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Riesz Triple Fuzzy Ideal of Almost Lacunary Cesàro C_{111} statistical convergence of χ^3 defined by a Musielak Orlicz function

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Abstract.

In this paper we introduce a new concept for Riesz Fuzzy ideal of almost lacunary Cesàro statistical convergence of χ^3 sequence spaces strong *P*- convergent to zero with respect to an Orlicz function and examine some properties of the resulting sequence spaces. We also introduce and study statistical convergence of Riesz Fuzzy ideal of almost lacunary Cesàro of χ^3 sequence spaces and also some inclusion theorems are discussed.

Keywords: Analytic sequence; Orlicz function; triple sequences; chi sequence; Riesz space; statistical convergence; Cesàro $C_{1,1,1}$ -statistical convergence.

Mathematics Subject Classification: 40F05, 40J05, 40G05.

1 Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write w^3 for the set of all complex triple sequences (x_{mnk}), where $m, n, k \in \mathbb{N}$, the set of positive integers. Then, w^3 is a linear space under the coordinate wise addition and scalar multiplication.

We can represent triple sequences by matrix. In case of double sequences we write in the form of a square. In the case of a triple sequence it will be in the form of a box in three dimensional case.

Some initial work on double series is found in Apostol [1] and double sequence spaces is found in Hardy [7], Mursaleen et al. [8-10], Subramanian et al. [11], Deepmala et al. [12,13] and many others. Later on investigated by some initial work on triple sequence spaces is found in Sahiner et al. [14], Esi et al. [2-5], Savas et al. [6], Subramanian et al. [15], Prakash et al. [16-17] and many others [18-22].

Let (x_{mnk}) be a triple sequence of real or complex numbers. Then the series $\sum_{m,n,k=1}^{\infty} x_{mnk}$ is called a triple series. The triple series $\sum_{m,n,k=1}^{\infty} x_{mnk}$ give one space is said to be convergent if and only if the triple sequence $(S_{mnk})$is convergent, where

$$S_{mnk} = \sum_{i,j,q=1}^{m,n,k} x_{ijq}(m, n, k = 1, 2, 3, ...) .$$

A sequence $x = (x_{mnk})$is said to be triple analytic if

$$sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty.$$

The vector space of all triple analytic sequences are usually denoted by Λ^3. A sequence $x = (x_{mnk})$ is called triple entire sequence if

$$|x_{mnk}|^{\frac{1}{m+n+k}} \to 0 \text{ as } m, n, k \to \infty.$$

The vector space of all triple entire sequences are usually denoted by Γ^3. Let the set of sequences with this property be denoted by Λ^3 and Γ^3 is a metric space with the metric

$$d(x, y) = sup_{m,n,k} \left\{ |x_{mnk} - y_{mnk}|^{\frac{1}{m+n+k}} : m, n, k : 1, 2, 3, ... \right\}, \tag{1.1}$$

forall $x = \{x_{mnk}\}$ and $y = \{y_{mnk}\}$ $in\,\Gamma^3$. Let $\phi = \{finite\ sequences\}$.

Consider a triple sequence $x = (x_{mnk})$. The $(m, n, k)^{th}$ section $x^{[m,n,k]}$ of the sequence is defined by $x^{[m,n,k]} = \sum_{i,j,q=0}^{m,n,k} x_{ijq} \Im_{ijq}$ for all $m, n, k \in \mathbb{N}$,

$$\Im_{ijq} = \begin{bmatrix} 0 & 0 & ...0 & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \\ . & & & & \\ . & & & & \\ . & & & & \\ 0 & 0 & ...1 & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \end{bmatrix}$$

with 1 in the $(i, j, q)^{th}$ position and zero otherwise.

A sequence $x = (x_{mnk})$ is called triple gai sequence if $((m + n + k)! |x_{mnk}|)^{\frac{1}{m+n+k}} \to 0$ as $m, n, k \to \infty$. The triple gai sequences will be denoted by χ^3.

2 Definitions and Preliminaries

A triple sequence $x = (x_{mnk})$ has limit 0 (denoted by $P - limx = 0$)
(i.e) $((m + n + k)! |x_{mnk}|)^{1/m+n+k} \to 0$ as $m, n, k \to \infty$. We shall write more briefly as $P - convergent\ to\ 0$.

2.1 Definition

A modulus function was introduced by Nakano [18]. We recall that a modulus f is a function from $[0, \infty) \to [0, \infty)$, such that
(1) $f(x) = 0$ if and only if $x = 0$
(2) $f(x + y) \leq f(x) + f(y)$, for all $x \geq 0, y \geq 0$,
(3) f is increasing,
(4) f is continuous from the right at 0. Since $|f(x) - f(y)| \leq f(|x - y|)$, it follows from here that f is continuous on $[0, \infty)$.

2.2 Definition

A triple sequence $x = (x_{mnk})$ of real numbers is called almost P- convergent to a limit 0 if

$$P - lim_{p,q,u\to\infty} sup_{r,s,t\geq 0} \frac{1}{pqu} \sum_{m=r}^{r+p-1} \sum_{n=s}^{s+q-1} \sum_{k=t}^{t+u-1} \left((m+n+k)! \, |x_{mnk}|\right)^{1/m+n+k} \to 0.$$

that is, the average value of (x_{mnk}) taken over any rectangle $\{(m,n,k) : r \leq m \leq r+p-1, s \leq n \leq s+q-1, t \leq k \leq t+u-1\}$ tends to 0 as both p, q and u to ∞, and this P- convergence is uniform in i, ℓ and j. Let denote the set of sequences with this property as $\left[\widehat{\chi^3}\right]$.

2.3 Definition

Let $(q_{rst}), (\overline{q_{rst}}), (\overline{\overline{q_{rst}}})$ be sequences of positive numbers and

$$Q_r = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1s} & 0\ldots \\ q_{21} & q_{22} & \cdots & q_{2s} & 0\ldots \\ . & & & & \\ . & & & & \\ . & & & & \\ q_{r1} & q_{r2} & \cdots & q_{rs} & 0\ldots \\ 0 & 0 & \ldots 0 & 0 & 0\ldots \end{bmatrix} = q_{11} + q_{12} + \ldots + q_{rs} \neq 0,$$

$$\overline{Q}_s = \begin{bmatrix} \overline{q}_{11} & \overline{q}_{12} & \cdots & \overline{q}_{1s} & 0\ldots \\ \overline{q}_{21} & \overline{q}_{22} & \cdots & \overline{q}_{2s} & 0\ldots \\ . & & & & \\ . & & & & \\ . & & & & \\ \overline{q}_{r1} & \overline{q}_{r2} & \ldots & \overline{q}_{rs} & 0\ldots \\ 0 & 0 & \ldots 0 & 0 & 0\ldots \end{bmatrix} = \overline{q}_{11} + \overline{q}_{12} + \ldots + \overline{q}_{rs} \neq 0,$$

$$\overline{\overline{Q}}_t = \begin{bmatrix} \overline{\overline{q}}_{11} & \overline{\overline{q}}_{12} & \cdots & \overline{\overline{q}}_{1s} & 0\ldots \\ \overline{\overline{q}}_{21} & \overline{\overline{q}}_{22} & \cdots & \overline{\overline{q}}_{2s} & 0\ldots \\ . & & & & \\ . & & & & \\ . & & & & \\ \overline{\overline{q}}_{r1} & \overline{\overline{q}}_{r2} & \ldots & \overline{\overline{q}}_{rs} & 0\ldots \\ 0 & 0 & \ldots 0 & 0 & 0\ldots \end{bmatrix} = \overline{q}_{11} + \overline{q}_{12} + \ldots + \overline{q}_{rs} \neq 0.$$ Then the transformation is given by

