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Soft Real Analysis

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Abstract

The Soft Real number is a parameterized collection real numbers. And by this relation, every properties of real numbers can be discussed in soft real numbers. In this paper, we introduce the operations on soft real numbers and define countable and uncountable soft real sets. Also, some concepts of real numbers such as(upper bound, lower bound, supermum and infimum) are introduced.

Keywords: Soft sets; Soft real sets; Soft real numbers; countable soft sets; Soft supermum and Soft infimum.

1 Introduction

During our studying soft metric function using soft elements in [4], we had used the notion of soft real numbers in the definition of soft metric space. The soft real numbers were originally introduced by Das and S.K. Samanta in [3]. In this paper, we study the soft real analysis and the operations on soft real sets are also given. On the other hand, many concepts as Infimum, Supremum, Countable and uncountable soft sets are introduced and their properties are studied.

2 Preliminaries:

In this section, The basic definitions and results of soft set theory which will be needed in the sequel are presented, [1, 6, 7, 8]. Throughout this study, X refers to an initial universe $P(X)$ is the power set of X, E is a set of parameters and $A \subseteq E$.

Definition 2.1. *([6, 7, 8]) A soft set F_A on the universe X is defined as a set of ordered pairs*

$$F_A = \{(e, F_A(e)) : e \in E, F_A(e) \in P(X)\},$$

where $F_A : E \to P(x)$, such that $F_A(e) \neq \phi \; \forall e \in A$ and $F_A(e) = \phi$ if $e \notin A$, and A is called the support of F_A.
The collection of all soft sets with support A is denoted by $P(X)^E_A$.
For simplicity some times, we will use notation F instead of F_A to denote the soft set.

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Definition 2.2. *([7]) Let $F_A = (F, A)$ and $G_B = (G, B)$ be two soft sets over X*
(1) (F, A) is called a soft subset of (G, B) denoted $(F, A)\tilde{\subseteq}(G, B)$ if $F(e) \subseteq G(e)\forall, e \in A, A \subseteq B \subseteq E$.
(2) (F, A) and (G, B) are called equals if $(F, A)\tilde{\subseteq}(G, B)$ and $(G, B)\tilde{\subseteq}(F, A)$

Definition 2.3. *([7]) The union of two soft sets F_A and G_B over the common universe X denoted $F_A\tilde{\cup}G_B$ is the soft set $H_C = (H, C)$ where $C = A \cup B$ and $\forall c \in C$*

$$H(c) = \begin{cases} F(e) & if \;\; e \in A - B \\ G(e) & if \;\; e \in B - A \\ F(e) \cup G(e) & if \;\; e \in A \cap B \end{cases}$$

Definition 2.4. *([7]) The intersection of two soft sets $F_A = (F, A)$ and $G_B = (G, B)$ over the common universe U denoted by $F_A\tilde{\cap}G_B$ is the soft set $H_C = (H, C)$, where $C = A \cap B$ and $H(e) = F(e) \cap G(e)\ \forall e \in C$.*

Definition 2.5. *([7]) The difference (H, A) of two soft sets (F, A) and (G, A) over X denoted by $(F, A)\tilde{-}(G, A)$ is defined by $H(e) = F(e) - G(e)\ \forall e \in A$.*

Definition 2.6. *([7]) The complemmaent of a soft set $(F, A) = F_A\tilde{\in}P(X)^E_A$ is denoted by $(F_A)^c\tilde{\in}P(X)^E_A$ where $(F_A)^c : E \to P(X)$ is a function given by $F^c(e) = X - F(e)\ \forall e \in E$, clearly $((F_A)^c)^c = F_A$.*

Definition 2.7. *([7]) A soft set F_A over X is called a null soft set with support A, denoted by $\tilde{\Phi}_A$ if $F(e) = \phi\ \forall e \in A$ and is called absolute soft set with support A denoted By $\tilde{X}_A$ if $F(e) = X\ \ \forall e \in A, A \subseteq E$ clearly $(\tilde{\Phi})^c = \tilde{X}$ and $(\tilde{X})^c = \tilde{\Phi}$.*

