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The behavior of the zeros of analytic functions of finite quantum systems with physical Hamiltonians

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Abstract

The paper contains an investigation of the behavior of the Zeros of analytic theta functions. We have considered briefly an analytic representation of finite quantum systems Z_N. We note that analytic functions on a torus have exactly N zeros. A brief introduction to the zeros of analytic functions and their time evolution is given. We discuss the periodic finite quantum systems. Then we introduce the time evolution of finite quantum systems with the evolution operators which reflect physical significance. The general ideas are demonstrated with several examples.

1 Introduction

This Paper is devoted to the study behavior of paths of zeros in analytic representation of finite quantum systems on a torus. Analytic functions are very important tools in several branches of physical sciences. Refs.[1, 2, 3] studied analytic functions and used them widely in quantum mechanics. The analytic Bargmann function [4, 5, 6, 7, 8, 9, 10] is important in studying the overcompleteness of the coherent states. Refs [11, 12] has studied analytic representations of finite quantum systems on a torus. The analytic function representing a quantum state has exactly N zeros which define uniquely the quantum state. Ref [13] studied the motion of the N zeros on the torus. In the present paper we introduce special evolution operators and thier time evolution. We present the eigenvalues of the $d\times d$ matrices H_A, H_B, H_C . The path of zeros are functions of time. The path of this motion is a curve as long as functions $x(t)$ and $y(t)$ are continuous. We demonstrate these general ideas with various concrete examples.

2 Analytic representation of finite quantum systems

Let $\mathbb{H}$ be a d-dimensional Hilbert space. Let $|X_m\rangle, |P_m\rangle$, where m is the integer modulo n, be an orthonormal basis in this Hilbert space (position states and momentum states respectively). and

$$|P_m\rangle = \mathbb{F}|X_m\rangle = N^{-1/2}\sum_n (\exp\left[i\frac{2\pi m}{N}\right])|X_m\rangle, \tag{1}$$

and $\mathbb{F}$ is the Fourier operator given by

$$\mathbb{F} = N^{-1/2}\sum_{m,n}(\exp\left[i\frac{2\pi m}{N}\right])|X_m\rangle\langle X_n|. \tag{2}$$

Let x, p be the position and momentum operators and that they are given by

$$x = \sum_{n=0}^{N-1} n|X_n\rangle\langle X_n, \tag{3}$$

$$p \quad = \quad \mathbb{F}x\mathbb{F}^{\dagger} = \sum_{n=0}^{N-1} n|P_n\rangle\langle P_n| \tag{4}$$

We study an arbitrary normalized state $|\mathbb{F}\rangle$, such that

$$|\mathbb{F}\rangle = \sum_m \mathbb{F}_m|X_m\rangle; \qquad \sum_m |\mathbb{F}_m|^2 = 1, \tag{5}$$

Refering to ref[13] we represent the state $|\mathbb{F}\rangle$ in Eq.(5), with the analytic function

$$f(z) = \pi^{-1/4} \sum_{m=0}^{N-1} \mathbb{F}_m \vartheta_3[\pi m N^{-1} - \sqrt{\frac{z\pi}{2N}}; \frac{i}{N}] \tag{6}$$

which obeys the quasi-periodic relations

$$\begin{aligned} f\left[z+\sqrt{2\pi N}\right] &= f(z) \\ f\left[z+i\sqrt{2\pi N}\right] &= f(z)\exp\left[\pi N - i\sqrt{2\pi N}z\right], \end{aligned} \tag{7}$$

where ϑ_3 is Theta function defined as

$$\vartheta_3(u,\tau) = \sum_{n=-\infty}^{\infty} \exp(i\pi\tau n^2 + i2nu). \tag{8}$$

The analytic function $f(z)$ is defined on a cell $[a, a+\sqrt{2\pi N}) \times [b, b+\sqrt{2\pi N})$ (defined on a torus)
Example 1 below gives a simple related demonstration

Example 1

We consider the case where $N = 3$ and the state $|\mathbb{F}(0)\rangle$ at $t = 0$ is described through the coefficients

$$\begin{aligned} \mathbb{F}_0(0) \quad &= \quad 0.08 - 0.24i,\ \mathbb{F}_1(0) = 0.52 + 0.45i, \\ \mathbb{F}_2(0 \quad) &= \quad 0.55 + 0.37i. \end{aligned} \tag{9}$$

In Fig.1 we plot the real part of the function $f(z)$in Eq.(6).

