Journal of Progressive Research in Mathematics(JPRM)
ISSN: 2395-0218

SCITECH
RESEARCH ORGANISATION|

Volume 10, Issue 1
Published online: November 25, 2016|

Journal of Progressive Research in Mathematics
www.scitecresearch.com/journals

On the crossing number of join of a graph of order six nK_1 with path and cycle

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Abstract

It has been conjectured by Zarankiewicz that the crossing number of the complete bipartite graph $K_{m,n}$ equals $\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$. This conjecture has been verified by Kleitman [11] for $\min\{m,n\} \leq 6$. Using this result, we give the exact value of crossing number of the join of a certain graph G on six vertices with P_n and C_n, a path and a cycle respectively on n vertices.

Keywords: drawing, crossing number, union and join of graphs, path, cycle.

1 INTRODUCTION

Crossing number minimization is one of the fundamental optimization problems in the sense that it is related to various other widely used notions. Besides its mathematical interest, there are numerous applications, most notably those in VLSI design and in combinatorial geometry [1],[2] and [3]. The study of crossing numbers of graphs also finds applications in areas of network design and circuit layout. Minimizing the number of wire crossings in a circuit greatly reduces the chance of cross-talk in long crossing wires carrying the same signal and also allows for faster operation and less power dissipation.

Let $G = (V, E)$ be a simple connected undirected graph with vertex set V and edge set E. A drawing D of a graph G is a representation of G in the Euclidean plane R^2 where vertices are represented as distinct points and edges by simple polygonal arcs joining points that correspond to their end vertices. A drawing D is *good* or *clean* if it has the following properties:

1. No edge crosses itself.
2. No pair of adjacent edges cross.
3. Two edges cross at most once.
4. No more than two edges cross at one point.

The number of crossings of D is denoted by $cr(D)$ and is called the *crossing number* of the drawing D. The crossing number $cr(G)$ of a graph G is the minimum $cr(D)$ taken over all good or clean drawings D of G. If a graph G admits a drawing D with $cr(D) = 0$ then G is said to be *planar*; otherwise *non-planar*. It is well known that K_5, the complete graph on 5 vertices and $K_{3,3}$ the complete bipartite graph with 3 vertices in its classes are non-planar. According to Kuratowski's famous theorem, a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.

For an arbitrary graph computing $cr(G)$ is NP-hard [5]. The exact values of the crossing number is known only for a few specific family of graphs. The cartesian product is one of the few graph classes for which exact crossing number is known. The table in [6] shows the summary of known crossing numbers for cartesian products of connected graphs of order five with star. There are few results concerning crossing numbers of join of some graphs in [7] and [8].

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be any two graphs. Their *union* denoted by $G_1 \cup G_2$, is the graph $(V_1 \cup V_2, E_1 \cup E_2)$. For any two vertex disjoint graphs G_1 and G_2, their *join*, denoted by $G_1 + G_2$ is obtained by joining every vertex of G_1 to every vertex of G_2. For $|V(G_1)| = m$ and $|V(G_2)| = n$, the edge set of $G_1 + G_2$ is the union of disjoint edge sets of G_1 and G_2 and that of the complete bipartite graph $K_{m,n}$.

Let A and B be two disjoint subsets of E. In a drawing D of G, the number of crossings of edges in A with the edges in B is denoted by $cr_D(A, B)$. The number of crossings among the edges in A is denoted by $cr_D(A)$. The following result is used in the proofs of our theorems.

Lemma 1 *[7] Let A, B, C be mutually disjoint subsets of E. Then*

$$cr_D(A \cup B) = cr_D(A) + cr_D(B) + cr_D(A, B)$$
$$cr_D(A, B \cup C) = cr_D(A, B) + cr_D(A, C)$$

where D is a good drawing of G.

It has been conjectured that the crossing number of the complete bipartite graph $K_{m,n}$ equals the Zarankiewicz's number $Z(m,n) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$. This conjecture has been verified by Kleitman [11] for $\min\{m, n\} \leq 6$, which we state below.

Lemma 2 *[11] For min $\{m, n\} \leq 6$, $cr(K_{m,n}) = Z(m, n)$ where*

$$Z(m,n) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Kulli and Muddebihal [10] gave the characterization of all pairs of graphs whose join is a planar graph. Klesc [7] has given the exact values for the crossing number of join of two paths, join of two cycles and join of path and cycle. In addition, he has given the exact values for crossing numbers of $G + P_n$ and $G + C_n$ for all graphs G of order at most four and some graphs of order five and six in [7], [8] and [9].

In this paper we give the exact values of crossing numbers of join $G + P_n$ and $G + C_n$ where G is a graph of order six as shown in Figure 1.

