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On the Dihedral Cohomology of Graded Banach Algebras

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Abstract.

We are concerned with the dihedral cohomology of a unital $\mathbb{Z}/2$-graded Banach algebra A over $K=\mathbb{C}$ with a graded involution and study some properties of it. It is considered the prototype example of graded algebras with topology.

Keywords: Graded Banach algebras - dihedral cohomology.

1 $\mathbb{Z}/2$-graded Banach algebras with graded involutions.

In this section we introduce some basic concepts and facts concerning $\mathbb{Z}/2$-graded Banach algebras.

Definition (1.1) [11]:

A norm on a vector space V is a map $\|.\|:V \to \mathbb{R}$, such that :

1- $\|x\| \geq 0$ for all $x \in V$, and $\|x\| = 0 \Leftrightarrow x = 0$;

2- $\|\alpha x\| = |\alpha| \|x\|$ for all $x \in V$ and $\alpha \in \mathbb{C}$;

3- $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in V$.

Then $(V, \|.\|)$ is a normed space. A norm on V induces a metric d on V by $d(x,y) = \|x-y\|$. We say that $(V, \|.\|)$ is complete if the metric space (V, d) is (Cauchy) complete. Complete normed vector spaces are called Banach spaces. For a normed space V and $\lambda \in \mathbb{R}$,we write

$$V_{[\lambda]} = \{x \in V : \|x\| \leq \lambda\}.$$

Thus $V_{[1]}$ is the closed unit ball of V .

The maps between Banach spaces are the maps which preserve both the linear structure and the topology. Such maps are bounded linear maps ,or operators.

Lemma (1.2) [9]:

Let $T : V \to W$ be a linear map between Banach spaces, then :

1- T is continuous with respect to the norms on V and W ;

2- T is continuous at 0 ;

3- For some $\lambda \in \mathbb{R}$,we have $\|T(X)\| \le \lambda \|x\|$ for all $x \in V$. T is bounded such that :

$$\lambda_{\min} = \|T\| = \sup\{\|T(x)\| / \|x\| : x \in V, x \neq 0\}.$$

Definition (1.3) [3]:

A Banach algebra is an algebra A with a norm $\|.\|$ such that $(A, \|.\|)$ is a Banach space with the property,

$\|ab\| \le \|a\| \|b\|$ for all $a, b \in A$.

Definition (1.4) [2]:

A Banach algebra A is called a simplicially trivial if $H^n(A, A^*) = \{ \}$, for all $n = 0,1,.....$, where $A^* = Hom_K(A, K)$ is the topological dual space of A . C^*-algebras can be thought of as special Banach algebras [1].

Definition (1.5) [1]:

Let A be an algebra. A map $*: A \to A$, $a \mapsto a^*$, is an involution if :

1- $(\lambda a + b)^* = \bar{\lambda} a^* + b^*$ for all $\lambda \in \mathbb{C}$ and $a, b \in A$;

2- $(ab)^* = b^* a^*$ for all $a, b \in A$;

3- $(a^*)^* = a$ for all $a \in A$.

Remark :

A map $*: A \to A$; $a \mapsto a^*$,is an involution if $*^2 = id_A : A \to A$. For all

$a, b \in A$,we have $*(a) = b$ and $*(b) = a$,then $*^2(a) = *(*(a)) = *(b) = a$,i.e.

$*^2 = id_A$.

Definition (1.6) [1]:

Let A be a Banach algebra. The pair $(A, *)$ is called a C*-algebra if

$$\|a^* a\| = \|a\|^2 \text{ for all } a \in A .$$

Definition (1.7):

Let G be a finite group with unit e , and A be a unital complex Banach algebra. A G -graded structure for A is a decomposition $A = \underset{g \in G}{\oplus} A_g$, where $A_g \subseteq A$ and $A_g A_h \subseteq A_{g+h}$ for all $g, h \in G$, $\oplus$ is the direct sum.

An element $a \in A_g$ is a homogenous element of degree $|a| = g$, it is called nontrivial homogenous if $g \neq e$.