Theorem 2.1. *([7],[1]) Let I be an index set and $F_A, G_B, H_C, F_{iA}, G_{iB}\tilde{\in}P(X)^E_E\ \forall i \in I$*
Then we have the following propositionerties :
(1) $F_A\tilde{\cup}(G_B\tilde{\cup}H_C) = (F_A\tilde{\cup}G_B)\tilde{\cup}H_C$,
$F_A\tilde{\cap}(G_B\tilde{\cap}H_C) = (F_A\tilde{\cap}G_B)\tilde{\cap}H_C$,
(2) demorgan's laws:
$(F_A\tilde{\cup}G_B)^c = (F_A)^c\tilde{\cap}(G_B)^c$,
$(F_A\tilde{\cap}G_B)^c = (F_A)^c\tilde{\cup}(G_B)^c$,
(3) $F_A\tilde{\cap}(\tilde{\cup}_{i\in I}G_{iB}) = \tilde{\cup}_{i\in I}(F_A\tilde{\cap}G_{iB})$,
(4) $(\tilde{\cap}_{i\in I}F_{iA})^c = \tilde{\cup}_{i\in I}(F_{iA})^c$,
(5) If $F_A\tilde{\subseteq}G_B$ then $(G_B)^c\tilde{\subseteq}(F_A)^c$,
(6) $F_A\tilde{\cup}F_A = F_A, F_A\tilde{\cap}F_A = F_A$,
(7) $F_A\tilde{\cup}\tilde{\Phi} = F_A, F_A\tilde{\cap}\tilde{\Phi} = \tilde{\Phi}$,
(8) $F_A\tilde{\cup}\tilde{X} = \tilde{X}, F_A\tilde{\cap}\tilde{X} = F_A$,
(9) $F_A\tilde{\cup}(F_A)^c = \tilde{X}, F_A\tilde{\cap}(F_A)^c = \tilde{\Phi}$.

Definition 2.8. *([3],[4]) A soft element with support A is a soft set such that $F(e) = \{x_e\}$ is a singleton set $\ \forall e \in A$. The collection of all soft elements is denoted by $[X]^E_A$ where $[X] = \{\{x\} : x \in X\} \cup \{\phi\}$. The collection of all soft elements with support A is sometimes denoted by $S_{el}(X, A)$.*

Definition 2.9. *([10]) A soft singleton with support A is a soft set with support A such that $\exists x \in X, F(e) = \{x\} \forall e \in A$, $E(e) = \phi$ for $e \notin A$. The collection of all soft singletons on X with support A is denoted by $\{X\}_A^E$ and sometimes denoted by $S_s(X, A)$.*

Definition 2.10. *([4]) A soft point is a soft set with a singleton support and a singleton image such that $F(e) = \{x\}$, for some $e \in E$ and is denoted by P_e^x*
The collection of all soft points denoted by $E \times X$ or $S_p(X, E)$.

Definition 2.11. *([10]) A soft member is a soft set with singleton support and is denoted by $(e, F(e))$. The collection of all soft member is denoted by $P(X)_e^E$ or $S_m(X, E)_e$.*

For simplicity, we will use the notations $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\lambda}, \tilde{\mu}$, to denote the soft elements, for example

$$\tilde{\alpha} = \{(e, \{x_e\}), e \in A\} \tilde{\in} [X]_A^E.$$

Definition 2.12. *([10]) A soft element $\tilde{\alpha} = (e, \{x_e\}, e \in A)$, is said to belongs to a soft set F_A, denoted by $\tilde{\alpha} \tilde{\in} F_A$, if $\{x_e\} \subset F(e) \; \forall e \in A \subseteq E$.*

Lemma 2.1. *([10]) For any two soft sets F_A and $G_A, F_A \tilde{\subset} G_A$ iff $\tilde{\alpha} \tilde{\in} F_A \Rightarrow \tilde{\alpha} \tilde{\in} G_A$ for any soft element $\tilde{\alpha}$ and hence $F_A = G_A$ iff $\tilde{\alpha} \tilde{\in} F_A \Leftrightarrow \tilde{\alpha} \tilde{\in} G_A$, where $\tilde{\in}$ denote that $\tilde{\alpha}$ belongs to F_A as a soft subset.*

Proposition 2.1. *([10]) (i) Every soft subset $(F, A), A \subset E$ can be considered as a union of its soft elements. Specially it can be considered as a union of soft elements all with the same support A.*
(ii) Every soft element is a union of a collection of one point support soft elements.