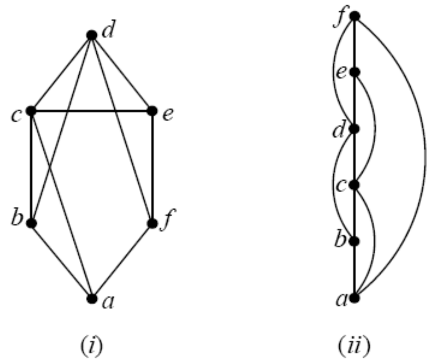

Figure 1: The graph G and a good drawing

2 The Graph $G + nK_1$

Let G be a graph as shown in Figure 1 with $V(G) = \{a, b, c, d, e, f\}$. Let $t_1, t_2, ..., t_n$ be the vertices of nK_1.

We begin with certain notations and terminology. Let H_n denote the graph $G + nK_1$. The edge set of H_n is the union of disjoint edge sets of G and that of the complete bipartite graph $K_{6,n}$. Let $T^i (i = 1, 2, ..., n)$ be the subgraph of H_n which consists of six edges incident with the vertex t_i. Then it is clear that

$$H_n = G + nK_1 = G \cup K_{6,n} = G \cup \left(\bigcup_1^n T^i \right).$$

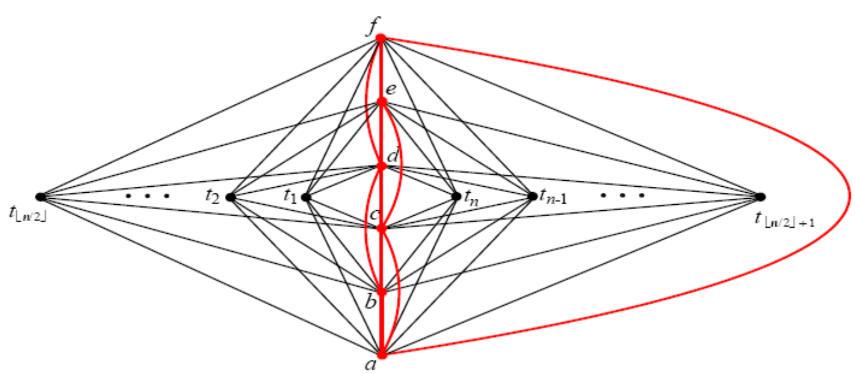

Figure 2: $G + nK_1$

Lemma 3 $cr(K_{m,n-1}, T^n) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor.$

Proof. We know that $K_{m,n} = K_{m,n-1} \cup T^n$. Therefore,

$$\begin{aligned} cr(K_{m,n}) &= cr(K_{m,n-1} \cup T^n) \\ &= cr(K_{m,n-1}) + cr(T^n) + cr(K_{m,n-1}, T^n) \\ &= cr(K_{m,n-1}) + cr(K_{m,n-1}, T^n) \text{ since } cr(T^n) = 0. \end{aligned}$$

Hence

$$\begin{aligned} cr(K_{m,n-1}, T^n) &= cr(K_{m,n}) - cr(K_{m,n-1}) \\ &= \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \\ &\quad - \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \\ &= \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\{ \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n-2}{2} \right\rfloor \right\} \\ &= \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \end{aligned}$$

which completes the proof.

Lemma 4 $cr(G + K_1) = 2$.

Proof. From Figure 3 (ii) it follows that, $cr(G + K_1) \leq 2$. Therefore it is enough to show that $cr(G + K_1) \geq 2$. Let T^i be the subgraph which consists of the edges $t_i a$, $t_i b$, $t_i c$, $t_i d$, $t_i e$, and $t_i f$, incident with the vertex t_i.The plane drawing of G (Figure 3 (i)) with 6 vertices and 10 edges, by Euler's formula has $f = 2 - n + \varepsilon = 6$ regions. Let us denote these regions by $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, β_1 and β_2. In each of the boundaries of the regions $\alpha_1, \alpha_2, \alpha_3$ and α_4 there are three vertices and the remaining three vertices lie outside the boundary. Similarly in each of the boundaries of the regions β_1 and β_2 there are four vertices and the remaining two vertices lie outside the boundary. Without loss of generality assume that t_i lies inside the region β_1(or β_2) which is the 4-cycle $abdfa$ as shown in Figure 3(ii). Also the vertices c and e lie outside β_1. Hence by Jordan curve theorem, the edges $t_i c$ and $t_i e$ contribute at least one crossing each with the edges of G and we have $cr(G + K_1) \geq 2$. ■

Theorem 1 $cr(H_n) = 6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n$.

