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First Digit Counting Compatibility II: Twin Prime Powers

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Abstract

The first digits of twin primes follow a generalized Benford law with size-dependent exponent and tend to be uniformly distributed, at least over the finite range of twin primes up to 10^m, $m = 5, \ldots, 16$. The extension to twin prime powers for a fixed power exponent is considered. Assuming the Hardy-Littlewood conjecture on the asymptotic distribution of twin primes, it is claimed that the first digits of twin prime powers associated to any fixed power exponent converge asymptotically to a generalized Benford law with inverse power exponent. In particular, the sequences of twin prime power first digits presumably converge asymptotically to Benford's law as the power exponent goes to infinity. Numerical calculations and the analytical first digit counting compatibility criterion support these conjectured statements.

Keywords: First digit; twin primes; Hardy-Littlewood conjecture; probabilistic number theory; asymptotic distribution; mean absolute deviation; probability weighted least squares.

Mathematics Subject Classification: 11A41, 11K99, 11N37, 11Y55, 62E20, 62F12.

1. Introduction

Newcomb [17] and Benford [3] observed that the first digits of many series of real numbers obey *Benford's law*

$$P^B(d) = \log_{10}(1+d) - \log_{10}(d), \quad d = 1, 2, \ldots, 9. \tag{1.1}$$

The increasing knowledge about Benford's law and its applications has been collected in various bibliographies by Hürlimann [10], Berger and Hill [4] and Beebe [2]. Two recent books are Berger and Hill [5], and Miller [15]. In Number Theory, it is known that for any fixed power exponent $s \geq 1$, the first digits of some integer sequences, like integer powers and prime powers, follow asymptotically a *Generalized Benford law* (GB) with exponent $\alpha = s^{-1} \in (0,1]$ (see Hürlimann [9,11]) such that

$$P_\alpha^{GB}(d) = \frac{(1+d)^\alpha - d^\alpha}{10^\alpha - 1}, \quad d = 1, 2, \ldots, 9. \tag{1.2}$$

Clearly, the limiting case $\alpha \to 0$, respectively $\alpha = 1$, of (1.2), converges weakly to Benford's law, respectively the uniform distribution.

As a follow-up to Hürlimann [11,12], we study the first digits of powers of the first prime in twin prime pairs using a numerical and an analytical method. Based on the numerical method we fit the GB to appropriate samples of first digits using two size-dependent goodness-of-fit measures, namely the ETA measure (derived from the mean absolute deviation) and the WLS measure (weighted least square measure derived from the chi-square statistics). In Section 2, we determine the minimum ETA and WLS estimators of the GB over finite ranges of twin primes up to 10^m, $m = 5, \ldots, 16$, which suggest convergence to the uniform distribution. Based on the Hardy-Littlewood conjectured twin prime counting function, the computation in Section 3 for twin prime powers with a fixed power exponent $s \geq 2$, illustrates convergence of the size-dependent GB with minimum ETA and WLS estimators to the

GB with exponent s^{-1}. Moreover, we show the existence of a one-parametric size-dependent exponent function that converges to these GB's and determine some approximate value that is close enough to the minimum ETA and WLS estimators to support the suggested convergence. Section 4 uses the analytical criterion of first digit counting compatibility introduced in [11,12]. In general, this criterion permits to decide whether or not a given size-dependent GB that belongs to the first digits of some integer sequence is compatible with the asymptotic counting function of this sequence, if it exists. Theorem 4.1 shows the existence of a parameter-free size-dependent GB for the sequence of twin prime powers that is first digit counting compatible with its conjectured asymptotic counting function. Besides the numerical support stated above, this result provides mathematical evidence for the assertion that the asymptotic distribution of the first digits of twin prime powers follows a GB with exponent s^{-1}.