Definition 2.13. *([10]) Consider the collection of all soft elements of X with support equal to or contained in E. A mapping $d_S : [X]_{\leq E}^E \times [X]_{\leq E}^E \to [R^+]_E^E$ is said to be soft metric or distance on the soft set (X, E) if d_S satisfies the following conditions*
(d_1) $d_S(\tilde{\alpha}, \tilde{\beta}) \tilde{\geq} \tilde{0} \; \forall \tilde{\alpha}, \tilde{\beta} \tilde{\in} [X]_{\leq E}^E$,
(d_2) $d_S(\tilde{\alpha}, \tilde{\beta}) = \tilde{0}$ *iff* $\tilde{\alpha} = \tilde{\beta}$
(d_3) $d_S(\tilde{\alpha}, \tilde{\beta}) = d_S(\tilde{\beta}, \tilde{\alpha}) \; \forall \tilde{\alpha}, \tilde{\beta} \tilde{\in} [X]_{\leq E}^E$,
(d_4) $d_S(\tilde{\alpha}, \gamma) \tilde{\leq} d_S(\tilde{\alpha}, \tilde{\beta}) \oplus d_S(\tilde{\beta}, \tilde{\gamma}) \; \forall \; \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} \tilde{\in} [X]_{\leq E}^E$.

3 Operations on Soft Real Numbers

In this section, the definitions of soft real sets and soft real numbers are presented and an operations on soft real numbers is introduced.
First we recall the definition of soft real sets:

Definition 3.1. *[3] A soft real set is a parameterized collection of bounded subsets of real numbers, $P_b(R)$. The collection of all soft real sets is denoted by $P_b(R)_A^E$.*

Definition 3.2. *By a soft real number we mean a soft element of real numbers with support A which is a parametrized collection of real numbers denoted by $\tilde{r}$ i.e. $\tilde{r}\tilde{\in}[R]_A^E$ and is called a soft +ve real number if $\tilde{r}\tilde{\in}[R^+]_A^E$ where $[R^+] = \{\{r\} : r \in R^+\}$.*
i.e. a soft real number with support A is given by

$$\tilde{r} = \{(e, \{r_e\}) : e \in A, r_e \in R\},\ r_e = 0\ for\ e \notin A$$

In the case of the real numbers as a universal set we have the following concepts:

Definition 3.3. *(i) A soft singleton +ve real number $\tilde{r}$ is a non-negative soft real number with support A such that*

$$\exists r \in R^+, \tilde{r}(e) = \begin{cases} \{r\} & \forall e \in A, \\ \{0\} & for\ e \notin A \end{cases}$$

and is denoted by $\{r\}_A^E$ for some $r \in R^+$.
(ii)([3],[4]) A soft point +ve real no. is a non-negative soft real number with a singleton support and singleton image such that $\tilde{r}(e) = \{r\}, e \in A$, $r \in R^+$ and the collection of all soft points +ve real numbers is denoted by $A \times R^+$.
(iii) A soft element of real number is a soft set $\tilde{r} = \{(e, \{r_e\}) : e \in A, r_e \in R\}$ with support $A \subset E$ (iv) A soft member +ve real no is a non-negative soft real number subset with a singleton support and is denoted by $P(R^+)_e^E$.

An ordered relation on the set of all soft elements real numbers is given in the following definition.

Definition 3.4. *For any two soft real numbers $\tilde{r} = \{\{r_e\} : e \in A\}\}, \tilde{l} = \{\{l_e\} : e \in B\}\}\tilde{\in}[R]_{\leq E}^E$*
(i) $\tilde{r} \leq \tilde{l}$ and $\tilde{l}\tilde{\geq}\tilde{r}$ if $r_e \leq l_e\ \forall e \in E$,
in this case $\tilde{r}$ is called smaller than or equal to $\tilde{l}$ and $\tilde{l}$ is greater than or equal to $\tilde{r}$. (ii) Define the operation $\oplus$,$\ominus$, $\odot$,respectivly on $[R]_{\leq E}^E$ by:

$$\tilde{r} \oplus \tilde{l} = r_e + l_e if\ e \in E, \tilde{r} \ominus \tilde{l} = r_e - l_e if\ e \in E, \tilde{r} \odot \tilde{l} = r_e \cdot l_e if\ e \in E.$$