$T_{rst} = \frac{1}{Q_r \overline{Q}_s \overline{\overline{Q}}_t} \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} q_m \overline{q}_n \overline{\overline{q}}_k \left((m+n+k)! \, |x_{mnk}|\right)^{1/m+n+k}$ is called the Riesz mean of triple sequence $x = (x_{mnk})$. If $P - lim_{rst} T_{rst}(x) = 0, 0 \in \mathbb{R}$, then the sequence $x = (x_{mnk})$ is said to be Riesz convergent to 0. If $x = (x_{mnk})$ is Riesz convergent to 0, then we write $P_R - lim x = 0$.

2.4 Definition

The triple sequence $\theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\}$ is called triple lacunary if there exist three increasing sequences of integers such that

$$m_0 = 0, h_i = m_i - m_{r-1} \to \infty \text{ as } i \to \infty \text{ and}$$
$$n_0 = 0, \overline{h_\ell} = n_\ell - n_{\ell-1} \to \infty \text{ as } \ell \to \infty.$$
$$k_0 = 0, \overline{h_j} = k_j - k_{j-1} \to \infty \text{ as } j \to \infty.$$

Let $m_{i,\ell,j} = m_i n_\ell k_j, h_{i,\ell,j} = h_i \overline{h_\ell h_j}$, and $\theta_{i,\ell,j}$ is determine by $I_{i,\ell,j} = \{(m,n,k) : m_{i-1} < m < m_i \, and \, n_{\ell-1} < n \leq n_\ell \, and \, k_{j-1} < k \leq k_j\}, q_k = \frac{m_k}{m_{k-1}}, \overline{q_\ell} = \frac{n_\ell}{n_{\ell-1}}, \overline{q_j} = \frac{k_j}{k_{j-1}}$.

Using the notations of lacunary Fuzzy sequence and Riesz mean for triple sequences.
$\theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\}$ be a triple lacunary sequence and $q_m\overline{q}_n\overline{\overline{q}}_k$ be sequences of positive real numbers such that $Q_{m_i} = \sum_{m\in(0,m_i]} p_{m_i}, Q_{n_\ell} = \sum_{n\in(0,n_\ell]} p_{n_\ell}, Q_{n_j} = \sum_{k\in(0,k_j]} p_{k_j}$ and $H_i = \sum_{m\in(0,m_i]} p_{m_i}, \overline{H} = \sum_{n\in(0,n_\ell]} p_{n_\ell}, \overline{\overline{H}} = \sum_{k\in(0,k_j]} p_{k_j}$. Clearly, $H_i = Q_{m_i} - Q_{m_{i-1}}, \overline{H}_\ell = Q_{n_\ell} - Q_{n_{\ell-1}}, \overline{\overline{H}}_j = Q_{k_j} - Q_{k_{j-1}}$. If the Riesz transformation of triple sequences is RH-regular, and $H_i = Q_{m_i} - Q_{m_{i-1}} \to \infty$ as $i \to \infty, \overline{H} = \sum_{n\in(0,n_\ell]} p_{n_\ell} \to \infty$ as $\ell \to \infty, \overline{\overline{H}} = \sum_{k\in(0,k_j]} p_{k_j} \to \infty$ as $j \to \infty$, then $\theta'_{i,\ell,j} = \{(m_i, n_\ell, k_j)\} = \left\{\left(Q_{m_i} Q_{n_j} Q_{k_k}\right)\right\}$ is a triple lacunary sequence. If the assumptions $Q_r \to \infty$ as $r \to \infty$, $\overline{Q}_s \to \infty$ as $s \to \infty$ and $\overline{\overline{Q}}_t \to \infty$ as $t \to \infty$ may be not enough to obtain the conditions $H_i \to \infty$ as $i \to \infty, \overline{H}_\ell \to \infty$ as $\ell \to \infty$ and $\overline{\overline{H}}_j \to \infty$ as $j \to \infty$ respectively.
For any lacunary sequences $(m_i), (n_\ell)$ and (k_j) are integers.
Throughout the paper, we assume that $Q_r = q_{11} + q_{12} + \ldots + q_{rs} \to \infty\,(r \to \infty), \overline{Q}_s = \overline{q}_{11} + \overline{q}_{12} + \ldots + \overline{q}_{rs} \to \infty\,(s \to \infty), \overline{\overline{Q}}_t = \overline{\overline{q}}_{11} + \overline{\overline{q}}_{12} + \ldots + \overline{\overline{q}}_{rs} \to \infty\,(t \to \infty)$, such that $H_i = Q_{m_i} - Q_{m_{i-1}} \to \infty$ as $i \to \infty, \overline{H}_\ell = Q_{n_\ell} - Q_{n_{\ell-1}} \to \infty$ as $\ell \to \infty$ and $\overline{\overline{H}}_j = Q_{k_j} - Q_{k_{j-1}} \to \infty$ as $j \to \infty$.
Let $Q_{m_i,n_\ell,k_j} = Q_{m_i}\overline{Q}_{n_\ell}\overline{\overline{Q}}_{k_j}, H_{i\ell j} = H_i\overline{H}_\ell\overline{\overline{H}}_j$,
$I'_{i\ell j} = \left\{(m,n,k) : Q_{m_{i-1}} < m < Q_{m_i}, \overline{Q}_{n_{\ell-1}} < n < Q_{n_\ell}\, and\, \overline{Q}_{k_{j-1}} < k < \overline{Q}_{k_j}\right\}$,
$V_i = \frac{Q_{m_i}}{Q_{m_{i-1}}}, \overline{V}_\ell = \frac{Q_{n_\ell}}{Q_{n_{\ell-1}}}$ and $\overline{\overline{V}}_j = \frac{Q_{k_j}}{Q_{k_{j-1}}}$. and $V_{i\ell j} = V_i\overline{V}_\ell\overline{\overline{V}}_j$.
If we take $q_m = 1, \overline{q}_n = 1\, and\, \overline{\overline{q}}_k = 1$ for all m, n and k then $H_{i\ell j}, Q_{i\ell j}, V_{i\ell j}$ and $I'_{i\ell j}$ reduce to $h_{i\ell j}, q_{i\ell j}, v_{i\ell j}$ and $I_{i\ell j}$.
Let $n \in \mathbb{N}$ and X be a real vector space of dimension m, where $n \leq m$. A real valued function $d_p(x_1, \ldots, x_n) = \|(d_1(x_1), \ldots, d_n(x_n))\|_p$ on X satisfying the following four conditions:
(i) $\|(d_1(x_1), \ldots, d_n(x_n))\|_p = 0$ if and and only if $d_1(x_1), \ldots, d_n(x_n)$ are linearly dependent,
(ii) $\|(d_1(x_1), \ldots, d_n(x_n))\|_p$ is invariant under permutation,
(iii) $\|(\alpha d_1(x_1), \ldots, \alpha d_n(x_n))\|_p = |\alpha|\, \|(d_1(x_1), \ldots, d_n(x_n))\|_p, \alpha \in \mathbb{R}$
(iv) $d_p\left((x_1, y_1), (x_2, y_2) \cdots (x_n, y_n)\right) = \left(d_X(x_1, x_2, \cdots x_n)^p + d_Y(y_1, y_2, \cdots y_n)^p\right)^{1/p} for 1 \leq p < \infty$;
(or)
(v) $d\left((x_1, y_1), (x_2, y_2), \cdots (x_n, y_n)\right) := \sup\left\{d_X(x_1, x_2, \cdots x_n), d_Y(y_1, y_2, \cdots y_n)\right\}$,
for $x_1, x_2, \cdots x_n \in X, y_1, y_2, \cdots y_n \in Y$ is called the p product metric of the Cartesian product of n metric spaces is the p norm of the n-vector of the norms of the n subspaces.