3 Zeros of the functions $f(z)$

Ref.[12] has proved that the sum of the zeros μ_n of $f(z)$, is gives as

$$\sum_{n=1}^{N} \mu_n = (2\pi)^{1/2}N^{3/2}(l+ir) + \left(\frac{\pi}{2}\right)^{1/2} N^{3/2}(1+i) \tag{10}$$

Again, according to refs.[12, 13] we construct the function $f(z)$ from its zeros μ_n which satisfy the relation of Eq.(10) as follows

$$\begin{aligned} f(z) \quad &= \quad q\exp\left[-i\left(\frac{2\pi}{N}\right)^{1/2} lz\right] \prod_{n=1}^{N} \vartheta_3\left[w_n(z);\ i\right] \\ w_n(z) \quad &= \quad \left(\frac{\pi}{2N}\right)^{1/2}(z-\mu_n) + \frac{\pi(1+i)}{2} \end{aligned} \tag{11}$$

where l is the integer relation of Eq.10; and q is fixed calculated from the normalization condition.
From ref.[13] the coefficients $\mathbb{F}_m$ were calculated from $f(z)$ as following.

4 paths of the zeros

Following ref.[13] we consider the state $|\mathbb{F}(0)\rangle = \sum \mathbb{F}_m(0)|X;m\rangle$ at $t=0$. Using the Hamiltonian H, the state $|\mathbb{F}(0)\rangle$ evolves with time t as:

$$|\mathbb{F}(t)\rangle = \exp(ith)|\mathbb{F}(0)\rangle = \sum_{m=0}^{N-1} \mathbb{F}_m(t)|X_m\rangle \tag{12}$$

Example 2

We consider the case where $N=3$ and that the state $|\mathbb{F}(0)\rangle$ at $t=0$ is described through the coefficients

$$\begin{aligned} \mathbb{F}_0(0) &= 0.9 - 0.008i, \quad \mathbb{F}_1(0) = 0.3 + 0.004i, \\ \mathbb{F}_2(0) &= 0.3 + 0.003i. \end{aligned} \tag{13}$$

We have calculate the coefficients $|\mathbb{F}(t)\rangle$ for the two cases of the Hamiltonians H_1, H_2

$$\begin{aligned} &H_1 = \frac{x^2}{2} + \frac{p^2}{2}, \\ &H_2 = -i\ln\left[\exp\left(\frac{ix^2}{2}\right)\exp\left(\frac{ip^2}{2}\right)\right]. \end{aligned} \tag{14}$$

Using MATLAB we calculate numerically the zeros μ_n of $f(z)$. In Fig.2 we present the three curves μ_n for the first Hamiltonian H_1 (dotted line), and the second Hamiltonian H_2(solid line) in Eq.14.

5 Periodicity of the zeros

Ref.[13] has discussed the Periodic finite quantum systems. In some cases d of the zeros follow the same path.Thence that we say this path has multiplicity d .

Example 3

Let

$$\begin{aligned} \mu_0(0) &= 1.37 + 2.29i; \; \mu_1(0) = 2.17 + 2.34i, \\ \mu_2(0) &= 3.02 + 1.94i \end{aligned} \tag{15}$$

be the zeros at $t=0$ and let

$$H = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{16}$$

be the Hamiltonian with eigenvalues $0, 1, 2$ with a period $\alpha = 2\pi$. Numerically we get

$$\mu_1(\alpha + t) = \mu_2(t), \quad \mu_2(\alpha + t) = \mu_1(t), \tag{17}$$

In this case and after one period the μ_1, μ_2 follow the same path and after another period they exchange position while μ_3 follows a closed path; i.e.

$$\mu_1(\alpha) = \mu_2(0), \;\; \mu_2(\alpha) = \mu_1(0), \;\; \mu_0(\alpha) = \mu_0(0), \tag{18}$$

In Fig.3 we plot the paths of these zeros.

