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GV-semigroups with their full subsemigroup lattices

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ABSTRACT.

The main aim of this paper is to study GV-semigroups whose full subsemigroups form distributive lattices. A sufficient and necessary condition for GV-semigroups to have distributive lattices of full subsemigroups is given. In particular, the structure of completely regular semigroups whose full subsemigroup lattices are distributive is characterized.

Keywords: GV-semigroups; full subsemigroup lattice; distributive lattice.

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1. Introduction

The subsemigroup lattices of semigroups have been the subject of continued investigation for many years. The main achievements in the area, accomplished by the mid-1990s,have been comprehensively reflected in the monograph [9]. Among the large fields of the investigation, much attention has been paid to the full regular subsemigroup lattices of regular semigroups. Recall that a subset of a semigroup is called full if it contains the set of all idempotents of the given semigroup.

Johnston and Jones researched regular semigroups with their full regular subsemigroup lattices in [2], which provides some interesting results. It is well known that full regular subsemigroups of inverse semigroups are just their full inverse subsemigroups.Thus the theory concerning full regular subsemigroup lattices has been extensively explored in the case of inverse semigroups. A series of papers have been devoted to the theme of describing the structure of inverse semigroups with various types of full regular subsemigroup lattices (see [3, 4, 5, 6, 7, 11]). Moreover, Jones and Tian generalized to eventually regular semigroups the study of the full regular subsemigroup lattices of regular semigroups and characterized the structure of eventually regular semigroups whose full eventually regular subsemigroup lattices are distributive lattices or chains in [8]. Recently, a program of studying the interrelationships of inverse semigroups and their full subsemigroup lattices has been presented in [10] by Tian, who established the structure of inverse semigroups with various assumptions on their full subsemigroup lattices.

In the present paper, the main purpose is to extend the study of the full subsemigroup lattices of inverse semigroups to that of GV-semigroups. We obtain a sufficient and necessary condition for GV-semigroups indicated in the title of the paper. Furthermore, completely regular semigroups with distributive full subsemigroup lattices are determined.

2. Preliminaries

Recall that a lattice $(L,\wedge,\vee)$ is distributive, if for any $a,b,c\in L, a\wedge(b\vee c)=(a\wedge b)\vee(a\wedge c)$. A semigroup S is called eventually regular if some power of each element of S is regular. If every regular element of an eventually regular semigroup S is completely regular, then S is called a GV-semigroup. For a completely regular semigroup S, as we all know, every regular element a of S exists and only exists an inverse of a which commutes with a. We usually denote the unique inverse of a by a^{-1}. Let S be a semigroup. As usual, E_S denotes the set of all idempotents of S, and $\mathrm{Re}\,gS$ denotes the set of all regular elements of S. In general, neither E_S nor $\mathrm{Re}\,gS$ is a subsemigroup of S. If a GV-semigroup S only has an idempotent, then it is called unipotent epigroup. Given $e\in E_S$, G_e denotes the maximal subgroup of a semigroup S containing e and we put

$$K_e=\{x\in S: x^n\in G_e \text{ for some } n\in Z^+\},$$

then we define the relation $k=\bigcup_{e\in E_S}(K_e\times K_e)$. As we all know, the relation k is an equivalent relation on a GV-semigroup and $K_e\in(e\in E_S)$ is called a unipotency class of S with $K_e=H_e^*$ for any $e\in E_S$.

For any subset A of a semigroup S, we denote by $\langle A\rangle$ the subsemigroup of S generated by A, $F\langle A\rangle$ the full subsemigroup of S generated by A, by A^* the set of all non-zero elements of A, by $\mathrm{Sub}S$ the lattice of all subsemigroups (including the empty set) of S, and by $\mathrm{Subf}S$ the lattice of all full subsemigroups of S. It is easy to show that the lattice $SubfS$ is a complete sublattice of the lattice $\mathrm{Sub}S$. In particular, for an inverse semigroup S, we shall denote by $\mathrm{Subfi}S$ the lattice of full inverse subsemigroups of S.