A trivial example of p product metric of n metric space is the p norm space is $X = \mathbb{R}$ equipped with the following Euclidean metric in the product space is the p norm:

$$\|(d_1(x_1), \ldots, d_n(x_n))\|_E = sup\left(|det(d_{mn}(x_{mn}))|\right) = sup\left[\begin{vmatrix} d_{11}(x_{11}) & d_{12}(x_{12}) & \ldots & d_{1n}(x_{1n}) \\ d_{21}(x_{21}) & d_{22}(x_{22}) & \ldots & d_{2n}(x_{1n}) \\ . & & & \\ . & & & \\ . & & & \\ d_{n1}(x_{n1}) & d_{n2}(x_{n2}) & \ldots & d_{nn}(x_{nn}) \end{vmatrix}\right]$$

where $x_i = (x_{i1}, \cdots x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots n$.
If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the p- metric. Any complete p- metric space is said to be p- Banach metric space.

2.5 Definition

A family $I \subset 2^{Y \times Y \times Y}$ of subsets of a non empty set Y is said to be an ideal in Y if
(1) $\phi \in I$
(2) $A, B \in I$ imply $A \bigcup B \in I$

(3) $A \in I, B \subset A$ imply $B \in I$.

while an admissible ideal I of Y further satisfies $\{x\} \in I$ for each $x \in Y$. Given $I \subset 2^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$ be a non trivial ideal in $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$. A sequence $(x_{mn})_{m,n,k\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$ in X is said to be I- convergent to $0 \in X$, if for each $\epsilon > 0$ the set $A(\epsilon) = \{m, n \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \|(d_1(x_1), \ldots, d_n(x_n)) - 0\|_p \geq \epsilon\}$ belongs to I.

2.6 Definition

A non-empty family of sets $F \subset 2^{X\times X\times X}$ is a filter on X if and only if
(1) $\phi \in F$
(2) for each $A, B \in F$, we have imply $A \bigcap B \in F$
(3) each $A \in F$ and each $A \subset B$, we have $B \in F$.

2.7 Definition

An ideal I is called non-trivial ideal if $I \neq \phi$ and $X \notin I$. Clearly $I \subset 2^{X\times X\times X}$ is a non-trivial ideal if and only if $F = F(I) = \{X - A : A \in I\}$ is a filter on X.

2.8 Definition

A non-trivial ideal $I \subset 2^{X\times X\times X}$ is called (i) admissible if and only if $\{\{x\} : x \in X\} \subset I$. (ii) maximal if there cannot exists any non-trivial ideal $J \neq I$ containing I as a subset.

If we take $I = I_f = \{A \subseteq \mathbb{N}\times\mathbb{N}\times\mathbb{N} : A\; is\; a\; finite\; subset\}$. Then I_f is a non-trivial admissible ideal of $\mathbb{N}$ and the corresponding convergence coincides with the usual convergence. If we take $I = I_\delta = \{A \subseteq \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \delta(A) = 0\}$ where $\delta(A)$ denote the asyptotic density of the set A. Then I_δ is a non-trivial admissible ideal of $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ and the corresponding convergence coincides with the statistical convergence.

Let D denote the set of all closed and bounded intervals $X = [x_1, x_2, x_3]$ on the real line $\mathbb{R}\times\mathbb{N}\times\mathbb{N}$. For $X, Y, Z \in D$, we define $X \leq Y \leq Z$ if and only if $x_1 \leq y_1 \leq z_1$, $x_2 \leq y_2 \leq z_2$ and $x_3 \leq y_3 \leq z_3$, $d(X, Y) = max\{|x_1 - y_1 - z_1|, |x_2 - y_2 - z_2|\}$, where $X = [x_1, x_2, x_3]$ and $Y = [y_1, y_2, y_3]$.

Then it can be easily seen that d defines a metric on D and (D, d) is a complete metric space. Also the relation $\leq$ is a partial order on D. A fuzzy number X is a fuzzy subset of the real line $\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ i.e. a mapping $X : R \to J(= [0, 1])$ associating each real number t with its grade of membership $X(t)$.

2.9 Definition

A fuzzy number X is said to be (i) convex if $X(t) \geq X(s) \wedge X(r) = min\{X(s), X(r)\}$, where $s < t < r$. (ii) normal if there exists $t_0 \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}$ such that $X(t_0) = 1$. (iii) upper semi-continuous if for each $\epsilon > 0, X^{-1}([0, a + \epsilon])$ for all $a \in [0, 1]$ is open in the usual topology of $\mathbb{R}\times\mathbb{R}\times\mathbb{R}$.

Let $R(J)$ denote the set of all fuzzy numbers which are upper semicontinuous and have compact support, i.e. if $X \in \mathbb{R}(J)\times\mathbb{R}(J)\times\mathbb{R}(J)$ the for any $\alpha \in [0, 1], [X]^\alpha$ is compact, where $[X]^\alpha = \{t \in \mathbb{R}\times\mathbb{R}\times\mathbb{R} : X(t) \geq \alpha,\; if\; \alpha \in [0, 1]\}$, $[X]^0$ =closure of $(\{t \in \mathbb{R}\times\mathbb{R}\times\mathbb{R} : X(t) > \alpha, if \alpha = 0\})$.

The set $\mathbb{R}$ of real numbers can be embedded $\mathbb{R}(J)$ if we define $\bar{r} \in \mathbb{R}(J)\times\mathbb{R}(J)\times\mathbb{R}(J)$ by

$$\bar{r}(t) = \begin{cases} 1, & \text{if } t = r : \\ 0, & \text{if } t \neq r \end{cases}$$

The absolute value, $|X|$ of $X \in \mathbb{R}(J)$ is defined by

$$|X|(t) = \begin{cases} max\{X(t), X(-t)\}, & \text{if } t \geq 0; \\ 0, & \text{if } t < 0 \end{cases}$$

Define a mapping $\bar{d} : \mathbb{R}(J) \times \mathbb{R}(J) \times \mathbb{R}(J) \to \mathbb{R}^+ \cup \{0\}$by

$$\bar{d}(X, Y) = sup_{0 \leq \alpha \leq 1} d([X]^\alpha, [Y]^\alpha, [Z]^\alpha).$$

It is known that $\left(\mathbb{R}(J), \bar{d}\right)$ is a complete metric space.