Example (1.8):

For the group $G = \mathbb{Z}_2 = \{0,1\}$ and $A = A_0 \oplus A_1$, we have $A_g A_h \subseteq A_{g+h} \mod 2$, for all $g, h \in G$, and

$$|a| = \begin{cases} 0 & if \quad a \in A_0 \quad (a \quad even) \\ 1 & if \quad a \in A_1 \quad (a \quad odd) \end{cases}.$$

For the reflexive group $G = \mathbb{Z}/2 = \{-1,+1\}$ of order 2 and $A = A^+ \oplus A^-$,

we have $A_g A_h \subseteq A_{g+h}$, for all $g,h \in G$.

Remark :

A morphism $f : A \to A$ is a graded if $f(A_g) \subseteq A_g$, for all $g \in G$. Firstly, we recall some definitions and facts we need here. See [8]. We set up the theory of $\mathbb{Z}/2$ -graded Banach spaces, complexes and algebras. Let K $(K = \mathbb{C})$ be a field such that $ch(K) = 0$, and $\alpha \in \{+,-\}$ which we identify with $\{+1,-1\}$. A Banach space V is a complete normed vector space $(V, \|.\|)$.

Definition (1.9) [8] :

A $\mathbb{Z}/2$-graded Banach space is a K - Banach space V equipped with an involution $\#: V \to V$, defined by $x \to \alpha x$, $(x \in V$, $\alpha = \pm)$. It is also, a $K[\mathbb{Z}/2]$-module V .

Lemma (1.10) [8]:

A $\mathbb{Z}/2$-graded K - Banach space V is trivially graded if :

(a) V is a trivial $K[\mathbb{Z}/2]$-module; or (b) $V = V^+$ or $\# = id_V$.

Proof :

If $V = V^+$, then $x \in V^+$, $|x| = 0$ and $\alpha = +$, hence $\#(x) = \alpha x = x$, i.e. $\# = id_V$.

If $\# = id_V$, then $\#(x) = x$, hence $\#^2(x) = \#(\#(x)) = \#(x) = x$, i.e. $\alpha = +$ or $V = V^+$

Remark:

A map of $\mathbb{Z}/2$-graded Banach spaces is a bounded linear map (continuous) $f : V \to W$, i.e. $V^\alpha \mapsto W^\alpha$, $\alpha = \pm$, or commutes with $\#$ or a map of $K[\mathbb{Z}/2]$-modules.

Definition (1.11)[8]:

A positively graded complex of Banach spaces

$V_* = \{\cdots \longrightarrow V_2 \xrightarrow{d} V_1 \xrightarrow{d} V_0 \longrightarrow 0\}$ is a $\mathbb{Z}/2$-graded complex if all Banach spaces V_i $(i \geq 0)$ are $\mathbb{Z}/2$-graded and all differentials $d : V_i \to V_{i-1}$ are maps of $\mathbb{Z}/2$-graded spaces.

Now, we can define the $\mathbb{Z}/2$-graded Banach algebra.

Definition (1.12) [8]:

A $\mathbb{Z}/2$-graded Banach algebra is an associative unital K - Banach algebra A such that the multiplication is a map $\pi : A \otimes A \to A$ of

$\mathbb{Z}/2$-graded Banach spaces. That is:

(a) the involution $\#: A \to A$ is a homomorphism of Banach algebras : $\#(ab) = \#(a)\#(b)$ for all $a,b \in A$, or

(b) $A^{\alpha}A^{\beta} \subset A^{\alpha\beta}, (\alpha, \beta \in \{\pm 1\})$.

Remark:

The multiplication $\pi : A \otimes A \to A$ is a bounded linear map (continuous) as $\|\pi(a \otimes b)\| \mapsto \|ab\| \le \|a\|\|b\|$ for all $a, b \in A$.

Example (1.13) [8]:

Any $\mathbb{Z}$-graded Banach algebra $A = \bigoplus_{n \in \mathbb{Z}} A_n$ gives rise to a $\mathbb{Z}/2$-graded Banach algebra B defined by $B = B^{+} + B^{-}$,where, $B^{+} = \bigoplus_{n} A_{2n}$ and $B^{-} = \bigoplus_{n} A_{2n+1}$.