Example 4

Let

$$\begin{aligned}\mu_0(0) &= 0.7+2.6i,\ \mu_1(0)=2.1+4.3i,\\ \mu_2(0) &= 3.7+1.1i,\ \mu_3(0)=3.7+2.2i,\end{aligned} \tag{19}$$

be the zeros at $t=0$ and let

$$H = \begin{bmatrix} 1 & -i & 0 & 0\\ i & 1 & 0 & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 2\end{bmatrix} \tag{20}$$

be the Hamiltonian with eigenvalues $0,2,2,2$. Using Matlab we calculate the paths of the zeros to find that

$$\begin{aligned}\mu_0(\alpha+t) &= \mu_1(t),\ \mu_1(\alpha+t)=\mu_2(t),\\ \mu_2(\alpha+t) &= \mu_3(t),\ \mu_3(\alpha+t)=\mu_0(t)\end{aligned} \tag{21}$$

In this example all the zeros follow the same path and after a period of $\alpha=\pi$ we get that

$$\begin{aligned}\mu_0(\alpha) &= \mu_1(0),\ \mu_1(\alpha)=\mu_2(0),\\ \mu_2(\alpha) &= \mu_3(0),\ \mu_3(\alpha)=\mu_0(0).\end{aligned} \tag{22}$$

In Fig.4 we plot the paths of this zeros.

Hence by definition the zeros μ_1,μ_2 in example.3 have a multiplicity $d=2$ and the four zeros in example.7 have a multiplicity $d=4$.

6 The behavior of the zeros of analytic functions of systems with physical Hamiltonians

Let Ω_i (where $i=0,...,d-1$) be the eigenvalues of the Hamiltonian H of the system. If the ratios Ω_i/Ω_0 are rational numbers then the system is periodic. In this case the d paths of the zeros $\zeta_n(t)$ are in general closed curves on the torus. In some cases $\mathfrak{M}$ of the zeros follow the same path(Ref. [13]). We say that this path has multiplicity $\mathfrak{M}$. Below we give examples with $\mathfrak{M}=2,3$. We also discuss how a perturbation of the initial values of the zeros splits a path with multiplicity $\mathfrak{M}$ into $\mathfrak{M}$ different paths. We have seen that in the special case of periodic systems the zeros follow closed paths. We have also studied the case of paths with multiplicity $\mathfrak{M}$ and shown that the zeros obey relations of type (17)and(21). After one period the zeros exchange their positions (Eqs(18),(22)).

Now, flowing ref. [13] we study the time evolution of such systems with the evolution operators

$$\begin{aligned}U_A(t) &= \exp(itH_A);\quad H_A=\frac{x^2}{2}+\frac{p^2}{2}\\ U_B(t) &= \exp(itH_B);\quad H_B=-i\ln\left[\exp\left(\frac{ix^2}{2}\right)\exp\left(\frac{ip^2}{2}\right)\right]\\ U_C(t) &= \exp(itH_C);\quad H_C=FH_BF^\dagger=-i\ln\left[\exp\left(\frac{ip^2}{2}\right)\exp\left(\frac{ix^2}{2}\right)\right]\end{aligned} \tag{23}$$

All these evolution operators are the analogues of a harmonic oscillator evolution operator with Hamiltonian $H=\frac{1}{2x^2}+\frac{1}{2p^2}$. In the present context there is no analogous formula to the Baker-Campbell-Hausdorff relation, and therefore there is no simple relation between them.

Below we present the eigenvalues of the $d \times d$ matrices H_A, H_B, H_C. Note that H_B, H_C have the same eigenvalues because they are related with a unitary transformation.

Mathematically, the eigenvalues of H_B, H_C are defined modulo $2\pi N$, because there is a multivaluedness associated with the logarithms in H_B, H_C. Physically, the eigenvalues of H_B should be close to the eigenvalues of H_A, because in the semiclassical limit these Hamiltonians are the same. This physical requirement defines which of the logarithms should be chosen.

6.1 Eigenvalues of the Hamiltonians H_A, H_B, H_C

As in ref. [13] we calculate the eigenvalues of the Hamiltonians H_A, H_B, H_C for the case $d = 5$. The calculation of the eigenvalues A_i of H_A (labelled in ascending order) is straightforward and the results are presented in table 1. The eigenvalues B_i of H_B are defined modulo $2\pi N$, but as we explained we chose the ones which are close to the eigenvalues A_i of H_A, because in the semiclassical limit the corresponding Hamiltonians are the same.