A semigroup S is called a U-semigroup, if $xy\in\langle x\rangle\cup\langle y\rangle$ for any $x,y\in S$. If a semigroup S is a U-semigroup as well as a nilsemigroup, then it is called a U-nilsemigroup. Let a semigroup S be an ideal extension of a semigroup T by a U-nilsemigroup, then S is called a U-nilextension of T. A band S is named a left(right) zero band if $ab=a(ab=b)$ for any $ab\in S$. A GV-semigroup S is called a FU-band of GV-semigroups $S_\alpha(\alpha\in Y)$ if $S=\bigcup_{\alpha\in Y}S_\alpha$ is a band of S_α and $xy,yx\in F\langle x\rangle\cup F\langle y\rangle$ for any $x\in S_\alpha$, $y\in S_\beta$ with $\alpha\neq\beta$.

3. Main Results

We are to investigate GV-semigroups whose full subsemigroups form distributive lattices in this section. We first formulate the main result in this section.

Theorem 3.1 Let S be a GV-semigroup and $SubfS$ distributive if and only if S is a GU-band of unipotent epigroups $K_e(e\in E_S)$ which are U-nilextensions of locally cyclic groups G_e for all $e\in E_S$.

We shall prove Theorem 3.1 proceeded by a sequence of lemmas.

Lemma 3.2 Let S be a GV-semigroup. If $\mathrm{Subf}S$ is distributive, then $K_eK_f\subseteq K_e$ or $K_eK_f\subseteq K_f$ for any $e,f\in E_S$.

Proof. First, we put $A=F\langle ef\rangle$, $B=F\langle e\rangle=\{e\}$, $C=F\langle f\rangle=\{f\}$ for any $e,f\in E_S$ with $e\neq f$. If assume $A\cap B=A\cap C=\phi$, then $A\cap(B\vee C)=F\langle ef\rangle\cap F\langle e,f\rangle=A\cap B=\phi$ by distributivity of $SubfS$. Obviously, it leads to a contradiction. Thus $A\cap B\neq\phi$ or $A\cap C\neq\phi$.

Let $A\cap B\neq\phi$. *i.e.* $e\in F\langle ef\rangle$. And there exists $g\in E_S$ such that $ef\in K_g$ since S is a GV-semigroup. We first assume $g\neq e$. It is clear that

$$F\langle ef\rangle\subseteq K_g, e\in F\langle ef\rangle\subseteq K_g \text{ and } e\in K_e,$$

hence $e\in K_e\cap K_g$, and so it contradicts the fact $K_e\cap K_g=\phi$. Whence $e=g, \langle ef\rangle\subseteq F\langle ef\rangle\subseteq K_e$, thus $ef\in K_e$. Similarly, let $A\cap C\neq\phi$, we have $ef\in K_f$ by the similar method.

Suppose there exist $a \in K_e, b \in K_f$ such that $ab \notin K_e \cup K_f$ with $e, f \in E_S, e \neq f$. Then there exists $g \in E_S \setminus \{e, f\}$ such that $ab \in K_g$. Hence $(ab)^n \in K_g$, $(ab)^n \notin K_e \cup K_f$ for any $n \in Z^+$, and so

$$F\langle ab\rangle \cap F\langle a\rangle = F\langle ab\rangle \cap F\langle b\rangle = \phi .$$

Thus $F\langle ab\rangle \cap F\langle a,b\rangle = \phi$ by distributivity of $SubfS$. Obviously, it is a contradiction and so $K_eK_f \subseteq K_e \cup K_f$. Symmetrically, we can prove $K_fK_e \subseteq K_e \cup K_f$ for any $e, f \in E_S$.

Let $ef \in K_e$ for any $e, f \in E_S$ with $e \neq f$ and assume there exist $a \in K_e, b \in K_f$ such that $ab \in K_f$. Hence

$$af = (ab)b^{r(b)-1}(b^{r(b)})^{-1} \in K_fK_f \subseteq K_f, af = (af)f \in G_f$$

and since there exists $n \in Z^+$ such that $a^n \in G_e$, whence

$$a^n f = aa\cdots f^n = (af)^n, a^n = a^n e \text{ and } a^n ef = a^n f = (af)^n \in K_f .$$

Therefore $K_e \cap K_f \neq \phi$ by $a^n ef = a^n(ef) \in K_e$. Thus $K_eK_f \subseteq K_e$. Dually, we have $K_eK_f \subseteq K_f$ if $ef \in K_f$ for any $e, f \in E_S$ with $e \neq f$.