2. Size-dependent generalized Benford law for twin prime powers

To investigate the optimal fit of the GB to first digit sequences of twin prime powers, it is necessary to specify goodness-of-fit (GoF) measures according to which optimality should hold. For this purpose, Hürlimann [11] introduces and motivates the following two GoF measures. Let $\{x_n\} \subset [1,\infty)$, $n \ge 1$, be an integer sequence, and let d_n be the (first) significant digit of x_n. The number of x_n's, $n = 1,\ldots,N$, with significant digit $d_n = d$ is denoted by $X_N(d)$. The *ETA measure* for the GB is defined to be

$$ETA_N(\alpha) = \frac{9}{2\cdot N}\cdot MAD_N(\alpha), \quad MAD_N(\alpha) = \frac{1}{9}\cdot\sum_{d=1}^{9}\left|P_\alpha^{GB}(d) - \tfrac{X_N(d)}{N}\right|, \tag{2.1}$$

where $MAD_N(\alpha)$ is the *mean absolute deviation* measure. The *WLS measure* is defined by

$$WLS_N(\alpha) = \frac{1}{N}\cdot\sum_{d=1}^{9}\frac{(P_\alpha^{GB}(d) - \tfrac{X_N(d)}{N})^2}{P_\alpha^{GB}(d)}. \tag{2.2}$$

We consider now the sequence of twin prime powers $\{p^s,(p+2)^s\},\ p^s < 10^{s\cdot m}$, for a fixed exponent $s = 1, 2, 3, \ldots$, and arbitrary primes below 10^m, $m \ge 4$. Denote by $I_k^s(d)$ the number of twin prime powers below 10^k, $k \ge 1$, such that the first prime power in the twin prime power pair has first digit d. This number is defined recursively by the relationship

$$I_{k+1}^s(d) = \pi_2(\sqrt[s]{(d+1)\cdot 10^k}) - \pi_2(\sqrt[s]{d\cdot 10^k}) + I_k^s(d), \quad k = 1,2,\ldots, \tag{2.3}$$

where the counting function $\pi_2(x)$ yields the number of twin prime pairs below x. Therefore, with $N = \pi_2(10^m)$ one has $X_N(d) = I_{s\cdot m}^s(d)$ in (2.1)-(2.2). A list of the $I_m^1(d)$, $m = 5,\ldots,16$, together with the sample size $N = \pi_2(10^m)$, is provided in Table 5 of the Appendix. Based on this we have calculated the so-called minimum ETA and minimum WLS estimators, which minimize these GoF measures. The obtained optimal estimators are reported in Table 1 below. Note that the minimum WLS is a critical point of the equation

$$\begin{aligned} &\frac{\partial}{\partial\alpha}WLS_N(\alpha) = \frac{1}{N}\cdot\sum_{d=1}^{9}\frac{\partial P_\alpha^{GB}(d)}{\partial\alpha}\cdot\frac{P_\alpha^{GB}(d)^2 - (\tfrac{X_N(d)}{N})^2}{P_\alpha^{GB}(d)^2} = 0, \\ &\frac{\partial P_\alpha^{GB}(d)}{\partial\alpha} = \frac{(1+d)^\alpha\{\ln(\tfrac{1+d}{10})10^\alpha - \ln(1+d)\} - d^\alpha\{\ln(\tfrac{d}{10})10^\alpha - \ln(d)\}}{(10^\alpha - 1)^2}, \quad d = 1, 2, \ldots, 9. \end{aligned} \tag{2.4}$$

For comparison, the ETA and WLS measures for the following size-dependent GB exponents

$$\alpha_{LL1}(m)=1-(\ln(10^m)-\ln^{c_1}(10^m))^{-1},\quad c_1=0.8416781,$$
$$\alpha_{LL2}(m)=1-(\ln(10^m)-\ln^{c_2}(10^m))^{-1},\quad c_2=0.816203531, \tag{2.5}$$

called LL estimators, are listed. This type of estimator is named in honour of Luque and Lacasa [14] who introduced a variant of it in their GB prime number analysis. By construction the LL1 estimator matches the approximate minimum WLS for $N=\pi_2^{HL}(10^{31})$, where $\pi_2^{HL}(x)$ is the Hardy-Littlewood conjectured approximation to $\pi_2(x)$ (see the formulas (2.6)-(2.7) below). The LL2 estimator matches the exact minimum ETA for $N=\pi_2(10^{16})$.