In order to do this, we express A_i as

$$A_i = A_i' + 2\pi N_i \ (\text{mod } 2\pi); \qquad 0 \le A_i' < 2\pi \tag{24}$$

We then calculate the eigenvalues of the matrix $\exp(iH_B) = \exp\left(\frac{ix^2}{2}\right)\exp\left(\frac{ip^2}{2}\right)$ which are $\mathcal{B}_i = \exp(iB_i)$. From $\mathcal{B}_i$ we calculate the values of $B_i' = -i \ln \mathcal{B}_i$ such that $0 \le B_i' < 2\pi$ (the B_i' are labelled in ascending order). We then add $2\pi N_i$ (calculated in Eq.(24)) to B_i' and hence In table 1, we present the B_i which, as we explained, are eigenvalues of both H_B, H_C.

H_A	H_B, H_C
12.82	12.90
8.15	8.46
5.17	4.44
2.87	3.41
0.96	0.77

Table 1: The eigenvalues of H_A, H_B, H_C.

Example 5

We consider the Hamiltonian H_B

Let $\zeta_0(t), \zeta_1(t), \zeta_2(t)$ be the paths of the three zeros. We assume that at $t = 0$, we have that

$$\begin{aligned} \mu_0(0) &= 2.18 + 2.15i, \ \mu_1(0) = 2.18 + 2.22i, \\ \mu_2(0) &= 2.19 + 2.19i, \end{aligned} \tag{25}$$

These zeros obey the constraint of Eq.(10) and they are on a torus (i.e., they are defined modulo $(6\pi)^{1/2}$). In Fig.7 we present the paths of these zeros $\mathfrak{z}_0(t)$, $\mathfrak{z}_1(t)$, $\mathfrak{z}_2(t)$.
In the first figure, during one period the three zeros seem to followi a closed path. In fact these zeros do not follow a closed path,actually they do not come to their original position after one period , they come to a position very close to the original position. In the second figure the three zeros still do not follow a closed path, these zeros do not come to their original position after 4 period, they come to position very closed to the original position. In the third and fourth figures the three zeros still do not follow closed path. In this case the 3 paths of the zeros $\zeta_n(t)$ are not closed curves on the torus, and this system is not periodic.

Example 6

Now consider the Hamiltonian H_C, and

Let $\zeta_0(t), \zeta_1(t), \zeta_2(t)$ be the paths of the three zeros. We assume that at $t=0$ we have

$$\begin{aligned} \mu_0(0) &= 2.18 + 2.15i,\ \mu_1(0) = 2.18 + 2.22i, \\ \mu_2(0) &= 2.19 + 2.19i, \end{aligned} \tag{26}$$

These zeros obey the constraint of Eq.(10) and they are on a torus (i.e., they are defined modulo $(6\pi)^{1/2}$). In Fig.7 we present the path of these zeros $\mathfrak{z}_0(t)$, $\mathfrak{z}_1(t)$, $\mathfrak{z}_2(t)$.
In the first figure, during a period the three zeros seem to following closed path. In fact these zeros do not follow closed path, these zeros do not come to their original position after 1 period , they come to position very closed to the original position. In the second figure the three zeros still do not follow closed path. In this case the 3 paths of the zeros $\zeta_n(t)$ are not closed curves on the torus, and this system is not periodic.

Example 7

Consider the Hamiltonian H_A, and

Let $\zeta_0(t), \zeta_1(t), \zeta_2(t)$ be the paths of the three zeros. We assume that at $t=0$, we have

$$\begin{aligned} \mu_0(0) &= 2.18 + 2.15i,\ \mu_1(0) = 2.18 + 2.22i, \\ \mu_2(0) &= 2.19 + 2.19i, \end{aligned} \tag{27}$$

These zeros obey the constraint of Eq.(10) and they are on a torus (i.e., they are defined modulo $(6\pi)^{1/2}$). In Fig.7we present the path of these zeros $\mathfrak{z}_0(t)$, $\mathfrak{z}_1(t)$, $\mathfrak{z}_2(t)$.
The zeros do not follow closed paths. In this case the 3 paths of the zeros $\zeta_n(t)$ are not closed curves on the torus, and this system is not periodic.

7 Conclusion

The analytic representation of finite quantum systems have been studied. The zeros of analytic theta function and their time evolution have been considered. special examples which relect physical meaning, have been encountered, and where the paths of various zeros for selected Hamiltonians were calculated and plotted. A brief discussion to the Periodicity of the zeros has been given. It all depends on the ratios Ω_i/Ω_0 of the eigenvalues of the Hamiltonian of the system. Not that the special Hamiltonians H_A, H_B, H_C we see that The H_B and H_C have the same eigenvalues. This because they are related with a unitary transformation.
We should conclude that our study so far was dealt with a torous. In a future study we will study analytic functions on a unit disc.