2.10 Definition

A metric on $\mathbb{R}(J)$ is said to be translation invariant if $\bar{d}(X+Y, Y+Z) = \bar{d}(X, Z)$, for $X, Y, Z \in \mathbb{R}(J)$.

2.11 Definition

A sequence $X = (X_{mnk})$ of fuzzy numbers is said to be convergent to a fuzzy number X_0 if for every $\epsilon > 0$, there exists a positive integer n_0 such that $\bar{d}(X_{mnk}, X_0) < \epsilon$ for all $m, n, k \geq n_0$.

2.12 Definition

A sequence $X = (X_{mnk})$ of fuzzy numbers is said to be (i) I-convergent to a fuzzy number X_0 if for each $\epsilon > 0$ such that

$$A = \left\{m, n, k \in \mathbb{N} : \bar{d}(X_{mnk}, X_0) \geq \epsilon\right\} \in I.$$

The fuzzy number X_0 is called I-limit of the sequence (X_{mnk}) of fuzzy numbers and we write $I - lim X_{mnk} = X_0$. (ii) I-bounded if there exists $M > 0$ such that

$$\{m, n, k \in \mathbb{N} : d(X_{mnk}, \bar{0}) > M\} \in I.$$

2.13 Definition

Let d be a mapping from $R(I) \times R(I) \times R(I)$ into $R^*(I) \times R^*(I) \times R^*(I)$ and let the mappings $L, f : [0,1] \times [0,1] \times [0,1] \to [0,1] \times [0,1] \times [0,1]$ be symmetric, non-decreasing Musielak Orlicz in both arguments and satisfy $L \times L \times L(0,0,0) = 0$ and $f \times f \times f(1,1,1) = 1$. Denote $[d(X,Y,Z)]_\alpha = [\lambda_\alpha(X,Y,Z), (X,Y,Z)]$, for $X, Y \in R(I) \times R(I) \times R(I)$ and $0 < \alpha < 1$.
The $(R(I) \times R(I) \times R(I), d, L \times L \times L, f \times f \times f)$ is called a fuzzy p- metric space and d a fuzzy translation metric, if
(1) $d(X, Y) = \bar{0}$ if and only if $X = Y = Z$,
(2) $d(X, Y) = d(Y, Z)\, d(Z, X)$ for all $X, Y, Z \in X$,
(3) for all $X, Y, Z \in R(I) \times R(I) \times R(I)$,

(i) $d(X,Y,Z)(s+t+u) \geq L{\times}L{\times}L\,(d(X,Y)(s), d(Y,Z)(t), d(Z,X)(u))$ whenever $s \leq \lambda_1(X,Y), t \leq \lambda_1(Y,Z), u \leq \lambda_1(Z,X)$ and $(s+t+u) \leq \lambda_1(X,Y,Z)$,
(ii) $d(X,Y)(s+t+u) \leq f{\times}f{\times}f\,(d(X,Y)(s), d(Y,Z)(t), d(Z,X)(u))$ whenever $s \geq \lambda_1(X,Y), t \geq \lambda_1(Y,Z), u \geq \lambda_1(Z,X)$ and $(s+t+u) \leq \lambda_1(X,Y,Z)$. The following well-known inequality will be used throughout the article. Let $p = (p_{mnk})$ be any sequence of positive real numbers with $0 \leq p_{mnk} \leq sup_{mnk} p_{mnk} = G, D = max\{1, 2G-1\}$ then

$$|a_{mnk} + b_{mnk}|^{p_{mnk}} \leq D\left(|a_{mnk}|^{p_{mnk}} + |b_{mnk}|^{p_{mnk}}\right) \text{ for all } m,n,k \in \mathbb{N} \text{ and } a_{mnk}, b_{mnk} \in \mathbb{C}.$$

Also $|a_{mnk}|^{p_{mnk}} \leq max\left\{1, |a|^G\right\}$ for all $a \in \mathbb{C}$.

3 Fuzzy Ideal Lacunary CeAro C_{111}- statistical convergence of triple sequences

Let $A = \left[a_{mnk}^{pqr}\right]_{m,n,k=0}^{\infty}$ be a triple infinite matrix of real number for $p, q, r = 1, 2, \cdots$ forming the sum

$$\mu_{pqr}(X) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_{mnk}^{pqr}\left(((m+n+k)!X_{mnk})^{1/m+n+k}, \bar{0}\right) \tag{3.1}$$

Called the A means of the triple sequence X yielded a method of summability. We say that a sequence X is A summable to the limit 0 of the A mean exist for all $p, q, r = 0, 1, \cdots$ and converges.

$$lim_{uvw\to\infty} \sum_m^u \sum_n^v \sum_k^w a_{mnk}^{pqr}\left((m+n+k)!X_{mnk}\right)^{1/m+n+k} = \mu_{pqr}$$

and

$$\lim_{pqr\to\infty} \mu_{pqr} = 0$$

Define the means

$$\sigma_{pqr}^X = \frac{1}{pqr}\sum_{m=0}^p \sum_{n=0}^q \sum_{k=0}^r \left((m+n+k)!X_{mnk}\right)^{1/m+n+k}$$

and

$$A\sigma_{pqr}^X = \frac{1}{pqr}\sum_{m=0}^p \sum_{n=0}^q \sum_{k=0}^r a_{mnk}^{pqr}\left(((m+n+k)!X_{mnk})^{1/m+n+k}, \bar{0}\right).$$

We say that $X = (X_{mnk})$ is statistical summable $(C, 1, 1, 1)$ to 0, if the sequence $\sigma = \left(\sigma_{mnk}^X\right)$ is Fuzzy statistically convergent to $\bar{0}$, that is, $st_3 - lim_{pqr}\sigma_{pqr}^X = 0$. It is denoted by $C_{111}(st_3)$, the set of all triple sequence which one statistically summable $(C, 1, 1, 1)$.

The main aim of this article to introduce the following lacunary fuzzy sequence spaces and examine topological and algebraic properties of the resulting sequence spaces. Let $f = (f_{mnk})$ be a Musielak-Orlicz function, $\left(X, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)$ be a fuzzy $p-$metric space, and $A\sigma_{pqr}(X) \to \bar{0}$ and as $m, n, k \to \infty$ and $A\sigma_{pqr}(X) = sup_{mnk} a_{mnk}^{pqr}\left(X_{mnk}^{1/m+n+k}, \bar{0}\right) < \infty$, be a sequence of fuzzy numbers. Using the concept of fuzzy metric, we introduce the following class of sequences:

Let $q_m, \overline{q}_n$ and $\overline{\overline{q}}_k$ be sequences of positive numbers and $Q_r = q_{11} + \cdots q_{rs}$, $\overline{Q}_s = \overline{q}_{11}\cdots\overline{q}_{rs}$ and $\overline{\overline{Q}}_t = \overline{\overline{q}}_{11}\cdots\overline{\overline{q}}_{rs}$,

If we choose $q_m = 1, \overline{q}_n = 1$ and $\overline{\overline{q}}_k = 1$ for all m, n and k, then we obtain the following lacunary fuzzy sequence spaces.