Table 1 below displays exact results. The ETA (resp. WLS) measures are given in units of $10^{-(m+1)}$ (resp. $10^{-(m+5)}$). By trying to extend the results beyond $m=16$ one encounters at least two difficulties. The Table in Nicely [18], which is used to calculate Table 5, stops at $m=16$. At the cost of a slight loss in accuracy, one can overcome this difficulty by using an approximation formula for $\pi_2(x)$, for example the conjectured logarithmic integral approximation by Hardy-Littlewood [7] given by (see Hardy and Wright [8], Section 22.20, Riesel [19], Chapter 3, Shanks [20], Section 12, Narkiewicz [16], Section 6.7, Conjecture B, Crandall and Pomerance [6], Section 1.2.1, among others)

$$\pi_2^{HL}(x)=H_2\cdot Li2(x),\quad Li2(x)=\int_2^x \ln^{-2}(t)dt,\quad H_2=2\cdot\prod_{p\geq 3}\frac{p(p-2)}{(p-1)^2}=1.320323632. \tag{2.6}$$

So far, nobody has been able to prove this conjecture. However, based on Hardy-Littlewood's circle method, Lavrik [13] obtained an almost-all result, which has been derived recently with an elementary method by Baier [1]. Based on it we replace $N=\pi_2(10^m)$ and formula (2.3) by Hardy-Littlewood's approximations $N=\pi_2^{HL}(10^m)$ and

$$I_{k+1}^s(d)=\pi_2^{HL}(\sqrt[s]{(d+1)\cdot 10^k})-\pi_2^{HL}(\sqrt[s]{d\cdot 10^k})+I_k^s(d),\quad k=1,2,\ldots \tag{2.7}$$

In this way the Table 1 extends (here in single precision only) to Table 2.

Again, the ETA (resp. WLS) measures are given in units of $10^{-(m+1)}$ (resp. $10^{-(m+5)}$). Taking into account the decreasing units, one observes that the optimal ETA and WLS measures decrease with increasing sample size. While the LL2 estimator beats the LL1 estimator over the fixed ranges $[1,10^m]$, $m=5,\ldots,22$, the LL1 estimator is best for the higher fixed ranges $[1,10^m]$, $m=23,\ldots,31$. Moreover, the latter converges faster to the miminum ETA and WLS estimators than the LL2 estimator, at least over the displayed fixed ranges.

Table 1. GB fit for twin primes up to 10^m: ETA versus WLS criterion

	parameters		ETA GoF measures				WLS GoF measures			
m=	WLS	ETA	LL1	LL2	WLS	**ETA**	LL1	LL2	**WLS**	ETA
5	0.802466	0.793906	21.86	**19.00**	16.18	**15.88**	33992	**23218**	**17737**	17959
6	0.837296	0.825771	16.57	**11.05**	8.925	**8.652**	13965	**5874**	**2778**	3369
7	0.858052	0.861267	15.26	**9.532**	8.659	**8.582**	8361	**3031**	**1977**	2040
8	0.882252	0.885525	17.14	**8.327**	3.601	**3.586**	7136	**1665**	**260.3**	346.8
9	0.897119	0.896164	17.69	**7.746**	1.645	**1.637**	5852	**1098**	**48.51**	57.92
10	0.907801	0.907706	17.39	**6.635**	1.205	**1.204**	4534	**634.9**	**20.26**	20.38
11	0.916659	0.917041	17.27	**5.726**	1.350	**1.262**	3643	**380.6**	**19.92**	22.19
12	0.923951	0.924460	16.92	**4.601**	1.166	**1.106**	2929	**204.5**	**12.90**	17.72
13	0.930114	0.930510	16.56	**3.483**	1.121	**1.055**	2383	**99.73**	**9.947**	13.40
14	0.935324	0.935669	16.11	**2.359**	1.139	**1.072**	1929	**38.14**	**8.849**	11.89
15	0.939820	0.940123	15.58	**1.308**	1.139	**1.073**	1564	**10.85**	**7.658**	10.36
16	0.943730	0.943992	14.98	**1.066**	1.134	**1.066**	1265	**8.945**	**6.640**	8.945