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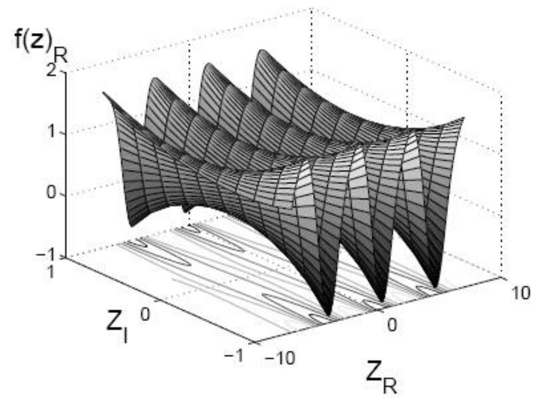

Figure 1: The real part of the function $f(z)$ in Eq.(6) where $N = 3$ and the $|F(t)\rangle$ at $t = 0$ is described through the coefficients in Eq.(13).

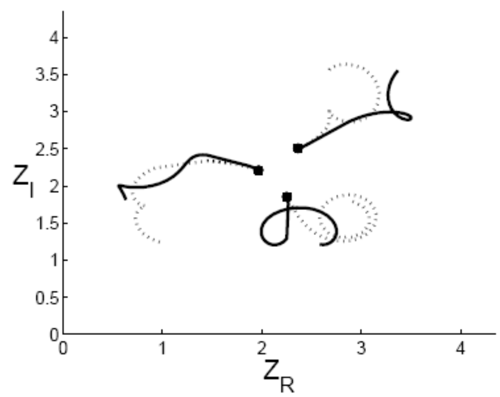

Figure 2: The distribution of the zeros $\mu_n(t)$ for the state $|\mathbb{F}(t)\rangle$ which at $t = 0$ is described in Eq.(13) for Hamiltonian H_1 (dotted line)and H_2 (solid line) of Eq.(14).

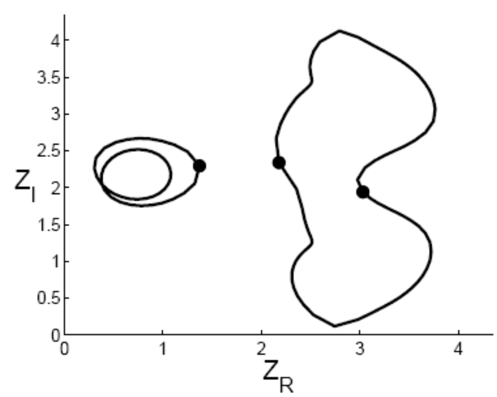

Figure 3: The path of the zeros μ_0, μ_1, μ_2 with the Hamiltonian of Eq.16. The initial values of the zeros are given in Eq.15

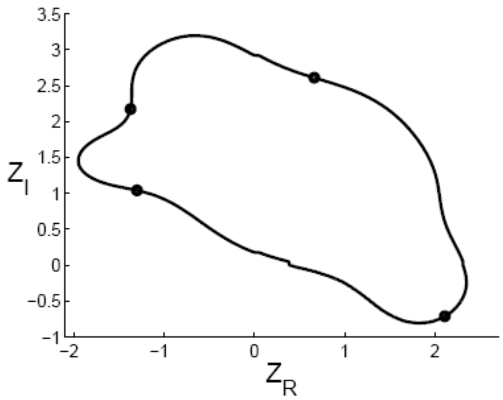

Figure 4: The path of the zeros $\mu_0, \mu_1, \mu_2, \mu_3$ for the Hamiltonian of Eq.20. The initial values of the zeros are given in Eq. 27