Remark:

The involution given on $C_n(A) = A^{\otimes(n+1)}, n = 0,1,..,$ by the grading over $\mathbb{Z}/2$ is given by: $\# : C_n(A) \to C_n(A)$, such that $\#(a_0 \otimes ..... \otimes a_n) = \#(a_0) \otimes ..... \otimes \#(a_n)$, for all $a_0, ..., a_n \in A$. It commutes with the differential $b_n : C_n(A) \to C_{n-1}(A)$, defined by

$$b_n(a_0 \otimes ...... \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes ... \otimes a_i a_{i+1} \otimes ... \otimes a_n$$

$$+(-1)^{n+|a_n|(|a_0|+........|a_{n-1}|)} a_n a_0 \otimes a_1 \otimes ... \otimes a_{n-1},$$

and cyclic operator $t_n : C_n(A) \to C_n(A)$,defined by

$$t_n(a_0 \otimes ... \otimes a_{n-1} \otimes a_n) = (-1)^{|a_n|(|a_0|+........|a_{n-1}|)} a_n \otimes a_0 \otimes ... \otimes a_{n-1}.$$

In other words, $\#b = b\#$ and $\#t = t\#$.

Now, we can define the reflexive operator r.

Definition (1.14):

Let $A = A^{+} \oplus A^{-}$ be a $\mathbb{Z}/2$-graded Banach $K-$algebra with a graded involution

$$\# : A \to A \quad ; a \mapsto \alpha a \quad , \alpha = \pm \quad ,\text{for all } a \in A.$$

The reflexive operator r acting on $C_n(A) = A^{\otimes(n+1)}, n = 0,1,..,$ by the graded involution $\#$ is given by

$$r : C_n(A) \to C_n(A)$$

such that

$$r(a_0 \otimes a_1 \otimes ... \otimes a_n) = \alpha(-1)^{\lambda(\lambda+1)/2} a_0^{\#} \otimes a_n^{\#} \otimes ... \otimes a_1^{\#} \cdots \qquad (1.1).$$

where $\alpha = \pm 1$, $a_i^{\#} = im(a_i)$ under the involution $\#$ and $\lambda = |a_0^{\#}| \sum_{i=1}^{n} |a_i^{\#}| = |a_0^{\#}|(|a_n^{\#}| + ... + |a_1^{\#}|)$. Since $|a_i^{\#}| = |\alpha a_i| = \alpha |a_i|, 0 \le i \le n$, then

$$\lambda = |a_0| \sum_{i=1}^{n} |a_i| = |a_0|(|a_n| + ... + |a_1|) \cdots \qquad (1.2).$$

Special cases:

(a) When $\alpha = +$, i.e. $a \in A = A^{+}$,we have $|a_i| = 0$ and $\lambda = 0$,in (1.1) we have

$$r(a_0 \otimes a_1 ... \otimes a_n) = a_0^{\#} \otimes a_n^{\#} \otimes ... \otimes a_1^{\#}.$$

(b) When $\alpha = -$, i.e. $a \in A = A^{-}$,we have $|a_i| = 1$ and $\lambda = |a_0| \sum_{i=1}^{n} 1 = n$, that is $\lambda(\lambda+1)/2 = n(n+1)/2$, in (1.1) we have

$$r(a_0 \otimes a_1 \otimes ... \otimes a_n) = -(-1)^{n(n+1)/2} a_0^{\#} \otimes a_n^{\#} \otimes ... \otimes a_1^{\#}.$$

For example, $r(a_0 \otimes a_1) = \alpha(-1)^{\lambda(\lambda+1)/2} a_0^{\#} \otimes a_1^{\#}$, where $\lambda = |a_0||a_1|, \alpha = \pm$.

Lemma (1.15):

The reflexive operator r , defined above satisfies the relation :

$$r^2 = 1,$$

where $1 = id_A : A \to A$ is the identity map of A .