Table 2. GB fit for first digits of twin primes with Hardy-Littlewood's approximation

	parameters		ETA GoF measures				WLS GoF measures			
m=	WLS	ETA	LL1	LL2	WLS	**ETA**	LL1	LL2	**WLS**	ETA
5	0.796621	0.791400	14.22	**7.841**	2.875	**2.546**	13891	**4500**	**544.9**	624.9
6	0.836692	0.835025	15.48	**8.183**	1.349	**1.325**	10841	**2992**	**77.19**	89.38
7	0.863351	0.863284	16.36	**8.129**	1.141	**1.140**	8665	**2110**	**39.08**	39.11
8	0.882316	0.882791	16.97	**7.869**	1.178	**1.127**	6927	**1447**	**30.21**	32.03
9	0.896542	0.896969	17.22	**7.276**	1.177	**1.116**	5547	**956.3**	**23.51**	25.39
10	0.907659	0.908000	17.26	**6.502**	1.167	**1.104**	4465	**608.3**	**18.55**	20.04
11	0.916597	0.916872	17.14	**5.594**	1.158	**1.095**	3609	**366.5**	**14.98**	16.16
12	0.923946	0.924448	16.90	**4.578**	1.151	**1.086**	2925	**203.3**	**12.36**	17.07
13	0.930097	0.930515	16.55	**3.470**	1.145	**1.079**	2375	**98.69**	**10.38**	14.22
14	0.935324	0.935677	16.11	**2.352**	1.141	**1.074**	1929	**38.10**	**8.840**	12.03
15	0.939820	0.940123	15.58	**1.306**	1.137	**1.069**	1565	**10.82**	**7.621**	10.33
16	0.943730	0.943993	14.98	**1.065**	1.134	**1.065**	1265	**8.948**	**6.640**	8.966
17	0.947162	0.947392	14.32	**2.012**	1.131	**1.062**	1019	**26.58**	**5.837**	7.844
18	0.950198	0.950401	13.60	**3.344**	1.129	**1.059**	815.6	**59.27**	**5.172**	6.929
19	0.952903	0.953084	12.82	**4.788**	1.126	**1.056**	647.6	**103.6**	**4.615**	6.164
20	0.95533	0.955491	11.99	**6.273**	1.125	**1.054**	508.8	**156.9**	**4.144**	5.520
21	0.957518	0.957663	11.11	**7.796**	1.123	**1.052**	394.5	**217.2**	**3.741**	4.976
22	0.959501	0.959633	10.19	**9.355**	1.121	**1.050**	300.7	**282.8**	**3.394**	4.507
23	0.961307	0.961426	**9.229**	10.95	1.120	**1.049**	**224.4**	352.6	**3.094**	4.087
24	0.962959	0.963068	**8.229**	12.57	1.119	**1.047**	**162.7**	425.4	**2.832**	3.733
25	0.964476	0.964576	**7.193**	14.22	1.118	**1.046**	**113.7**	500.5	**2.602**	3.436
26	0.965873	0.965964	**6.126**	15.92	1.113	**1.042**	**75.65**	576.9	**2.382**	3.122
27	0.967164	0.967249	**5.022**	17.67	1.115	**1.043**	**46.69**	655.1	**2.216**	2.911
28	0.968361	0.96844	**3.889**	19.44	1.115	**1.043**	**25.85**	733.7	**2.057**	2.705
29	0.969474	0.969547	**2.774**	21.24	1.114	**1.042**	**11.99**	812.6	**1.914**	2.514
30	0.970511	0.970579	**1.669**	23.06	1.113	**1.041**	**4.189**	891.7	**1.785**	2.340
31	0.97148	0.971543	**1.113**	24.90	1.113	**1.040**	**1.668**	970.6	**1.668**	2.183