Suppose $e = f$, then the assertion of the lemma now follows without difficulty since any unipotency of a GV-semigroup S is a subsemigroup of S.

From the proof of this lemma, we have the following corollary easily.

Corollary 3.3 Let S be a GV-semigroup. If $SubfS$ is distributive, then

(1) $k = \bigcup_{e\in E_S}(K_e \times K_e)$ is a congruence on S, that is, a band of unipotent epigroups $K_e(e \in E_S)$;

(2) $\{K_e, K_f\}$ is a left(right) zero band or chain for any $e, f \in E_S$.

Lemma 3.4 Let S be a GV-semigroup. If $SubfS$ is distributive, then $xy, yx \in F\langle x\rangle \cup F\langle y\rangle \cup G_e$ for any $x, y \in K_e, e \in E_S$.

Proof. Let S be a GV-semigroup. Then K_e is a subsemigroup of S, hence $xy \in K_e$ for any $x, y \in K_e$. Thus we can prove the lemma from two cases.

If $xy \in K_e \setminus G_e$. Then x, y both lie in $K_e \setminus G_e$ since G_e is an ideal of K_e. Next we denote by "$\circ$" the operation in K_e / G_e, denote by "$\cdot$" the operation in K_e. Hence we have $x \cdot y = x \circ y$ since in K_e / G_e the product of two elements in $K_e \setminus G_e$ is the same as their product in K_e if the product is in $K_e \setminus G_e$. Whence

$$x \circ y = x \cdot y = \langle\langle x\rangle\rangle \cup \langle\langle y\rangle\rangle$$

where $\langle\langle x\rangle\rangle$ or $\langle\langle y\rangle\rangle$ refers to the subsemigroup generated by x or y under the operation "$\circ$". Obviously, $x \circ y \neq 0$, hence

$$x \circ y \in (\langle\langle x\rangle\rangle \setminus \{0\}) \cup (\langle\langle y\rangle\rangle \setminus \{0\}) \text{ where } \langle\langle x\rangle\rangle = \{x, x \circ x, \cdots, 0\}$$

the order of $\langle\langle x\rangle\rangle$ is $r(x)$ which denotes the nilindex of x, and so all elements of $\langle\langle x\rangle\rangle$ except 0 can be found in $\langle x\rangle$ which is the subsemigroup generated by x under the operation "$\cdot$". Thus

$$x \cdot y = x \circ y \in (\langle\langle x\rangle\rangle \setminus \{0\}) \cup (\langle\langle y\rangle\rangle \setminus \{0\}) \subseteq \langle x\rangle \cup \langle y\rangle \subseteq F\langle x\rangle \cup F\langle y\rangle .$$

If $xy \in G_e$, then $xy \in F\langle x\rangle \cup F\langle y\rangle \cup G_e$, clearly. That $yx \in F\langle x\rangle \cup F\langle y\rangle \cup G_e$ can be proved dually.

Therefore, we have $xy, yx \in F\langle x\rangle \cup F\langle y\rangle \cup G_e$ for any $x, y \in K_e$ with $e \in E_S$.

In view of Lemmas 3.2 and 3.4, we are to prove the following lemma, which takes an important role in proving the necessity of Theorem 3.1.

Lemma 3.5 Let S be a GV-semigroup. If $SubfS$ is distributive, then S is a FU-band of unipotent epigroups $K_e(e \in E_S)$ and every K_e is a U-nilextension of a locally cyclic group G_e for any $e \in E_S$.

Proof. First by corollary 3.3, we have S is a band of unipotent epigroups $K_e(e \in E_S)$ every of which is a U-nilextensions of a locally cyclic group G_e for any $e \in E_S$. Next we only need to prove $xy, yx \in F\langle x\rangle \cup F\langle y\rangle$ for any $x \in K_e, y \in K_f$ with $e, f \in E_S$ and $e \neq f$. Put $A = F\langle xy, x\rangle$, $B = F\langle x\rangle$ and $C = F\langle y\rangle$. Hence

$$F\langle xy, x\rangle \cap F\langle x, y\rangle = F\langle xy, x\rangle = F\langle x\rangle \vee (A \cap C)$$

since $SubfS$ is distributive.

