ &= (\sigma_1\sigma_2\sigma_3)(\sigma_1\sigma_2\sigma_3\sigma_4)(\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7)^8\sigma_7^2(\sigma_7\sigma_6)^2 \end{aligned}$$



$(\sigma_1\sigma_2\sigma_3)(\sigma_1\sigma_2\sigma_3\sigma_4)$ $\qquad \Delta_8^2 \qquad$ $\sigma_7^2(\sigma_7\sigma_6)^2$

Fig. 3: a $\qquad$ *b* $\qquad$ *c*

Also we can recover the permutation from the invariant set, which comes directly from the stairs. The pair $(4,1)$ *means that the* 1*st strand from the left group passes over* 4 *strands in the right group, hence* $\pi_1 = 1+4 = 5$*. The pair* $(5,1)$ *means that the* 2*nd strand from the left group passes over* 5 *strands in the right group, hence* $\pi_2 = 2+5 = 7$*. The pair* $(8,5)$ *means that the the next five strands from the left group passes over* 8 *strands in the right group, hence* $\pi_3 = 3+8 = 11$*,* $\pi_4 = 4+8 = 12$*,* $\pi_5 = 5+8 = 13$*,* $\pi_6 = 6+8 = 14$*,* $\pi_7 = 7+8 = 15$*. The pair* $(10,1)$ *means that the* 8*th strand from the left group passes over* 10 *strands in the right group, hence*

$\pi_8 = 8 + 10 = 18$. *Finally the* $9th$ *and* $10th$ *strand from the left group each passes over* 12 *strands in the right group, so* $\pi_9 = 9 + 12 = 21, \pi_{10} = 10 + 12 = 22$. *Therefore* $\pi = (5\ 7\ 11\ 12\ 13\ 14\ 15\ 18\ 21\ 22\ \ldots\ldots\ldots\ldots\ldots)$. *At the end, complete the permutation by inserting the remaining numbers from* 1 *to* 22 *in an increasing order in the empty cells, hence* $\pi = (5\ 7\ 11\ 12\ 13\ 14\ 15\ 18\ 21\ 22\ 1\ 2\ 3\ 4\ 6\ 8\ 9\ 10\ 16\ 17\ 19\ 20)$

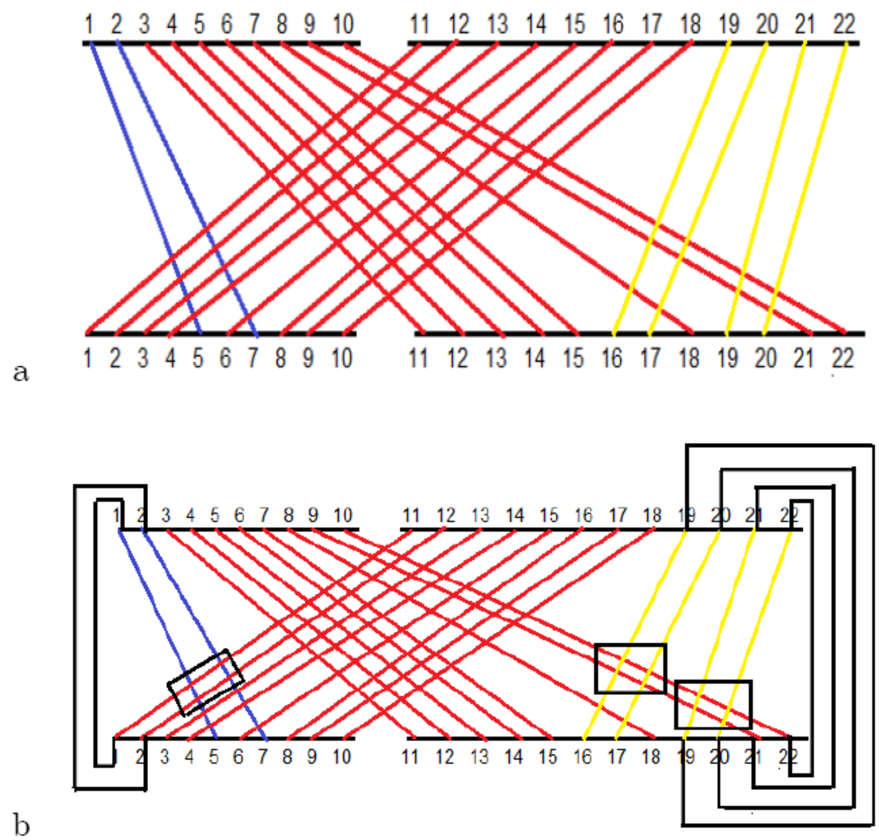


Fig. 4

Example 9 *Another example in [11], the Lorenz braid* $L(9, 8)$ *with the Lorenz permutation,*

$$\pi = (3\ 4\ 5\ 6\ 8)\,9\ 13\ 16\ 17\ 1\ 2\ 7\ 10\ 11\ 12\ 14\ 15) \text{ in } S_{17}.$$

Then with a direct application to our algorithm, we have

- *The associated Lorenz matrix is*

$$\begin{bmatrix}
0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0\\
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0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix}$$

- *Its invariant set is* $S_\pi = \{(2,4), (3,2), (6,1), (8,2)\}$

- *Its trip number is* 3, *with minimal braid representative*

$$\begin{aligned}
b_\pi &= (\sigma_1)(\sigma_1)(\sigma_1)(\sigma_1)(\sigma_1\sigma_2)(\sigma_1\sigma_2).\Delta_3^2.\sigma_2.\sigma_2(\sigma_2\sigma_1)(\sigma_2\sigma_1)(\sigma_2\sigma_1)\\
&= (\sigma_1)^4(\sigma_1\sigma_2)^2.(\sigma_1\sigma_2)^3.\sigma_2^2(\sigma_2\sigma_1)^3
\end{aligned}$$

3 Conclusion

We applied matrices of inversions for permutations for solving the recognition problem of Lorenz braids. Each Lorenz braid is uniquely determined by a specific binary strictly upper triangle matrix. We got a simple algorithm for counting the trip number (Minimal braid index). These binary matrices might be help for providing a good understanding of the relation between modular knots, Lorenz knots and twisted knots. It might help for computing polynomial invariants of such these knots, where S_n represents a basis for some Hecke algebra. Also for counting the Lorenz permutations in each S_n.

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