3. Size-dependent generalized Benford law for twin prime squares and higher powers

The results of the preceding Section are extended to twin prime power sequences $\{p^s,(p+2)^s\}$, $p^s<10^{s\cdot m}$, for a fixed power $s=1,2,3,\ldots$, and arbitrary primes below $10^m, m\geq 5$. In the next Section, we provide analytical support for the affirmation that the first digits of twin prime powers $p^s<10^{s\cdot m}, m\geq 5$, are approximately GB distributed with size-dependent exponent of the form

$$\alpha(N,s,c)=s^{-1}\cdot\{1-(\ln(N)-\ln^c(N))^{-1}\},\quad N=10^m,\, c\in(0,1), \tag{3.1}$$

and converge asymptotically to the GB with exponent s^{-1} provided the Hardy-Littlewood conjecture is true. This extends Theorem 4.1 in Hürlimann [11] from prime powers to twin prime powers. In particular, asymptotically as the power $s\to\infty$ the sequences of twin prime powers presumably obey Benford's law. Moreover, similarly to Luque and Lacasa [14], Section 5(a), we develop from (3.1) the asymptotic twin prime counting function (4.7) (with optimal parameter $a=1$) of the same asymptotic order as the Hardy-Littlewood logarithmic integral approximation (2.6) to the twin prime counting function $\pi_2(x)$.

The extension of the results from Section 2 is first illustrated at twin prime squares with fixed $s=2$. A Table of twin prime squares count in form $I_{2\cdot m}^{2}(d),\ m=5,\ldots,16$, does not seem to be readily available, but the Hardy-Littlewood values (2.7) suffice for the present purpose. Table 3 is similar to Table 2 and holds in single precision.

Table 3. GB fit for first digit of twin prime squares with Hardy-Littlewood's approximation

	parameters		ETA GoF measures				WLS GoF measures			
m=	WLS	ETA	LL1	LL2	WLS	**ETA**	LL1	LL2	**WLS**	ETA
5	0.405660	0.403749	107.0	**106.5**	55.2	**54.42**	82626	**81811**	**27139**	27259
6	0.420091	0.419201	87.11	**86.35**	13.67	**12.82**	35818	**35204**	**928.3**	967.7
7	0.432107	0.431809	91.24	**90.34**	5.573	**5.397**	25846	**25342**	**99.41**	105.6
8	0.441621	0.441588	95.41	**94.41**	3.457	**3.443**	21087	**20648**	**24.91**	25.01
9	0.448654	0.448607	97.17	**96.08**	3.195	**3.170**	17005	**16625**	**16.29**	16.55
10	0.454141	0.454102	97.63	**96.45**	3.126	**3.101**	13747	**13415**	**12.51**	12.74
11	0.458555	0.458523	97.17	**95.90**	3.104	**3.080**	11152	**10861**	**10.11**	10.30
12	0.462188	0.462160	95.98	**94.62**	3.093	**3.069**	9072	**8816**	**8.365**	8.536
13	0.465232	0.465207	94.15	**92.70**	3.085	**3.061**	7392	**7166**	**7.046**	7.196
14	0.467819	0.467798	91.78	**90.25**	3.079	**3.056**	6025	**5825**	**6.019**	6.151
15	0.470047	0.470029	88.94	**87.32**	3.074	**3.051**	4905	**4728**	**5.203**	5.320
16	0.471985	0.471969	85.66	**83.97**	3.070	**3.048**	3983	**3827**	**4.544**	4.648
17	0.473687	0.473673	82.01	**80.23**	3.067	**3.045**	3222	**3084**	**4.002**	4.096
18	0.475194	0.475181	78.00	**76.15**	3.065	**3.043**	2591	**2469**	**3.553**	3.638
19	0.476537	0.476525	73.68	**71.75**	3.063	**3.041**	2069	**1962**	**3.175**	3.252
20	0.477741	0.477731	69.06	**67.05**	3.061	**3.039**	1636	**1542**	**2.855**	2.925
21	0.478828	0.478819	64.17	**62.09**	3.059	**3.038**	1278	**1196**	**2.581**	2.643
22	0.479814	0.479805	59.03	**56.87**	3.058	**3.036**	982.7	**911.9**	**2.345**	2.403
23	0.480712	0.480703	53.65	**51.41**	3.056	**3.035**	741.0	**680.3**	**2.140**	2.194
24	0.481533	0.481525	48.05	**45.74**	3.055	**3.034**	544.6	**493.3**	**1.961**	2.010
25	0.482287	0.48228	42.24	**39.85**	3.054	**3.034**	387.0	**344.4**	**1.803**	1.848
26	0.482982	0.482976	36.26	**33.80**	3.096	**3.076**	262.9	**228.4**	**1.706**	1.745
27	0.483624	0.483618	30.05	**27.52**	3.058	**3.037**	167.1	**140.2**	**1.544**	1.584
28	0.484219	0.484214	23.68	**21.07**	3.052	**3.032**	96.39	**76.41**	**1.430**	1.467
29	0.484773	0.484768	17.14	**14.47**	3.051	**3.031**	47.20	**33.78**	**1.330**	1.365
30	0.485289	0.485285	10.64	**8.050**	3.051	**3.031**	16.77	**9.493**	**1.241**	1.273
31	0.485772	0.485767	4.193	**3.030**	3.050	**3.030**	2.689	**1.192**	**1.161**	1.192