If $\{K_e, K_f\}$ is a left zero band, then $xy \in K_e K_f \subseteq K_e$, thus $A \cap C = \phi$. Hence $F\langle xy, x\rangle = F\langle x\rangle$, and so

$$xy \in F\langle x\rangle \text{ , i.e. } xy \in F\langle x\rangle \cup F\langle y\rangle .$$

If $\{K_e, K_f\}$ is a right zero band, then $xy \in K_e K_f \subseteq K_f$, thus $A \cap C \neq \phi$. Put $A \cap C = E \in SubfS$ and $F\langle xy\rangle = D$, then $F\langle xy, x\rangle = F\langle x\rangle \vee E$, that is, $F\langle x\rangle \vee F\langle xy\rangle = F\langle x\rangle \vee E = F\langle x\rangle \vee D$, hence

$$(B \vee D) \wedge D = (B \vee E) \wedge D \text{ with } B \wedge D = \phi$$

and $SubfS$ is distributive, and so

$$(B \vee D) \wedge D = (B \vee D) \vee D = D = (B \wedge D) \vee (E \wedge D) = E \wedge D,$$

therefore $D \subseteq E$, *i.e.* $F\langle xy\rangle \subseteq E \subseteq F\langle y\rangle$, thus $xy \in F\langle y\rangle$.

If $\{K_e, K_f\}$ is a chain. It is easy to prove $xy \in F\langle x\rangle \cup F\langle y\rangle$.

Lemma 3.6 Let S be a GV-semigroup and $S = \bigcup_{\alpha \in Y} S_\alpha$ a FU-band of GV-semigroups $S_\alpha(\alpha \in Y)$, then $F(A, B) = A \cup B$ for any $A \in SubfS_\alpha$, $B \in SubfS_\beta$ where $\alpha, \beta \in Y$ with $\alpha \neq \beta$.

Proof. For any $a \in A \subseteq S_\alpha, b \in B \subseteq S_\beta$, hence $ab, ba \in F\langle a\rangle \cup F\langle b\rangle \subseteq A \cup B$ since S is a FU band of $S_\alpha(\alpha \in Y)$, and so $\langle A, B\rangle = A \cup B$. Let any $a \in \langle A, B\rangle \cap \operatorname{Reg} S = (A \cap \operatorname{Reg} S) \cup (B \cap \operatorname{Reg} S)$, then $a^{-1} \in A \cup B$ for $A, B \in SubfS$. Thus $F\langle A, B\rangle = A \cup B$.

Lemma 3.7 Let S be a GV-semigroup and K_e a U-nilextension of G_e for any $e \in E_S$, then $SubfK_e$ is embedded into the direct product of $SubfK_e / G_e$ and $SubfG_e$.

Proof. We first make a map $\psi : SubfK_e \to SubfK_e / G_e \times SubfG_e$, given by $\psi(A) = ((A \cup G_e)/G_e, A \cap G_e)$.For any $A, B \in SubfK_e$, we have $A \cup G_e \in SubfK_e$, $A \cap G_e \in SubfG_e$ and ψ is injective, $\psi(A \wedge B) = \psi(A) \wedge \psi(B)$. Next we only need to prove $\psi(A \vee B) = \psi(A) \vee \psi(B)$.

Notice

$$\psi(A \vee B) = ((F\langle A, B\rangle \cup G_e)/G_e, F\langle A, B\rangle \cap G_e),$$

$$\psi(A) \vee \psi(B) = (F\langle (A \cup G_e)/G_e, (B \cup G_e)/G_e\rangle, F\langle A \cap G_e, B \cap G_e\rangle).$$

Clearly, $(F\langle A, B\rangle \cup G_e)/G_e = F\langle (A \cup G_e)/G_e, (B \cup G_e)/G_e\rangle$ since K_e is a U-nilextension of G_e. Now we show $F\langle A, B\rangle \cap G_e = F\langle A \cap G_e, B \cap G_e\rangle$.For any $x \in \langle A, B\rangle \cap G_e$, then there exist $x_1, x_2, \cdots, x_n \in A \cup B (n \in Z^+)$ such that $x = x_1 x_2 \cdots x_n$. Hence $x = xe = x_1 x_2 \cdots x_n e$ by $x \in G_e$ and $x_i e = e x_i \in G_e (i \in n)$ by $x_1, x_2, \cdots, x_n \in K_e$, and so $x = (x_1 e)(x_2 e) \cdots (x_n e)$. And since