Proof :

$$r^2(a_0 \otimes a_1 \otimes ... \otimes a_n) = r(r(a_0 \otimes a_1 \otimes ... \otimes a_n))$$
$$= \alpha(-1)^{\lambda(\lambda+1)/2} r(a_0^{\#} \otimes a_n^{\#} \otimes ... \otimes a_1^{\#})$$
$$= \alpha^2(-1)^{\lambda(\lambda+1)} a_0^{\#\#} \otimes a_1^{\#\#} \otimes ... \otimes a_n^{\#\#}.$$

Since $a_i^{\#\#} = (a_i^{\#})^{\#} = a_i$ and $\alpha^2 = 1$ as $\alpha = \pm$,and also $(-1)^{\lambda(\lambda+1)} = 1$ as $\lambda(\lambda+1)$ is always even if λ is even or odd.

Thus, we have $r^2(a_0 \otimes a_1 \otimes ... \otimes a_n) = a_0 \otimes a_1 \otimes ... \otimes a_n$,i.e. $r^2 = id_A$.

2 Dihedral cohomology of $\mathbb{Z}/2$-graded Banach algebras.

We use the references [4], [5], and [6]. For the basic concepts and constructions of dihedral homology and cohomology, see [10].

Let $A = A^{+} \oplus A^{-}$ be a $\mathbb{Z}/2$-graded Banach algebra over K. The topological dual space of A ,$A^{*} = Hom(A,K)$ will be given over the $\mathbb{Z}/2$- grading by $(A^{*})^{\alpha} = (A^{\alpha})^{*}$, $\alpha = \pm$ .The associated involution on A^{*} is the transpose map of involution on A ,that is

$$\#: A \to A; a \mapsto \alpha a \Rightarrow \#^{*}: A^{*} \to A^{*}; (a)^{*} \mapsto (\alpha a)^{*} = \alpha a^{*}.$$

Let A be a unital $\mathbb{Z}/2$-graded Banach algebra over K with a graded involution $\#: A \to A$; $a \mapsto \alpha a$, $\alpha = \pm$.

Consider the codihedral $K[\mathbb{Z}/2]$-module $C(A) = (C^n(A), b^n, t^n, r^n)$,

where $C^n(A) = Hom_K(A^{\otimes n+1}, K), n \ge 0,$ is the $\mathbb{Z}/2$-graded Banach space of $(n+1)$-graded bounded linear maps (continuous) from A to K .These maps are called cochain maps $f : A^{\otimes n+1} \to K$,$C^0(A) = Hom_K(A,K) = A^{*}$, where $b^n: C^n(A) \to C^{n+1}(A)$, given by:

$$b^n f(a_0 \otimes ..... \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i f(a_0 \otimes ... \otimes a_i a_{i+1} \otimes ... \otimes a_n)$$

$$+(-1)^{n+|a_n|(|a_0|+........|a_{n-1}|)} f\,(a_n a_0 \otimes a_1 \otimes ... \otimes a_{n-1}) \qquad (2.1),$$

cyclic operator $t^n : C^n(A) \to C^n(A)$, given by

$$t^n f\,(a_0 \otimes ... \otimes a_{n-1} \otimes a_n) = (-1)^{|a_n|(|a_0|+........|a_{n-1}|)} f\,(a_n \otimes a_0 \otimes ... \otimes a_{n-1}) \qquad (2.2),$$

and reflexive operator $r^n : C^n(A) \to C^n(A)$, given by

$$r^n f\,(a_0 \otimes a_1 \otimes ... \otimes a_n) = \alpha(-1)^{\lambda(\lambda+1)/2} f\,(a_0^{\#} \otimes a_n^{\#} \otimes ... \otimes a_1^{\#}),$$

where $\alpha = \pm 1$, $a_i^{\#} = im(a_i)$ under the involution $\#$ and $\lambda = |a_0|(|a_n| + ... + |a_1|)$.

The complexes $CC^n(A)$, ${}_{\alpha}CR^n(A)$ and ${}_{\alpha}CD^n(A)$ are called the cyclic, reflexive and dihedral complexes and their cohomologies give the cyclic reflexive and dihedral cohomology groups: $HC^n(A)$, ${}_{\alpha}HR^n(A)$ and, ${}_{\alpha}HD^n(A)$ respectively. the relation between the cyclic and dihedral cohomologies of Banach algebra is given by the following assertion [6].

