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Solving Ordinary differential equations with variable coefficients

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Abstract
ZZ transform, whose fundamental properties are presented in this paper, is still not widely known, nor used, ZZ transform may be used to solve problems without resorting to a new frequency domain. In this paper, we use ZZ Transform to solve Ordinary Differential equationswith variable coefficients.The result reveals that the proposed method is very efficient, simple and can be applied to linear and nonlinear differential equations.

Keyword: ZZ transform- differential equations.

Introduction

Integral transforms [1-2] play an important role in many fields of science. In literature, integral transforms are widely used in mathematical physics, optics, engineering mathematics and, few others. Among these transforms which were extensively used and applied on theory and applications areLaplace Transform, Fourier Transform, SumuduTransform [3], Elzaki Transform [4-6], ZZ Transform, Natural Transform, and Aboodh Transform [7-9. Of these the most widely used transform is Laplace Transform. New integraltransform, named as ZZ Transformation [10-13] introduce by ZainUlAbadinZafar[2016], ZZ transform was successfully applied to integral equations, partial differential equations, ordinary differential equationsand system of all these equations. The main objective is to introduce solution ofOrdinary Differential Equation with Variable Coefficients by using a ZZtransform .The plane of the paper is asfollows: In section 2, we introduce the basic idea of ZZtransform, application in 3 and conclusion in 4, respectively.

2. Definitions and Standard Results

The ZZ Transform:

Definition: Let (t) be a function defined for all $t \geq 0$. The ZZ transform of $f(t)$is the function

$Z(u,s)$Defined by

$Z(u,s) = H\{f(t)\} = s\int_0^\infty f(ut)e^{-st}dt$ (1)

Provided the integral on the right side exists. The unique function $f(t)$ in (1) is called the inverse

Transform of $Z(u,s)$ is indicated by

$$f(t) = H^{-1}\{Z(u,s)\}$$

Equation (1) can be written as

$H\{f(t)\} = \frac{s}{u}\int_0^\infty f(t)e^{-\frac{s}{u}t}dt$ (2)

ZZ transform of some functions:

$H\{1\} = 1$, $H\{t^n\} = n!\frac{u^n}{s^n}$, $H\{e^{at}\} = \frac{s}{s-ua}$

$H(\sin(at)) = \frac{aus}{s^2+a^2u^2}$, $H(\cos(at)) = \frac{s^2}{s^2+a^2u^2}$.

ZZ transform of derivatives:

Theorem I

If ZZ transform of the function $f(t)$ given by$H[f(t)] = Z(u,s)$, then:

1) let $H\{f(t)\} = Z(u,s)$ then

$$H\{f^{(n)}(t)\} = \frac{s^n}{u^n}Z(u,s) - \sum_{k=0}^{n-1}\frac{s^{n-k}}{u^{n-k}}f^{(k)}(0)$$

2) (i) $H\{tf(t)\} = \frac{u^2}{s}\frac{d}{du}\left(Z(u,s)\right) + \frac{u}{s}Z(u,s)$

(ii) $H\{tf'(t)\} = \frac{u^2}{s}\frac{d}{du}\left(\frac{s}{u}Z(u,s)\right) + Z(u,s)$

(iii) $H\{tf''(t)\} = s\frac{d}{du}\left(Z(u,s)\right) - \frac{s}{u}Z(u,s) + \frac{s}{u}f(0)$

Proof :