Proof. Consider a good drawing of $G + nK_1$ as shown in Figure 2. This contains $K_{6,n}$ as a subgraph. Also the edges bd, df, ac and ce contribute for crossings. As bd and df are to the left of y-axis and there are $\left\lfloor \frac{n}{2} \right\rfloor$ edges of $K_{6,n}$, each edge contributes $\left\lfloor \frac{n}{2} \right\rfloor$ to the crossing number. Similarly there

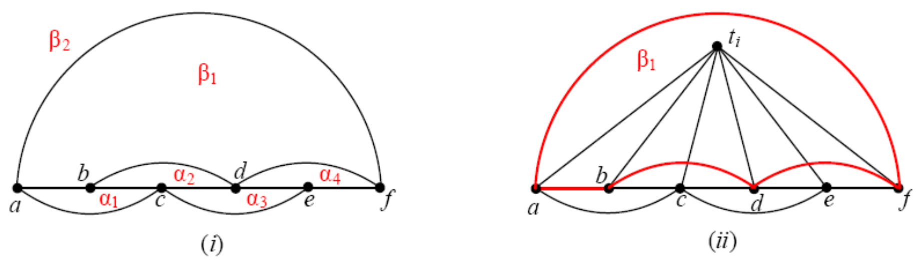

Figure 3: (i) Plane drawing of G and its faces (ii) $G+K_1$

are $\lceil \frac{n}{2} \rceil$ edges of $K_{6,n}$ to the right of y-axis, each of the edges ac and ce contributes $\lceil \frac{n}{2} \rceil$ crossings. Hence

$$\begin{aligned} cr(H_n) &\leq cr(K_{6,n}) + 2\left\lfloor \frac{n}{2} \right\rfloor + 2\left\lceil \frac{n}{2} \right\rceil \\ &= 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\{ \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil \right\} \\ &= 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n \end{aligned}$$

We next claim that $cr(H_n) \geq 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n$. We prove this result by induction. For $n = 1$, the result is true by Lemma 4. Assume that the result is true for H_k, $1 < k < n$ ie.,

$$cr(H_{n-1}) \geq 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 2(n-1)$$

We know that $H_n = H_{n-1} \cup T^n = G \cup \left(\bigcup_1^{n-1} T^i \right) \cup T^n$. Hence,

$$\begin{aligned} cr(H_n) &= cr(G \cup \left(\bigcup_1^{n-1} T^i \right) \cup T^n) \\ &= cr((G \cup K_{6,n-1}) \cup T^n) \\ &= cr(G \cup K_{6,n-1}) + cr(T^n) + cr(G \cup K_{6,n-1}, T^n), \text{ by Lemma 1} \\ &= cr(H_{n-1}) + 0 + cr(G, T^n) + cr(K_{6,n-1}, T^n) \\ &\geq 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 2(n-1) \\ &\quad +2 + 6\left\lfloor \frac{n-1}{2} \right\rfloor, \text{ by induction assumption, Lemmas 3 and 4} \\ &= 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n. \end{aligned}$$

which completes the proof. ■

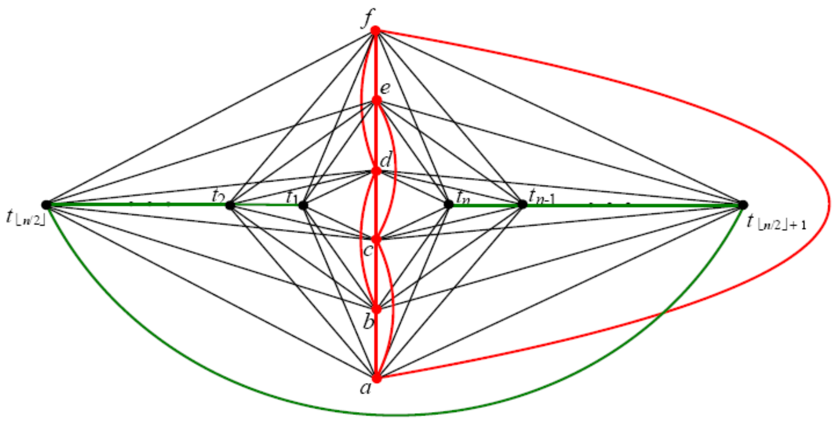

Figure 4: $G + P_n$

Theorem 2 $cr(G + P_n) = 6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n + 1.$

Proof. It is easily seen that it is possible to add $n - 1$ edges which form the path P_n on n vertices of nK_1, with 1 crossing. See Figure 4. Hence,

$$cr(G + P_n) \leq 6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n + 1$$

Suppose this graph has fewer than $6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n + 1$ crossings, then removal of the edge $t_{\lfloor \frac{n}{2} \rfloor} t_{\lfloor \frac{n}{2} \rfloor + 1}$ which contributes one crossing will result in $G + nK_1$ with fewer than $6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n$ crossings which is a contradiction to theorem 1. Therefore

$$cr(G + P_n) = 6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n + 1$$

■

It is also possible to add n edges which form the cycle C_n on n vertices of nK_1, with 4 crossings. Proceeding as in Theorems 1 and 2 we can prove the following result.

Theorem 3 $cr(G + C_n) = 6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n + 4.$

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