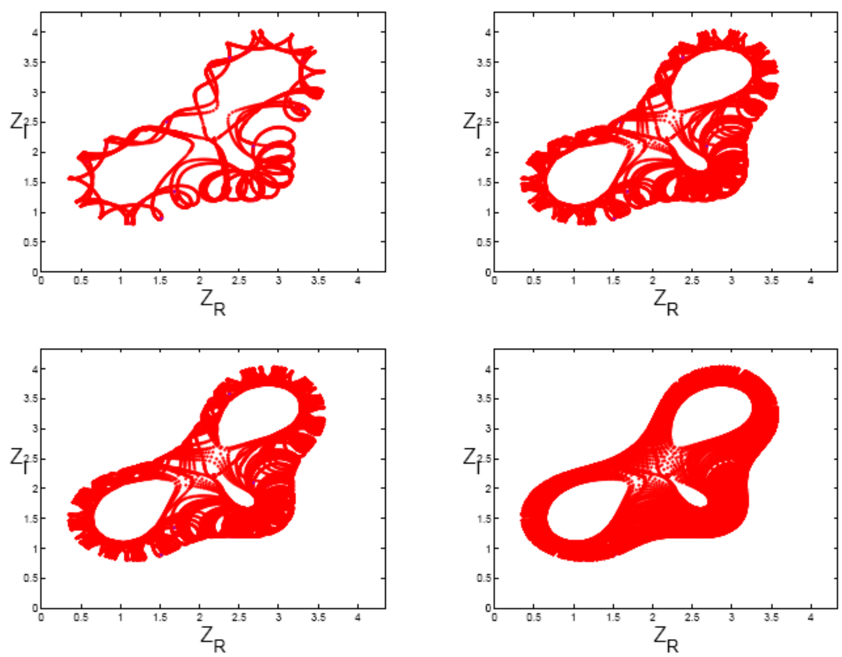

Figure 5: The paths of the zeros $\mathfrak{z}_0(t)$, $\mathfrak{z}_1(t)$, $\mathfrak{z}_2(t)$ for the system with the Hamiltonian H_B after 1 period, after 4 periods, after 8 periods and after 10 periods. At $t = 0$ the zeros $\mathfrak{z}_0(0)$, $\mathfrak{z}_1(0)$, $\mathfrak{z}_2(0)$ are given in Eq.(27)

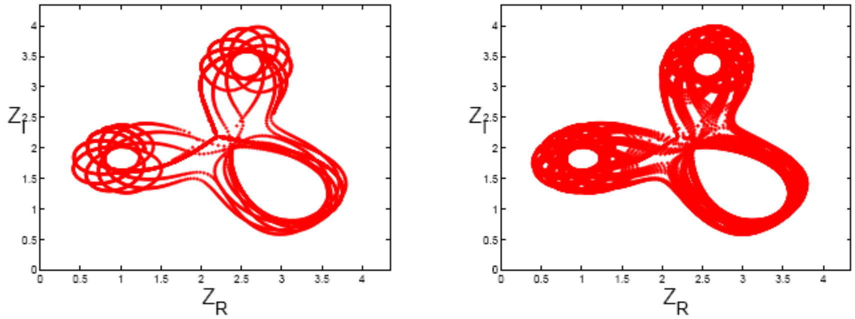

Figure 6: The paths of the zeros $\mathfrak{z}_0(t)$, $\mathfrak{z}_1(t)$, $\mathfrak{z}_2(t)$ for the system with the Hamiltonian H_C after 1 period and after 8 periods. At $t=0$ the zeros $\mathfrak{z}_0(0)$, $\mathfrak{z}_1(0)$, $\mathfrak{z}_2(0)$ are given in Eq.(27)

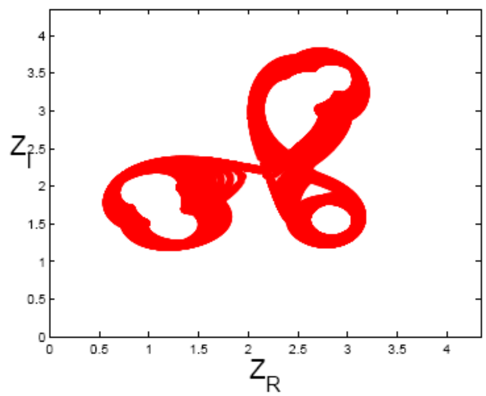

Figure 7: The paths of the zeros $\mathfrak{z}_0(t)$, $\mathfrak{z}_1(t)$, $\mathfrak{z}_2(t)$ for the system with the Hamiltonian H_A after 8 periods. At $t=0$ the zeros $\mathfrak{z}_0(0)$, $\mathfrak{z}_1(0)$, $\mathfrak{z}_2(0)$ are given in Eq.(27)