2) (i) $z(u,s) = H\{f(t)\} = \frac{s}{u}\int_0^\infty f(t)e^{-\frac{s}{u}t}dt$

$$\frac{d}{du}Z(u,s) = \frac{d}{du}\left(\frac{s}{u}\int_0^\infty f(t)e^{-\frac{s}{u}t}dt\right) = \frac{s}{u}\int_0^\infty \frac{\partial}{\partial u}\left(e^{-\frac{s}{u}t}\right)f(t)dt - \frac{s}{u^2}\int_0^\infty f(t)e^{-\frac{s}{u}t}dt$$

$$\frac{d}{du}Z(u,s) = \frac{s}{u}.\frac{s}{u^2}\int_0^\infty tf(t)e^{-\frac{s}{u}t}dt - \frac{s}{u^2}\int_0^\infty f(t)e^{-\frac{s}{u}t}dt$$

$$\frac{d}{du}Z(u,s)=\frac{s}{u^2}H\{tf(t)\}-\frac{1}{u}\ z(u,s)$$

$$\frac{s}{u^2}H\{tf(t)\}=\frac{d}{du}Z(u,s)+\frac{1}{u}\ z(u,s)$$

$$H\{tf(t)\}=\frac{u^2}{s}\frac{d}{du}Z(u,s)+\frac{u}{s}\ z(u,s)$$

The proof of (ii) and (iii) are similar to the Proof of (i).

Now we apply the above theorem to find ZZtransform for some differential equations:

4 Application :

Example 4.1

Solve the differential equation:

$y'' + ty' - y = 0$ (3)

With the initial condition $,y(0)=0\ ,\ y'(0)=1$ (4)

solution

Using the differential property of ZZ transform Eq.(3) can be written as

$$\frac{s^2}{u^2}Z(u,s)-\frac{s}{u}+\frac{u^2}{s}\frac{s}{u}\frac{dZ(u,s)}{du}+\frac{u^2}{s}\left(-\frac{s}{u^2}\right)Z(u,s)+Z(u,s)-Z(u,s)=0$$

$$u\frac{d}{du}Z(u,s)+\left(\frac{s^2}{u^2}-1\right)Z(u,s)=\frac{s}{u}$$

$\frac{d}{du}Z(u,s)+\left(\frac{s^2}{u^3}-\frac{1}{u}\right)Z(u,s)=\frac{s}{u^2}$ (5)

This is a linear differential equation for unknown function $Z(u,s)$, have the Solution in the form

$Z(u,s)=\frac{u}{s}+\frac{c}{u}$ and $C=0$, then:$Z(u,s)=\frac{u}{s}$ (6)

By using the inverse ZZ transform we obtain the Solution in the form of

$y(t)=t$ (7)

Example 4.2

Solve the differential equation:

$y'' + 2ty' - 4y = 6$ (8)

With the initial condition $,y(0)=0\ ,\ y'(0)=0$ (9)

Solution

Using the differential property of ZZ transform Eq.(8) can be written as

$$\frac{s^2}{u^2}Z(u,s)+2\frac{u^2}{s}\frac{s}{u}\frac{dZ(u,s)}{du}-2\frac{u^2}{s}\frac{s}{u^2}Z(u,s)+2Z(u,s)-4Z(u,s)=6$$

$$\frac{s^2}{u^2}Z(u,s) + 2u\frac{d}{du}Z(u,s) - 4Z(u,s) = 6$$

$\frac{d}{du}Z(u,s) - \left(\frac{s^2}{2u^3} - \frac{2}{u}\right)Z(u,s) = \frac{3}{u}$ (10)

This is a linear differential equation for unknown function $Z(u,s)$, have the Solution in the form

$Z(u,s) = 6\frac{u^2}{s^2} + Ce^{-\frac{1}{4}\frac{s^2}{u^2}}$ and $C = 0$, then: $Z(u,s) = \frac{u^2}{s^2}$ (11)

By using the inverse ZZ transform we obtain the Solution in the form of

$y(t) = 3t^2$ (12)

Example 4.3 Consider the second-order differential equation

$ty''(t) + (t+1)y'(t) + 2y(t) = e^{-t}$ (13)

With the initial condition $, y(0) = 0 \ , \ y'(0) = 4$ (14)

Solution:

Applying the ZZ transform of both sides of Eq. (13),:

H $ty''(t)$ − H{$(t+1)y'(t)$} + H{$2y$} = H{0} , so (15)

Using the differential property of ZZ transform Eq.(15) can be written as:

$s\frac{d}{du}Z(u,s) - \frac{s}{u}Z(u,s) + \frac{s}{u}y(0) + \frac{u^2}{s}\frac{d}{du}\left(\frac{s}{u}Z(u,s)\right) + Z(u,s) + \frac{s}{u}Z(u,s) - \frac{s}{u}y(0) +$
$2Z(u,s) = \frac{s}{s+u}$ (16)

Using initial condition (14), Eq. (16) can be written as

$$(s+u)\frac{d}{du}Z(u,s) + 2Z(u,s) = \frac{s}{s+u}$$

$\frac{d}{du}Z(u,s) + \frac{2}{s+u}Z(u,s) = \frac{s}{(s+u)^2}$ (17)

This is a linear differential equation for unknown function $Z(u,s)$, have the Solution in the form

$Z(u,s) = \frac{su}{(s+u)^2} + c$, and $c = 0$

By using the inverse ZZ transform we obtain the Solution in the form of

Y(t) = te^{-t} (18)

Example 4.4

Consider the initial value problem

$ty''(t) + y'(t) + ty(t) = 0$ (19)

With the initial conditions

$y(0) = 1 \ , \ y'(0) = 0$ (20)

Solution:

Applying the ZZ transform to both sides of (19) we have

$$H\{t\,y''(t)\} + H\{y'(t)\} + H\{ty(t)\} = 0 \quad (21)$$

Using the differential property of ZZ transform Eq.(21) can be written as:

$$s\frac{d}{du}Z(u,s) - \frac{s}{u}Z(u,s) + \frac{s}{u}y(0) + \frac{s}{u}Z(u,s) - \frac{s}{u}y(0) + \frac{u^2}{s}\frac{d}{du}Z(u,s) + \frac{u}{s}Z(u,s) = 0 \quad (22)$$

Now applying the initial condition to obtain

$$s\frac{d}{du}Z(u,s) + \frac{u^2}{s}\frac{d}{du}Z(u,s) = -\frac{u}{s}Z(u,s)$$

$$\frac{d}{du}Z(u,s) = \frac{-u}{s^2+u^2}Z(u,s)$$

Therefore $Z(u,s) = \frac{c}{\sqrt{s^2+u^2}}$ (23)

Now applying the inverse ZZ transform, we get

$$\mathrm{y(t)} = J_0(t) \quad (24)$$

Example 4.5

Consider the initial value problem

$$ty''(t) - ty'(t) + y(t) = 2 \quad (25)$$

With the initial conditions

$$y(0) = 2\ ,\ y'(0) = -1 \quad (26)$$

Solution:

Applying the ZZ transform to both sides of (25) we have

$$H\{t\,y''(t)\} - H\{ty'(t)\} + H\{y(t)\} = H\{2\} \quad (27)$$

Using the differential property of ZZ transform Eq.(27) can be written as

$$s\frac{d}{du}Z(u,s) - \frac{s}{u}Z(u,s) + \frac{s}{u}f(0) + \frac{u}{s}\frac{d}{du}\frac{s}{u}Z(u,s) + Z(u,s) + Z(u,s) = 2 \quad (28)$$

Now applying the initial condition to obtain

$$sZ'(u,s) - \frac{s}{u}Z(u,s) + \frac{2s}{u} - \frac{u^2}{s}\left(\frac{s}{u}Z'(u,s)\right) - \frac{s}{u^2}Z(u,s) = 2$$

$$Z'(u,s) - \frac{1}{u}Z(u,s) = -\frac{2}{u} \quad (29)$$

Equation (29) is a linear differential equation, which has solution in the form

$$Z(u,s) = 2 + cu \quad (30)$$

Now applying the inverse ZZ transform, we get

$$y(t) = 2 + ct \quad (31)$$

Conclusion

. In this paper, we apply a new integral transform "ZZ transform" to solve some ordinary differential equation with variable coefficients, The result reveals that the proposed method is very efficient, simple and can be applied to linear and nonlinear differential equations.

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