$\left[\chi_{q,R_f}^{3I(F)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right] =$
$P-lim_{r,s,t\to\infty} \frac{1}{Q_r\overline{Q}_s\overline{\overline{Q}}_t}\sum_{m=1}^r \sum_{n=1}^s \sum_{k=1}^t q_m\overline{q}_n\overline{\overline{q}}_k \left[f\left(A\sigma_{pqr}^X, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right] = 0 \in$
I, uniformly in i, ℓ and j.

$\left[\Lambda_{q,R_f}^{3I(F)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right] =$
$P - sup_{r,s,t} \frac{1}{Q_r\overline{Q}_s\overline{\overline{Q}}_t}\sum_{m=1}^r \sum_{n=1}^s \sum_{k=1}^t q_m\overline{q}_n\overline{\overline{q}}_k \left[f\left(A\sigma_{pqr}^X, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right] < \infty \in$
I, uniformly in i, ℓ and j.

3.1 Definition

Let f be an Musielak Orlicz function and
$\theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\}$ be a triple lacunary fuzzy sequence $\left[\chi_{q,\theta C_f}^{3I(F)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right] =$

$P-lim_{i,\ell,j}\frac{1}{h_{i\ell j}}\sum_{m\in I_{i,\ell,j}}\sum_{n\in I_{i,\ell,j}}\sum_{k\in I_{i,\ell,j}}\left[f\left(A\sigma^{X}_{pqr},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]=0\in I,$ uniformly in i,ℓ and j.

We shall denote $\left[\chi^{3I(F)}_{q,\theta C_f},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right]$ as $\left[\chi^{3I(F)}_{q,\theta C},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right]$ respectively when $p_{mnk}=1$ for all m,n and k If X is in $\left[\chi^{3I(F)}_{q,\theta C},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right]$, we shall say that X is Fuzzy almost lacunary χ^3 strongly $p-$convergent with respect to the Musielak Orlicz function f. Also note if $f(X)=X$ for all m,n and k then $\left[\chi^{3I(F)}_{q,\theta C_f},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right]=\left[\chi^{3I(F)}_{q,\theta C},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right]$ which are defined as follows:

$\left[\chi^{3I(F)}_{q,\theta C},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right]=$

$P-lim_{i,\ell,j}\frac{1}{h_{i\ell j}}\sum_{m\in I_{i,\ell,j}}\sum_{n\in I_{i,\ell,j}}\sum_{k\in I_{i,\ell,j}}\left[f\left(A\sigma^{X}_{pqr},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]=0\in I,$ uniformly in i,ℓ and j.

3.2 Definition

Let f be an Musielak Orlicz function, we define the following sequence space:

$\left[\chi^{3I(F)}_{f},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right]=$

$P-lim_{r,s,t\to\infty}\sum_{m=1}^{r}\sum_{n=1}^{s}\sum_{k=1}^{t}q_m\left[f\left(A\sigma^{X}_{pqr},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]=0\in I,$ uniformly in i,ℓ and j.

If we take $f(x)=x$ for all m,n and k then $\left[\chi^{3I(F)}_{f},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right]=\left[\chi^{3I(F)},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right].$

3.3 Definition

Let $\theta_{i,\ell,j}$ be a triple lacunary fuzzy sequence; the triple sequence X is $C_{111}\left(\widehat{S_{\theta i,\ell,j}}\right)-p$- convergent to 0 then

$P-lim_{i,\ell,j}\frac{1}{h_{i,\ell,j}}max_{i,\ell,j}$

$\left|\left\{(m,n,k)\in I_{i,\ell,j}:\left[f\left(A\sigma^{X}_{pqr},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]=0\in I\right\}\right|=0.$

In this case we write $C_{111}\left(\widehat{S_{\theta i,\ell,j}}\right)-lim\left(f\left(m+n+k\right)!\left|x_{m+i,n+\ell,k+j}-0\right|\right)^{1/m+n+k}=0.$

4 Main Results

4.1 Theorem

Let f be an Musielak Orlicz function and $\theta_{i,\ell,j}=\{(m_i,n_\ell,k_j)\}$ be a triple lacunary sequence $\left[\chi^{3I(F)}_{q,\theta C_f},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right]$ is linear space

Proof: The proof is easy. Theorefore omit the proof.

4.2 Theorem

For any Musielak Orlicz function f, we have

$\left[\chi^{3I(F)}_{q,\theta C},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right]\subset\left[\chi^{3I(F)}_{q,\theta C_f},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right]$

Proof: Let $x\in\left[\chi^{3I(F)}_{q,\theta C},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right]$

$\left[\chi^{3I(F)}_{q,\theta C},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right]=$

$P-lim_{i,\ell,j}\frac{1}{pqr,h_{i\ell j}}\sum_{m=0,m\in I_{i,\ell,j}}^{p}\sum_{n=0,n\in I_{i,\ell,j}}^{q}\sum_{k=0,k\in I_{i,\ell,j}}^{r}$
$\left[\left(A\sigma_{pqr}^{X},\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right)\right]=0\in I$, uniformly in i,ℓ and j.
Since f is continuous at zero, for $\varepsilon>0$ and choose δ with $0<\delta<1$ such that $f(t)<\epsilon$ for every t with $0\le t\le\delta$. We obtain the following,
$\frac{1}{h_{i\ell j}}(h_{i\ell j}\epsilon)+\frac{1}{pqr,h_{i\ell j}}\sum_{m=0,m\in I_{i,\ell,j}}^{p}\sum_{n=0,n\in I_{i,\ell,j}}^{q}\sum_{k=0,k\in I_{i,\ell,j}\ and\ |x_{m+i,n+\ell,k+j}-0|>\delta}^{r}$
$\left[f\left(A\sigma_{pqr}^{X},\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right)\right]$
$\frac{1}{h_{i\ell j}}(h_{i\ell j}\epsilon)+\frac{pqr}{h_{i\ell j}}K\delta^{-1}a_{mnk}^{pqr}f(2)h_{i\ell j}\left[\chi_{q,\theta C}^{3I(F)},\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right]$

Hence i,ℓ and j goes to infinity, we are granted $x\in\left[\chi_{q,\theta C_f}^{3I(F)},\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right]$

4.3 Theorem

Let $\theta_{i,\ell,j}=\{m_i,n_\ell,k_j\}$ be a triple lacunary fuzzy sequence with $liminf_i q_i>1$, $liminf_\ell\overline{q_\ell}>1$ *and* $liminf_j q_j>1$ then for any Musielak Orlicz function f,
$\left[\chi^{3I(F)},\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right]\subset\left[\chi_{q,\theta C_f}^{3I(F)},\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right]$