The following isomorphism holds :

$$HC^n(A) \approx {}_{-}HD^n(A) \oplus {}_{+}HD^n(A).$$

<u>Notes</u> :

1- When $\alpha = +$,we have $|a_i| = 0$,then $\varepsilon = |a_n|(|a_0| + \cdots\cdots + |a_{n-1}|) = 0$,in (2.1), we have

$$b^n f\,(a_0 \otimes ..... \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i f\,(a_0 \otimes ... \otimes a_i a_{i+1} \otimes ... \otimes a_n) + (-1)^n f\,(a_n a_0 \otimes a_1 \otimes ... \otimes a_{n-1})$$ which is a trivially graded case $(A = A^+)$ .See [9].

When $\alpha = -$, $(A = A^-)$ we have $|a_i| = 1$,then $\varepsilon = |a_n|(|a_0| + \cdots\cdots + |a_{n-1}|) = n$,and $\varepsilon + n = 2n$ .Since $(-1)^{\varepsilon+n} = (-1)^{2n} = 1$ as $2n$ is always even ,in (2.1), we have

$$b^n f\,(a_0 \otimes ...... \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i f\,(a_0 \otimes ... \otimes a_i a_{i+1} \otimes ... \otimes a_n) + f\,(a_n a_0 \otimes a_1 \otimes ... \otimes a_{n-1}).$$

2- When $\alpha = +$, $(A = A^+)$,we have $|a_i| = 0$,then $\varepsilon = |a_n|(|a_0| + \cdots\cdots + |a_{n-1}|) = 0$,

in (2.1), we have

$$t^n f\,(a_0 \otimes ... \otimes a_{n-1} \otimes a_n) = f\,(a_n \otimes a_0 \otimes ... \otimes a_{n-1}).$$

When $\alpha = -$, $(A = A^-)$ we have $|a_i| = 1$,then $\varepsilon = |a_n|(|a_0| + \cdots\cdots + |a_{n-1}|) = n$,

in (2.1), we have

$$t^n f\,(a_0 \otimes ... \otimes a_{n-1} \otimes a_n) = (-1)^n f\,(a_n \otimes a_0 \otimes ... \otimes a_{n-1}),$$

which is a trivially graded case .See [9].

2.1 Low-Dimensional Computations.

Example (2.1.1)[4]:

$H^0(A, A^*)$ is the space of all bounded graded traces, i.e.

$$H^{0}(A,A^{*})=\{ \text{ all } f \in Hom_{K}(A,K)|f =0 \text{ on } [A,A]_{gr} \},$$

which is the dual space of $H_{0}(A,A)=A/[A,A]_{gr}$.

Proof :

Consider the Hochschild complex :

$$0\longrightarrow C^{0}(A,A^{*})\xrightarrow{b_1}C^{1}(A,A^{*})\longrightarrow\cdots, \text{ i.e.}$$

$$0\longrightarrow Hom_{K}(A,K)\xrightarrow{b_1}Hom_{K}(A\otimes A,K)\longrightarrow\cdots .$$

$H^{0}(A,A^{*})=\ker(b_{1})$,where

$b_{1}f(a_{0}\otimes a_{1})=f(a_{0}a_{1})+(-1)^{1+|a_1||a_0|}f(a_{1}a_{0})=f(a_{0}a_{1}-(-1)^{|a_1||a_0|}a_{1}a_{0})=f([a_{0},a_{1}]_{gr})$.

A cocycle $f \in Hom_{K}(A,K)=A^{*}$ is inside $\ker(b_{1})$ if

$$0=b_{1}f(a_{0}\otimes a_{1})=f([a_{0},a_{1}]),$$

i.e. f vanishes on the subgroup $[A,A]_{gr}$,i.e. $f:A\to K$ is a bounded graded trace. Then

$$H^{0}(A,A^{*})=\ker(b_{1})=\{ \text{ all } f \in Hom_{K}(A,K)|f =0 \text{ on } [A,A]_{gr} \}$$

$$=C^{0}(A,A^{*})=Hom_{K}(A,K)$$

where, $C^{q}(A,A^{*})=Hom_{K}(A^{\otimes q},A^{*})=Hom_{K}(A^{\otimes(q+1)},K)$.