Here, we compare the size-dependent exponent (3.1) with $c_1 = 0.8416781$ as in (2.5), called LL1 estimator, with the size-dependent exponent $c_2 = 0.841244875548$, called LL2 estimator, that by construction matches the minimum ETA for twin prime squares over the range $[1,10^{62}]$. The ETA (resp. WLS) measures are given in the somewhat changed units of $10^{-(m+2)}$ (resp. $10^{-(m+6)}$). One observes that the LL2 estimator yields optimal size-dependent exponents that outperform uniformly the ones from the LL1 estimator over the fixed ranges $[1,10^{2\cdot m}],\ m=5,\ldots,31$.

For higher twin prime powers the convergence of the size-dependent GB with minimum ETA and WLS estimators to the GB with exponent s^{-1} is illustrated in Table 3.2. Here, the ETA (resp. WLS) GoF measures are given in units of $10^{-(m+2)}$ (resp. $10^{-(m+8)}$). Over the finite ranges $[1,10^{s\cdot m}],\ m=10,15,20,25,30,\ s=3,4,5,8$, the size-dependent minimum WLS and ETA exponents increase to the expected limiting GB exponent s^{-1}, and the fit in the WLS and ETA GoF measures becomes better as s increases.

Table 4. GB fit for first digit of higher twin prime powers

s=	3	4	5	8	3	4	5	8
m =	minimum WLS exponents				minimum ETA exponents			
10	0.302802	0.227111	0.181692	0.113561	0.302806	0.227062	0.181661	0.113549
15	0.313383	0.235042	0.188035	0.117523	0.313380	0.235021	0.188022	0.117518
20	0.318504	0.238881	0.191106	0.119442	0.318503	0.238869	0.191098	0.119439
25	0.321531	0.241150	0.192921	0.120576	0.321530	0.241143	0.192916	0.120574
30	0.323531	0.242649	0.19412	0.121325	0.323530	0.242644	0.194117	0.121324
m =	WLS GoF measures				ETA GoF measures			
10	247.27	77.480	33.030	4.7037	1.3862	0.7592	0.4857	0.1793
15	103.82	32.888	13.464	2.0493	1.3626	0.7595	0.4784	0.1821
20	57.055	18.083	7.4042	1.1271	1.3575	0.7579	0.4772	0.1816
25	36.061	11.432	4.6816	0.7127	1.3554	0.7573	0.4773	0.1816
30	24.847	7.8787	3.2265	0.4913	1.3542	0.7574	0.4769	0.1815

4. Analytical first digit counting compatibility for twin prime powers

The Tables 3 and 4 provide numerical support for the analytical approximation $\frac{I_{s\cdot m}^{s}(d)}{\pi_2^{HL}(10^{5m})} \approx P_{\alpha(10^m,s,c)}^{GB}(d)$, which holds with increased precision by growing value of m. Since $\alpha(10^m,s,c) \to s^{-1}\ (m \to \infty)$ this approximation suggests the asymptotic convergence $\frac{I_{s\cdot m}^{s}(d)}{\pi_2^{HL}(10^{m})} \to P_{s^{-1}}^{GB}(d) \quad (m \to \infty)$. With this, the relative density of the first digits of twin prime powers converges asymptotically to a GB with exponent s^{-1}. Unfortunately, a rigorous proof of this statement is not available, even conditionally on the truth of the Hardy-Littlewood conjecture. However, it is possible to support its validity through application of the first digit counting compatibility criterion introduced and applied in [11,12].