(iii) The additive inverse of a soft element real number $\tilde{r}$ is denoted by $\tilde{-r}$ and given by

$$(\tilde{-r})(e) = \{\{-r_e\}, e \in A\}$$

(iv) The multiplication inverse of a soft element real number $\tilde{r}$ which is totally different than $\tilde{0}$ is denoted by $\tilde{r}^{-1}$ and given by

$$(\tilde{r})^{-1}(e) = \frac{1}{\tilde{r}(e)} = \{\{\frac{1}{r_e}\}, e \in A\}$$

Example 3.1. *Consider $E = \{x, y, z, w, p, q\}$ and let the soft elements real numbers $\{\tilde{r}, \tilde{l}, \tilde{m}\}$ given as in the table 1:*
We will see that $(\tilde{r} \oplus \tilde{l})(x) = 1 + \frac{-3}{4} = \frac{1}{4}$ and $(\tilde{l} \odot \tilde{m})(w) = -4 \cdot \frac{1}{3} = \frac{-4}{3}$ and so on.

Table 1: Soft real numbers

$[R^+]_A^E$ E	x	y	z	w	p	q
$\tilde{r}$	1	2	-1	$\frac{1}{2}$	3	-7
$\tilde{l}$	$\frac{-3}{4}$	0	5	-4	$\frac{2}{5}$	3
$\tilde{m}$	6	2	-10	$\frac{1}{3}$	0	8

Proposition 3.1. *Consider the operation* $*$ *stand for* $\oplus$ *or* $\odot$ *.For any three soft elements real numbers* $\tilde{r}, \tilde{l}, \tilde{m} \tilde{\in} [R]_{\leq E}^E$, $\tilde{r} = \{\{r_e\}, e \in E\}, \tilde{l} = \{\{l_e\}, e \in E\}, \tilde{m}\{\{m_e\}, e \in E\}$, *the following laws are satisfied*

1. *(identity law)* $\tilde{r} \oplus \tilde{0} = \tilde{0} \oplus \tilde{r} = \tilde{r}$ *and* $\tilde{r} \odot \tilde{1} = \tilde{1} \odot \tilde{r} = \tilde{r}$,*where* $\tilde{0}$ *is the identity of the operation* $\oplus$ *and* $\tilde{1}$ *is the identity in* $\odot$, *where* $\tilde{0}, \tilde{1}$ *are the soft singletons* $\tilde{0}(e) = 0,\ \tilde{1}(e) = 1\ \forall e \in E$.
2. *(commutative law)* $\tilde{r} * \tilde{l} = \tilde{l} * \tilde{r}$
3. *(associative law)* $\tilde{r} * (\tilde{l} * \tilde{m}) = (\tilde{r} * \tilde{l}) * \tilde{m}$
4. *(Distributive law)* $\tilde{r} \odot (\tilde{l} \oplus \tilde{m}) = (\tilde{r} \odot \tilde{l}) \oplus (\tilde{r} \odot \tilde{m})$

Theorem 3.1. *The set of all soft elements real numbers* $[R]_{\leq E}^E$ *forms a field*

Proof. In $([R]_{\leq E}^E, \oplus)$

1. $([R]_{\leq E}^E, \oplus)$ is closed (i.e. $\tilde{r} \oplus \tilde{l} \in [R^+]_A^E,\ \forall \tilde{r}, \tilde{l} \tilde{\in} [R]_{\leq E}^E$)
2. $([R]_{\leq E}^E, \oplus)$ has identity (i.e. $\tilde{r} \oplus \tilde{0} = \tilde{0} \oplus \tilde{r} = \tilde{r},\ \forall \tilde{r} \tilde{\in} [R]_{\leq E}^E$)
3. $([R]_{\leq E}^E, \oplus)$ has inverse (i.e. $\tilde{r} \oplus \tilde{-r} = \tilde{-r} \oplus \tilde{r} = \tilde{0},\ \forall \tilde{r} \tilde{\in} [R]_{\leq E}^E$
4. $([R]_{\leq E}^E, \oplus)$ is associative (i.e. $\tilde{r} \oplus (\tilde{l} \oplus \tilde{m}) = (\tilde{r} \oplus \tilde{l}) \oplus \tilde{m},\ \forall \tilde{r}, \tilde{l}, \tilde{m} \tilde{\in} [R]_{\leq E}^E$)
5. $([R]_{\leq E}^E, \oplus)$ is commutative (i.e. $\tilde{r} \oplus \tilde{l} = \tilde{l} \oplus \tilde{r},\ \forall \tilde{r}, \tilde{l} \tilde{\in} [R]_{\leq E}^E$)

$\Rightarrow ([R^+]_A^E, \oplus)$ is a commutative group.