Remark :

Since $f:A\to K$ is a bounded graded trace, then $[A,A]_{gr}=0$, so , $H_{0}(A,A)=A$ and $Hom_{K}(H_{0}(A,A),K)=Hom_{K}(A,K)=A^{*}=H^{*}(A,A^{*})$.

More generally, Since K is a field, it is clear that $H^{n}(A,A^{*})$ is the dual space of $H_{n}(A,A)$.Hence $H^{*}(A,A^{*})=H_{*}(A,A)^{*}$.

Lemma (2.1.2) [5]:

For any unital $\mathbb{Z}/2$-graded Banach algebra A ,we show that $H^{1}(A,A^{*})=Der_{gr}(A,A^{*})/$ { Continuous graded inner derivations }, where $Der_{gr}(A,A^{*})$ is the K-module of all continuous graded derivations from $A\otimes A$ to K $(f \in Hom_{K}(A\otimes A,K)$.

Proof :

Consider the Hochschild complex :

$$0\longrightarrow Hom_{K}(A,K)\xrightarrow{b_1}Hom_{K}(A\otimes A,K)\xrightarrow{b_2}Hom_{K}(A\otimes A\otimes A,K)\longrightarrow\ldots .$$

We know that $H^{0}(A,A^{*})=\dfrac{\ker(b_{2})}{im(b_{1})}$, and $b_{1}f(a_{0}\otimes a_{1})=f([a_{0},a_{1}]_{gr})$.

The coboundaries in degree 1 are maps $f: A\otimes A \to K$,defined by $f(a_0\otimes a_1)\mapsto f([a_0,a_1]_{gr})$ or

$a_0\otimes a_1 \mapsto [a_0,a_1]_{gr} = a_0a_1-(-1)^{|a_1||a_0|}a_1a_0 \in K$.

These functions are continuous graded K -derivations which are called continuous graded inner derivations.
Then:

$im(b_1)=\{$ all $f: A\otimes A \to K : a_0\otimes a_1 \mapsto [a_0,a_1]_{gr}$ $\}=$

{Continuous graded inner derivations }.

$b_2 f(a_0\otimes a_1\otimes a_2) = f(a_0a_1\otimes a_2) - f(a_0\otimes a_1a_2) + (-1)^{|a_2|(|a_0|+|a_1|)} f(a_2a_0\otimes a_1)$, since f is a cocycle in degree 1 ,then $b_2 f(a_0\otimes a_1\otimes a_2)=0$, i.e.

$$f(a_0\otimes a_1a_2) = f(a_0a_1\otimes a_2) + (-1)^{|a_2|(|a_0|+|a_1|)} f(a_2a_0\otimes a_1) \qquad (2.3) .$$

Hence, $f: A\otimes A \to K$ is a continuous graded derivation .Thus we have

$\ker(b_2)=\{$ all $f \in Hom_K(A\otimes A, K): f$ is a continuous graded derivation $\}$

$= Der_{gr}(A\otimes A, K) = Der_{gr}(A, A^*)$, as

$$Der_{gr}(A,A^*) = Der_{gr}(A, Hom_K(A,K)) = Der_{gr}(A\otimes A, K),$$

thus we get

$H^1(A,A^*) = Der_{gr}(A,A^*)$ / { Continuous graded inner derivations } .

Note that all bounded linear maps ,or operators are continuous.

Remark [2]:

A $\mathbb{Z}/2$ -graded Banach algebra A is weakly amenable if $H^1(A,A^*)=\{0\}$.

For any unital $\mathbb{Z}/2$ -graded Banach algebra A over K .

We expect that : $HR^0(A)\cong HC^0(A)\cong HD^0(A) \cong H^0(A,A^*)$, the space of all bounded graded traces, i.e. $H^0(A,A^*)=\{$ all $f\in Hom_K(A,K) | f=0$ on $[A,A]_{gr}$ $\}$, which is the dual space of $H_0(A,A) = A/[A,A]_{gr}$.

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