Proof: Suppose $liminf_i q_i>1$, $liminf_\ell\overline{q_\ell}>1$ and $liminf_j q_j>1$ then there exists $\delta>0$ such that $q_i>1+\delta$, $\overline{q_\ell}>1+\delta$ and $q_j>1+\delta$. This implies $\frac{h_i}{m_i}\ge\frac{\delta}{1+\delta}$, $\frac{h_\ell}{n_\ell}\ge\frac{\delta}{1+\delta}$ and $\frac{h_j}{k_j}\ge\frac{\delta}{1+\delta}$ Then for $x\in\sigma^{\chi_f^3}(P)$, we can write for each r,s and u.
$B_{i,\ell,j}=\frac{1}{pqr,h_{i\ell j}}\sum_{m=0,m\in I_{i,\ell,j}}^{p}\sum_{n=0,n\in I_{i,\ell,j}}^{q}\sum_{k=0,k\in I_{i,\ell,j}}^{r}\left[f\left(A\sigma_{pqr}^{X},\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right)\right]$
$=\frac{1}{pqr,h_{i\ell j}}\sum_{m=0}^{p_i}\sum_{n=0}^{q_\ell}\sum_{k=0}^{r_j}\left[f\left(A\sigma_{pqr}^{X},\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right)\right]-$
$\frac{1}{pqr,h_{i\ell j}}\sum_{m=0}^{p_{i-1}}\sum_{n=0}^{q_{\ell-1}}\sum_{k=0}^{r_{i-1}}\left[f\left(A\sigma_{pqr}^{X},\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right)\right]-$
$\frac{1}{pqr,h_{i\ell j}}\sum_{m=m_i+0}^{p_i}\sum_{n=0}^{q_{\ell-1}}\sum_{k=0}^{r_{j-1}}\left[f\left(A\sigma_{pqr}^{X},\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right)\right]-$
$\frac{1}{pqr,h_{i\ell j}}\sum_{k=k_j+0}^{r_j}\sum_{n=n_{\ell-0}+0}^{q_\ell}\sum_{m=1}^{p_{k-1}}\left[f\left(A\sigma_{pqr}^{X},\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right)\right]=$
$\frac{m_in_\ell k_j}{pqr,h_{i\ell j}}\left(\frac{1}{m_in_\ell k_j}\sum_{m=0}^{p_i}\sum_{n=0}^{q_\ell}\sum_{k=0}^{r_j}\left[f\left(A\sigma_{pqr}^{X},\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right)\right]\right)-$
$\frac{m_{k-1}n_{\ell-1}k_{j-1}}{pqr,h_{i\ell j}}$
$\left(\frac{1}{m_{i-1}n_{\ell-1}k_{j-1}}\sum_{m=0}^{p_{i-0}}\sum_{n=0}^{q_{\ell-0}}\sum_{k=0}^{r_{j-0}}\left[f\left(A\sigma_{pqr}^{X},\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right)\right]\right)-$
$\frac{k_{j-1}}{pqr,h_{i\ell j}}\left(\frac{1}{k_{j-1}}\sum_{m=m_{i-0}+0}^{p_i}\sum_{n=0}^{q_{\ell-0}}\sum_{k=0}^{r_j}\left[f\left(A\sigma_{pqr}^{X},\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right)\right]\right)-\frac{n_{\ell-1}}{pqr,h_{i\ell j}}\left(\frac{1}{n_{\ell-1}}\sum_{m=m_{k-0}+0}^{p_k}\right.$
$\frac{m_{k-1}}{pqr,h_{i\ell j}}\left(\frac{1}{m_{k-1}}\sum_{k=0}^{r_j}\sum_{n=n_{\ell-0}+0}^{q_\ell}\sum_{m=0}^{p_{k-0}}\left[f\left(A\sigma_{pqr}^{X},\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right)\right]\right).$
Since $x\in\left[\chi^{3I(F)},\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right]$ the last three terms tend to zero uniformly in m,n,k in the sense, thus, for each r,s and u
$B_{i,\ell,j}=\frac{m_in_\ell k_j}{pqr,h_{i\ell j}}\left(\frac{1}{m_in_\ell k_j}\sum_{m=0}^{p_i}\sum_{n=0}^{q_\ell}\sum_{k=0}^{r_j}\left[f\left(A\sigma_{pqr}^{X},\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right)\right]\right)-$
$\frac{m_{i-1}n_{\ell-1}k_{j-1}}{pqr,h_{i\ell j}}\left(\frac{1}{m_{i-1}n_{\ell-1}k_{j-1}}\sum_{m=0}^{p_{i-0}}\sum_{n=0}^{q_{\ell-0}}\sum_{k=0}^{r_{j-0}}\left[f\left(A\sigma_{pqr}^{X},\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right)\right]\right)+O(1).$
Since $h_{i\ell j}=m_in_\ell k_j-m_{i-1}n_{\ell-1}k_{j-1}$ we are granted for each i,ℓ and j the following

$$\frac{m_in_\ell kj}{pqr,h_{i\ell j}}\le\frac{1+\delta}{\delta}\text{ and }\frac{m_{i-0}n_{\ell-0}k_{j-0}}{pqr,h_{i\ell j}}\le\frac{1}{\delta}.$$

The terms
$\left(\frac{1}{m_in_\ell k_j}\sum_{m=0}^{p_i}\sum_{n=0}^{q_\ell}\sum_{k=0}^{r_j}\left[f\left(A\sigma_{pqr}^{X},\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right)\right]\right)$ and
$\left(\frac{1}{m_{i-1}n_{\ell-1}k_{j-1}}\sum_{m=0}^{p_{i-0}}\sum_{n=0}^{q_{\ell-0}}\sum_{k=0}^{r_{j-0}}\left[f\left(A\sigma_{pqr}^{X},\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right)\right]\right)$ are both gai sequences for all i,ℓ and j. Thus $B_{i\ell j}$ is a gai sequence for each i,ℓ and j.

Hence $x\in\left[\chi_{q,\theta C_f}^{3I(F)},\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right].$

4.4 Theorem

Let $\theta_{i,\ell,j} = \{m, n, k\}$ be a triple lacunary fuzzy sequence with $limsup_\eta q_\eta < \infty$ and $limsup_i \overline{q_i} < \infty$ then for any Musielak Orlicz function f,
$\left[\chi^{3I(F)}_{q,\theta C_f}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right] \subset \left[\chi^{3I(F)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]$.