$$x_i e \in (A \cup B) \cap G_e = (A \cap G_e) \cup (B \cap G_e) \text{ for } e \in A \cap B,$$

therefore $x \in \langle A \cap G_e, B \cap G_e \rangle$, *i.e.* $\langle A, B \rangle \cap G_e \subseteq \langle A \cap G_e, B \cap G_e \rangle$. Obviously $\langle A \cap G_e, B \cap G_e \rangle \subseteq \langle A, B \rangle \cap G_e$. Hence $\langle A, B \rangle \cap G_e = \langle A \cap G_e, B \cap G_e \rangle$, and so $GV(\langle A, B \rangle \cap G_e) = F\langle A, B \rangle \cap G_e = F\langle A \cap G_e, B \cap G_e \rangle$. Thus we conclude that ψ is a monomorphism from $SubfK_e$ to the direct product of $SubfK_e / G_e$ and $SubfG_e$.

Lemma 3.8 Let S be a GV-semigroup and $S = \bigcup_{\alpha \in Y} S_\alpha$ a FU-band of GV-semigroups. $S_\alpha (\alpha \in Y)$, then $SubfS \cong \prod_{\alpha \in Y} SubfS_\alpha$.

Proof. First we make a map $\psi : SubfS \to \prod_{\alpha \in Y} SubfS_\alpha$ and define $\psi(A) = (\cdots, A \cap S_\alpha, \cdots)$ for any $A, B \in SubfS$. Clearly, ψ is a injective and $\psi(A \cap B) = \psi(A) \cap \psi(B)$. Let any

$$(\cdots, H_\alpha, \cdots) \in \prod_{\alpha \in Y} SubfS_\alpha,\ H_\alpha \in SubfS_\alpha \text{ and put } A = F\langle H_\alpha / \alpha \in Y \rangle.$$

By lemma 3.6, we get $A = \bigcup_{\alpha \in Y} H_\alpha$, hence $\psi(A) = (\cdots, A \cap S_\alpha, \cdots) == (\cdots, H_\alpha, \cdots)$, thus ψ is surjective. Next we prove $\psi(A) \vee \psi(B) = \psi(A \vee B)$. Notice

$$\psi(A) \vee \psi(B) = (\cdots, A \cap S_\alpha, \cdots) \vee (\cdots, B \cap S_\alpha, \cdots) = (\cdots, F\langle A \cap S_\alpha, B \cap S_\alpha \rangle, \cdots),$$

$\psi(A \vee B) = (\cdots, F\langle A, B \rangle \cap S_\alpha, \cdots)$. From lemma 3.6, we get

$$F\langle A, B \rangle = F\langle \bigcup_{\alpha \in Y} (A \cap S_\alpha), \bigcup_{\alpha \in Y} (B \cap S_\alpha) \rangle = (\cdots, F\langle A \cap S_\alpha, B \cap S_\alpha \rangle, \cdots) = \bigcup_{\alpha \in Y} F\langle A \cap S_\alpha, B \cap S_\alpha \rangle,$$

hence $F\langle A, B \rangle \cap S_\alpha = F\langle A \cap S_\alpha, B \cap S_\alpha \rangle$, *i.e.* $\psi(A \vee B) = \psi(A) \vee \psi(B)$. Thus $SubfS \cong \prod_{\alpha \in Y} SubfS_\alpha$.

From the above lemmas, we can prove the main result in the section.

Proof. The necessity can be proved by lemma 3.5. To prove the sufficiency, suppose S is a FU-band of unipotent epigroups $K_e (e \in E_S)$ and every K_e is a U-nilextension of a locally cyclic group G_e for any $e \in E_S$. By lemma 3.7, we have $SubfK_e$ is an isomorphism to a sublattice of the direct product of $SubfK_e / G_e$ and $SubfG_e$. And since $SubgG_e = SubfG_e$ is distributive by G_e is a locally cyclic group and $SubK_e / G_e = SubfK_e / G_e$ is distributive, hence $SubfK_e$ is distributive. Thus we have $SubfS$ is distributive by lemma 3.8.

The following result for completely regular semigroups follows from Theorem 3.1 immediately.

Theorem 3.9 Let S be a completely regular semigroup and $SubfS$ distributive if and only if S is a FU-band of locally cyclic groups G_e for all $e \in E_S$.

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