In $([R^*]_{\leq E}^E, \odot)$, where $[R^*]_{\leq E}^E$ is the collection of all soft elements real numbers totally different than $\tilde{0}$, i.e. $\tilde{r} \tilde{\in} [R^*]_{\leq E}^E$ if $\tilde{r}(e) \neq 0\ \forall e \in E$

1. $([R]_{\leq E}^E, \odot)$ is closed (i.e. $\tilde{r} \odot \tilde{l} \in [R^+]_A^E,\ \forall \tilde{r}, \tilde{l} \tilde{\in} [R^*]_{\leq E}^E$)
2. $([R]_{\leq E}^E, \odot)$ has identity (i.e. $\tilde{r} \odot \tilde{1} = \tilde{1} \odot \tilde{r} = \tilde{r},\ \forall \tilde{r} \hat{\in} [R^*]_{\leq E}^E$)
3. $([R^*]_{\leq E}^E, \odot)$ has inverse (i.e. $\tilde{r} \odot \tilde{(r)}^{-1} = r^{\tilde{-1}} \odot \tilde{r} = \tilde{1},\ \forall \tilde{r} \tilde{\in} [R^*]_{\leq E}^E$
4. $([R]_{\leq E}^E, \odot)$ is associative (i.e. $\tilde{r} \odot (\tilde{l} \odot \tilde{m}) = (\tilde{r} \odot \tilde{l}) \odot \tilde{m},\ \forall \tilde{r}, \tilde{l}, \tilde{m} \tilde{\in} [R^*]_{\leq E}^E$)

5. $([R]^E_{\leq E}, \odot)$ is commutative (i.e. $\tilde{r} \odot \tilde{l} = \tilde{l} \odot \tilde{r}, \ \forall \tilde{r}, \tilde{l} \tilde{\in} [R^*]^E_{\leq E}$)

$\Rightarrow$ $([R^*]^E_{\leq E}, \odot)$ is a commutative group.
$\Rightarrow$ $([R]^E_A, \oplus, \odot)$ satisfy the distributive law (i.e. $\tilde{r} \odot (\tilde{l} \oplus \tilde{m}) = (\tilde{r} \odot \tilde{l}) \oplus (\tilde{r} \odot \tilde{m}), \quad \forall \tilde{r}, \tilde{l}, \tilde{m} \tilde{\in} [R]^E_{\leq E}$).
Hence $([R]^E_{\leq E}, \oplus, \odot)$ form a field. □

Definition 3.5. *For any soft element soft real number $\tilde{r} \tilde{\in} [R]^E_A$ if $\tilde{r} = \{(e_a, \{r_{e_a}\}) : a \in A \subset E\} \cup \{(e_b, \{0\}) : b \in E - A\}$. Then the restricted soft element soft real number on A is defined by*

$$\tilde{r}/_A = \{(e_a, \{r_{e_a}\}) : a \in A\}$$

Remark 3.1. *Every soft subset is the union of their soft elements in general , but in the soft real numbers any soft subset of real numbers is the union of its restricted soft element soft real number as it shown in the following example:*

Example 3.2. *Let $E = \{x, y, z, w, p, q\}, \ A = \{x, y, z\} \subset E$ and F_A defined by*

$$F_A(x) = [5, 100], \quad F_A(y) = \{1, 2, 3, -7\}, \quad F_A(z) = \{7n : n \in N\},$$

then F_A is a union of soft elements soft real numbers restricted on A i.e.