Recall its definition. Let $\{x_n\}, n \geq 1$, be an arbitrary integer sequence, and suppose that the asymptotic counting function $Q(N)$ as $N \to \infty$ of this sequence exists. Further, let $\alpha(N) \in [0,1]$ be a size-dependent exponent such that the sequence of numbers generated by the power-law density $x^{-\alpha(N)}$, has a GB first digit distribution $P_{1-\alpha(N)}^{GB}(d)$ with exponent $1-\alpha(N)$.

Definition 4.1. The generalized Benford law $P_{1-\alpha(N)}^{GB}(d)$ is *counting compatible* with the counting function $Q(N)$ if there exists a constant $c(N)$ such that the generalized Benford counting function defined by $c(N)\cdot \int_2^N x^{-\alpha(N)}dx$ is asymptotically equivalent to $Q(N)$.

Let us apply this criterion to the sequence of twin prime powers. Starting point is the asymptotic counting function (2.6) for twin primes, which give their total number in the inteval $[1,N]$, denoted by $Q(N)$. It is given by

$$Q(N) = H_2 \cdot N / \ln^2(N), \quad (N \to \infty), \quad H_2 = 2 \cdot \prod_{p\geq 3} \tfrac{p(p-2)}{(p-1)^2} = 1.320323632 . \tag{4.1}$$

Similarly, for any fixed positive integer $s \geq 1$, the number of twin prime powers p^s in the interval $[1,N^s]$, denoted by $Q_s(N^s)$, follows the same asymptotic distribution

$$Q_s(N^s) = H_2 \cdot N / \ln^2(N), \quad (N \to \infty). \tag{4.2}$$

This follows from the fact that $p^s < N^s$ if, and only if, one has $p < N$. In the notation of Definition 4.1, consider the following slightly modified parametric GB size-dependent exponent that corresponds to (3.1), namely

$$\tilde{\alpha}(N,s,c,a) = \frac{s-1+\tilde{\beta}(N,c,a)}{s}, \quad \tilde{\beta}(N,c,a) = \frac{a}{\ln(N)-\ln^c(N)}, \; a>0, \; c \in (0,1). \tag{4.3}$$

Theorem 4.1 (*Counting compatibility of the GB for twin prime powers*). For any fixed $c \in (0,1)$, any fixed positive integer $s \geq 1$, and any $m \geq 1$, set

$$\alpha(m,s,c,a) = 1-\tilde{\alpha}(10^m,s,c,a) = \frac{1}{s}\left(1-a\cdot(\ln(10^m - \ln^c(10^m))^{-1}\right). \tag{4.4}$$

Then, the generalized Benford law $P^{GB}_{\alpha(m,s,c,a)}(d),\; d=1,\ldots,9$, is counting compatible with the twin prime power counting function (4.2) if, and only if, the parameter $a=1$. More precisely, the choice of the constant

$$c(N,s) = \frac{e \cdot H_2}{s \cdot \ln^2(N)} \tag{4.5}$$

implies that the generalized Benford counting function $L_s(N^s) = c(N,s) \cdot \int_2^{N^s} x^{-\tilde{\alpha}(s,N,c,a)}dx$ is asymptotically equivalent to $Q_s(N^s) \sim H_2 \cdot N / \ln^2(N)\; (N \to \infty)$ if, and only if, one has $a=1$.