Proof: Since $limsup_i q_i < \infty$ and $limsup_i \overline{q_i} < \infty$ there exists $H > 0$ such that $q_i < H$, $\overline{q_\ell} < H$ and $q_j < H$ for all i, ℓ and j. Let $x \in \left[\chi^{3I(F)}_{q,\theta C_f}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]$. Also there exist $i_0 > 0, \ell_0 > 0$ and $j_0 > 0$ such that for every $a \geq i_0$ $b \geq \ell_0$ and $c \geq j_0$ and i, ℓ and j.

$$A'_{abc} = \frac{1}{pqr, h_{abc}} \sum_{m=0, m\in I_{a,b,c}}^{p} \sum_{n=0, n\in I_{a,b,c}}^{q} \sum_{k=0, k\in I_{a,b,c}}^{r} \left[f\left(A\sigma^X_{pqr}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right] \to 0\, as\, m, n, k \to \infty.$$

Let $G' = max\left\{A'_{a,b,c} : 1 \leq a \leq i_0,\ \ 1 \leq b \leq \ell_0\ \ and\ \ 1 \leq c \leq j_0\right\}$ and p, q and r be such that $m_{i-1} < p \leq m_i$, $n_{\ell-1} < q \leq n_\ell$ and $m_{j-1} < r \leq m_j$. Thus we obtain the following:

$$\frac{1}{pqr} \sum_{m=0}^{p} \sum_{n=0}^{q} \sum_{k=1}^{r} \left[f\left(A\sigma^X_{pqr}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]$$
$$\leq \frac{1}{pqr, m_{i-1}n_{\ell-1}k_{j-1}} \sum_{m=0}^{p_i} \sum_{n=0}^{q_\ell} \sum_{k=0}^{r_j} \left[f\left(A\sigma^X_{pqr}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]$$
$$\leq \frac{1}{pqr, m_{i-1}n_{\ell-1}k_{j-1}} \sum_{a=1}^{i} \sum_{b=1}^{\ell} \sum_{c=1}^{j}$$
$$\left(\sum_{m=0, m\in I_{a,b,c}}^{p} \sum_{n=0, n\in I_{a,b,c}}^{q} \sum_{k=0, k\in I_{a,b,c}}^{r} \left[f\left(A\sigma^X_{pqr}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]\right)$$
$$= \frac{1}{pqr, m_{i-1}n_{\ell-1}k_{j-1}} \sum_{a=1}^{i_0} \sum_{b=1}^{\ell_0} \sum_{c=1}^{j_0} h_{a,b,c} A'_{a,b,c} + \frac{1}{m_{k-1}n_{\ell-1}k_{j-1}} \sum_{(i_0<a\leq i)\bigcup(\ell_0<b\leq\ell)\bigcup(j_0<c\leq j)} h_{a,b,c} A'_{a,b,c}$$
$$\leq \frac{G'}{pqr, m_{i-1}n_{\ell-1}k_{j-1}} \sum_{a=1}^{i_0} \sum_{b=1}^{\ell_0} \sum_{c=1}^{j_0} h_{a,b,c} + \frac{1}{pqr, m_{i-1}n_{\ell-1}k_{j-1}} \sum_{(i_0<a\leq i)\bigcup(\ell_0<b\leq\ell)\bigcup(j_0<c\leq J)} h_{a,b,c} A'_{a,b,c}$$
$$\leq \frac{G' m_{i_0} n_{\ell_0} k_{j_0} i_0 \ell_0 j_0}{pqr, m_{i-1}n_{\ell-1}k_{j-1}} + \frac{1}{pqr, m_{i-1}n_{\ell-1}j_{j-1}} \sum_{(i_0<a\leq i)\bigcup(\ell_0<b\leq\ell)\bigcup(j_0<c\leq j)} h_{a,b,c} A'_{a,b,c}$$
$$\leq \frac{G' m_{i_0} n_{\ell_0} k_{j_0} i_0 \ell_0 j_0}{pqr, m_{i-1}n_{\ell-1}k_{j-1}} + \left(sup_{a\geq i_0 \bigcup b\geq \ell_0 \bigcup c\geq j_0} A'_{a,b,c}\right) \frac{1}{pqr, m_{i-1}n_{\ell-1}k_{j-1}} \sum_{(i_0<a\leq i)\bigcup(\ell_0<b\leq\ell)\bigcup(j_0<c\leq j)} h_{a,b,c}$$
$$\leq \frac{G' m_{i_0} n_{\ell_0} k_{j_0} i_0 \ell_0 j_0}{pqr, m_{i-1}n_{\ell-1}k_{j-1}} + \frac{\epsilon}{pqr, m_{i-1}n_{\ell-1}k_{j-1}} \sum_{(i_0<a\leq i)\bigcup(\ell_0<b\leq\ell)\bigcup(j_0<c\leq j)} h_{a,b,c}$$
$$\leq \frac{G' m_{i_0} n_{\ell_0} k_{j_0} i_0 \ell_0 j_0}{pqr, m_{i-1}n_{\ell-1}k_{j-1}} + \epsilon H^3.$$

Since m_i, n_ℓ and k_j both approaches infinity as both p, q and r approaches infinity, it follows that

$$\frac{1}{pqr} \sum_{m=0}^{p} \sum_{n=0}^{q} \sum_{k=0}^{r} \left[\left(A\sigma^X_{pqr}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right] = 0,\ uniformly\, in\, i, \ell\, and\, j.$$

Hence $x \in \left[\chi^{3I(F)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]$.

4.5 Theorem

Let $\theta_{i,\ell,j}$ be a triple lacunary fuzzy sequence then
(i) $(x_{mnk}) \xrightarrow{P} \left[\chi^{3I(F)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right] C_{111}\left(\widehat{S_{\theta_{i,\ell,j}}}\right)$
(ii) $\left[\left(A\sigma^X_{pqr}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]$ is a proper subset of $C_{111}\left(\widehat{S_{\theta_{i,\ell,j}}}\right)$
(iii) If $x \in \Lambda^3$ and $(x_{mnk}) \xrightarrow{P} \left[\chi^{3I(F)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right] C_{111}\left(\widehat{S_{\theta_{i,\ell,j}}}\right)$ then $(x_{mnk}) \xrightarrow{P} \left[\chi^{3I(F)}_{q,\theta C_f}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]$
(iv) $\left[\chi^{3I(F)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right] C_{111}\left(\widehat{S_{\theta_{i,\ell,j}}}\right) \bigcap \Lambda^3 = \left[\chi^{3I(F)}_{q,\theta C_f}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right] \bigcap \Lambda^3$.

Proof: **(i)** Since for all i, ℓ and j

$\left|\left\{(m,n,k)\in I_{i,\ell,j}:\left[\left(A\sigma^X_{pqr},\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right)\right]\right\}=0\right|\leq$
$\sum_{m=0,m\in I_{i,\ell,j}}^{p}\sum_{n=0,n\in I_{i,\ell,j}}^{q}\sum_{k=0,k\in I_{i,\ell,j}\ and\ |x_{m+i,n+\ell,k+j}|=0}^{r}\left[\left(A\sigma^X_{pqr},\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right)\right]$
$\leq\sum_{m=0,m\in I_{i,\ell,j}}^{p}\sum_{n=0,n\in I_{i,\ell,j}}^{q}\sum_{k=0,k\in I_{i,\ell,j}}^{r}\left[\left(A\sigma^X_{pqr},\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right)\right]$, for all i,ℓ and j
$P-lim_{i,\ell,j}\frac{1}{pqr,h_{i\ell j}}\sum_{m=0,m\in I_{i,\ell,j}}^{p}\sum_{n=0,n\in I_{i,\ell,j}}^{q}\sum_{k=0,k\in I_{i,\ell,j}}^{r}$
$\left[\left(A\sigma^X_{pqr},\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right)\right]=0$
This implies that for all i,ℓ and j
$P-lim_{i,\ell,j}\frac{1}{pqr,h_{i,\ell,j}}\left|\left\{(m,n,k)\in I_{i,\ell,j}:\left[\left(A\sigma^X_{pqr},\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right)\right]\right\}\right|=0.$
(ii)let $x=(x_{mnk})$ be defined as follows:

$$(x_{mnk})=\begin{bmatrix}1 & 2 & 3 & \ldots\frac{\left[\sqrt[4]{pqr,h_{i,\ell,j}}\right]^{m+n+k}}{(m+n+k)!} & 0 & \ldots\\ 1 & 2 & 3 & \ldots\frac{\left[\sqrt[4]{pqr,h_{i,\ell,j}}\right]^{m+n+k}}{(m+n+k)!} & 0 & \ldots\\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 1 & 2 & 3 & \ldots\frac{\left[\sqrt[4]{pqr,h_{i,\ell,j}}\right]^{m+n+k}}{(m+n+k)!} & 0 & \ldots\\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 0 & 0 & 0 & \ldots 0 & 0 & \ldots\\ . & & & & & \\ . & & & & & \\ . & & & & & \end{bmatrix};$$

Here x is an trible sequence and for all i,ℓ and j
$P-lim_{i,\ell,j}\frac{1}{pqr,h_{k,\ell,j}}\left|\left\{(m,n,k)\in I_{i,\ell,j}:\left[\left(A\sigma^X_{pqr},\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right)\right]=0\right\}\right|=$
$P-lim_{i,\ell,j}\frac{1}{pqr,h_{i,\ell,j}}\left(\frac{(m+n+k)!\left[\sqrt[4]{pqr,h_{i,\ell,j}}\right]^{m+n+k}}{(m+n+k)!}\right)^{1/m+n+k}=0.$
Therefore $(x_{mnk})\xrightarrow{P}\left[\chi^{3I(F)},\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right]C_{111}\left(\widehat{S_{\theta_{i,\ell,j}}}\right)$. Also
$P-lim_{i,\ell,j}\frac{1}{pqr,h_{i\ell j}}\sum_{m=0,m\in I_{i,\ell,j}}^{p}\sum_{n=0,n\in I_{i,\ell,j}}^{q}\sum_{k=0,k\in I_{i,\ell,j}}^{r}$
$\left[\left(A\sigma^X_{pqr},\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right)\right]=$
$P-\frac{1}{2}\left(lim_{i,\ell,j}\frac{1}{pqr,h_{i,\ell,j}}\left(\frac{(m+n+k)!\left[\sqrt[4]{pqr,h_{i,\ell,j}}\right]^{m+n+k}\left[\sqrt[4]{pqr,h_{i,\ell,j}}\right]^{m+n+k}\left[\sqrt[4]{pqr,h_{i,\ell,j}}\right]^{m+n+k}}{(m+n+k)!}\right)^{1/m+n+k}+1\right)=$
$\frac{1}{2}$.
Therefore $(x_{mnk})\overset{P}{\nrightarrow}\left[\chi^{3I(F)}_{q,\theta C_f},\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right]$.
(iii) If $x\in\Lambda^3$ and $(x_{mnk})\xrightarrow{P}\left[\chi^{3I(F)},\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right]C_{111}\left(\widehat{S_{\theta_{i,\ell,j}}}\right)$ then $(x_{mnk})\xrightarrow{P}$ $\left[\chi^{3I(F)}_{q,\theta C_f},\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right]$.
Suppose $x\in\Lambda^3$ then for all i,ℓ and j, $\left[\left(A\sigma^X_{pqr},\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right)\right]\leq M$ for all m,n,k. Also for given $\epsilon>0$ and i,ℓ and j large for all i,ℓ and j we obtain the following:
$\frac{1}{pqr,h_{i\ell j}}\sum_{m=0,m\in I_{i,\ell,j}}^{p}\sum_{n=0,n\in I_{i,\ell,j}}^{q}\sum_{k=0,k\in I_{i,\ell,j}}^{r}\left[\left(A\sigma^X_{pqr},\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right)\right]=$
$\frac{1}{pqr,h_{i\ell j}}\sum_{m=0,m\in I_{i,\ell,j}}^{p}\sum_{n=0,n\in I_{i,\ell,j}}^{q}\sum_{k=0,k\in I_{i,\ell,j}\ and\ |x_{m+i,n+\ell,k+j}|\geq 0}^{r}$

$\left[\left(A\sigma^{X}_{pqr}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right] +$
$\frac{1}{pqr,h_{i\ell j}} \sum_{m=0,m\in I_{i,\ell,j}}^{p} \sum_{n=0,n\in I_{i,\ell,j}}^{q} \sum_{k\in I_{i,\ell,j}\ and\ |x_{m+i,n+\ell,k+j}|\leq 0}^{r}$
$\left[\left(A\sigma^{X}_{pqr}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]$
$\leq \frac{M}{pqr,h_{i\ell j}} \left|\left\{(m,n,k) \in I_{i,\ell,j} : \left[\left(A\sigma^{X}_{pqr}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]\right\} = 0\right| + \epsilon.$
Therefore $x \in \Lambda^3$ then $(x_{mnk}) \xrightarrow{P} \left[\chi^{3I(F)}_{q,\theta C_f}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]$.
(iv) $\left[\chi^{3I(F)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right] C_{111}\left(\widehat{S_{\theta_{i,\ell,j}}}\right) \bigcap \Lambda^3 =$
$\left[\chi^{3I(F)}_{q,\theta C_f}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right] \bigcap \Lambda^3$. follows from (i),(ii) and (iii).

4.6 Theorem

If f be any Musielak Orlicz function then
$\left[\chi^{3I(F)}_{q,\theta C_f}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right] \notin \left[\chi^{3I(F)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right] C_{111}\left(\widehat{S_{\theta_{i,\ell,j}}}\right)$
Proof: Let $x \in \left[\chi^{3I(F)}_{q,\theta C_f}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]$, for all i, ℓ and j.
Therefore we have
$\frac{1}{pqr,h_{i\ell j}} \sum_{m=0,m\in I_{i,\ell,j}}^{p} \sum_{n=0,n\in I_{i,\ell,j}}^{q} \sum_{k=0,k\in I_{i,\ell,j}}^{r} \left[f\left(A\sigma^{X}_{pqr}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right] \geq$
$\frac{1}{pqr,h_{i\ell j}} \sum_{m=0,m\in I_{i,\ell,j}}^{p} \sum_{n=0,n\in I_{i,\ell,j}}^{q} \sum_{k=0,k\in I_{i,\ell,j}\ and\ |x_{m+r,n+s,k+u}|=0}^{r}$
$\left[f\left(A\sigma^{X}_{pqr}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right] >$
$\frac{1}{pqr,h_{i\ell j}} \left[f\left(A\sigma^{X}_{pqr}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]$
$\left|\left\{(m,n,k) \in I_{i,\ell,j} : \left[f(0)\left(A\sigma^{X}_{pqr}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right] = 0\right\}\right|.$
Hence $x \notin \left[\chi^{3I(F)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right] C_{111}\left(\widehat{S_{\theta_{i,\ell,j}}}\right)$.

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