Proof. Counting compatibility holds provided the following limiting condition holds:

$$\lim_{N\to\infty} \frac{L_s(N^s)}{H_2 \cdot N / \ln^2(N)} = 1. \tag{4.6}$$

Using (4.4) one obtains the equivalent asymptotic formula

$$\begin{aligned} L_s(N^s) &\sim \frac{e\cdot H_2}{\ln^2(N)\cdot s\cdot(1-\tilde{\alpha}(N,s,c,a))} N^{s\cdot(1-\tilde{\alpha}(N,s,c,a))} = \frac{e\cdot H_2}{\ln^2(N)\cdot(1-\tilde{\beta}(N,c,a))} N^{1-\tilde{\beta}(N,c,a)} \\ &= \frac{H_2\cdot N}{\ln^2(N)} \cdot \frac{\ln(N)-\ln^c(N)}{\ln(N)-\ln^c(N)-a} \cdot \exp\left\{-\frac{(a-1)\ln(N)+\ln^c(N)}{\ln(N)-\ln^c(N)}\right\} \end{aligned} \tag{4.7}$$

Clearly, the factor

$$f_N(a,c) = \frac{L_s(N^s)}{H_2\cdot N/\ln^2(N)} \sim \frac{\ln(N)-\ln^c(N)}{\ln(N)-\ln^c(N)-a} \cdot \exp\left\{-\frac{(a-1)\ln(N)+\ln^c(N)}{\ln(N)-\ln^c(N)}\right\}$$

converges to 1 as $N \to \infty$ for any fixed $c \in (0,1)$ if, and only if, one has $a=1$, and in this case counting compatibility holds. Moreover, the form (4.4) of the GB exponent in Definition 4.1 follows by setting $N = 10^s$ in Equation (4.3). The result is shown. ◊

Good values of $c \in (0,1)$ can be obtained through optimization. As an example, the size-dependent exponent (4.3) with $c = 0.8416781$ in (2.5) does the job. As shown in Table 2, this estimator is reasonable over the finite ranges of twin primes $[1,10^m]$, $m=5,\ldots,31$. No attempt has been made to find similar good values of $c\in(0,1)$ for twin prime powers higher than twin prime squares in Section 3.

Appendix: Tables of first digit distributions for the first prime in twin prime pairs

Based on the recursive relation (2.3), the computation of $I_m^1(d)$, $m=5,\ldots,16$, is straightforward by using the Table from Nicely [18]. These numbers are listed in Table 5.

Table 5. First digit distribution of twin primes up to 10^k, $k=5,\ldots,16$

k sample size	5 1'224	6 8'169	7 58'980	8 440'312	9 3'424'506	10 27'412'679	11 224'376'048
/ first digit							
1	172	1'108	7'810	56'237	429'296	3'392'831	27'489'251
2	151	985	7'046	52'531	405'640	3'227'743	26'274'262
3	148	958	6'886	50'747	392'000	3'126'294	25'527'383
4	141	902	6'505	48'853	381'290	3'055'018	25'001'993
5	128	894	6'347	47'804	373'935	3'000'178	24'590'893
6	116	846	6'189	47'097	367'664	2'953'416	24'254'048
7	116	821	6'180	46'164	362'047	2'916'062	23'976'946
8	129	835	6'084	45'724	358'235	2'885'269	23'739'770
9	123	820	5'933	45'155	354'399	2'855'868	23'521'502

k sample size	12 1'870'585'220	13 15'834'664'872	14 135'780'321'665	15 1'177'209'242'304	16 10'304'195'697'298
/ first digit					
1	227'197'856	1'909'383'579	16'273'581'482	140'351'660'071	1'222'900'721'441
2	218'075'309	1'839'065'151	15'718'887'019	135'901'489'797	1'186'660'986'967
3	212'459'401	1'795'530'692	15'374'094'333	133'127'936'873	1'164'011'766'240
4	208'406'589	1'764'067'516	15'125'101'703	131'120'433'445	1'147'594'079'302
5	205'285'512	1'739'634'993	14'931'051'942	129'553'790'751	1'134'760'621'160
6	202'731'495	1'719'763'349	14'772'776'796	128'272'594'921	1'124'253'391'604
7	200'581'005	1'702'963'537	14'638'994'161	127'190'543'411	1'115'373'873'144
8	198'729'069	1'688'474'319	14'523'564'595	126'256'201'836	1'107'697'639'212
9	197'118'984	1'675'781'736	14'422'269'634	125'434'591'199	1'100'942'618'228

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