$$\tilde{r} = \{(x, \{5\}), (y, \{-7\}), (z, \{4\}), (w, \{0\}), (p, \{0\}), (q, \{0\})\} \tilde{\notin} F_A,$$

$$\tilde{r}/_A = \{(x, \{5\}), (y, \{-7\}), (z, \{4\})\} \tilde{\in} F_A$$

4 Least Upper Bound and Greatest Lower Bound in Soft Real Analysis

Definition 4.1. *A soft subset of soft elements real numbers $F_A \subset (R, E)$ is called soft bounded from above iff there exists a soft real number $\tilde{r} = \{(e, \{r_e\}), e \in E\}$ such that*

$$\tilde{\alpha} = \{(e, \{x_e\}), e \in E\} \tilde{\leq} \tilde{r} \ \forall \tilde{\alpha} \tilde{\in} F_A.$$

I.e.

$$\forall e \in E \ F_A(e) \leq r_e$$

A soft subset of soft elements real numbers $F_A \subset (R, E)$ is called soft bounded from below iff there exists a soft real number $\tilde{l} = \{(e, \{l_e\}), e \in E\}$ such that

$$\tilde{\beta} = \{(e, \{y_e\}), e \in E\} \tilde{\geq} \tilde{l} \ \forall \tilde{\beta} \tilde{\in} F_A.$$

I.e.

$$\forall e \in E \ F_A(e) \geq l_e$$

A soft subset of soft elements real numbers $F_A \subset (R, E)$ is called soft bounded if it is both soft bounded from above and from below.

Definition 4.2. *Let $F_A \subset (R, E)$ be a soft subset of soft elements real numbers which is bounded from above. The soft least upper bound $\tilde{r}/_A = \{(e, \{r_e\})\}$ of F_A which is denoted by* $\sup F_A$ *is a soft element real number satisfying the following two conditions:*

1. $\tilde{\alpha}/_A \tilde{\leq} \tilde{r}/_A \; \forall \tilde{\alpha}/_A \tilde{\in} F_A$
2. *If* $\tilde{\alpha}/_A \tilde{\leq} \tilde{l}/_A \; \forall \tilde{\alpha}/_A \tilde{\in} F_A$, *then* $\tilde{r}/_A \tilde{\leq} \tilde{l}/_A$

These two conditions can be formulated in another equivalent form to give another equivalent definition for $\sup F_A$:

Proposition 4.1. *For a bounded from above soft subset F_A of soft element real number,* $\tilde{r}/_A = \sup F_A$ *iff*

1. $\tilde{r}/_A$ *is an upper bound for* F_A.
2. $\forall \tilde{\varepsilon}/_A = \{(e_a, \{\varepsilon_{e_a}\})\} \tilde{>} \tilde{0}$ *the soft element real number* $\tilde{r}/_A \ominus \tilde{\varepsilon}/_A$ *is not an upper bound for* F_A

Proof. Straightforward □

Another equivalent form of the definition:

Proposition 4.2. *For a bounded from above soft subset F_A of soft element real numbers,* $\tilde{r}/_A = \sup F_A$ *iff*

1. $\tilde{r}/_A$ *is an upper bound for* F_A.
2. $\forall \tilde{\varepsilon}/_A \tilde{>} \tilde{0} \; \exists$ *a soft element real number* $\tilde{\beta}/_A \tilde{\in} F_A$ *such that* $\tilde{\beta}/_A \tilde{>} \tilde{r}/_A \ominus \tilde{\varepsilon}/_A$ *is not an upper bound for* F_A

Proof. Sraightforward □

Definition 4.3. *For a bounded from above soft subset F_A of soft element real number if*

$$\sup F_A \tilde{\in} F_A$$

then we write

$$\sup F_A = \max F_A$$

In this case the maximum of the soft real subset F_A exists.

Remark 4.1. *1. We notice that the soft least upper bound of a bounded from above soft subset is one of its upper bounds and therefore we can say that it is the minimum of its upper bounds and write*

$$\sup F_A = \min\{\tilde{l}/_A : \tilde{\alpha}/_A \tilde{\leq} \tilde{l}/_A \; \forall \tilde{\alpha}/_A \tilde{\in} F_A\}$$

As a dual of the soft least upper bound of soft subset of soft real numbers , we have the concept of the greatest soft lower bound.

Definition 4.4. *Let $F_A \subset (R, E)$ be a soft subset of soft elements real numbers which is bounded from below. The soft greatest lower bound $\tilde{m}/_A = \{(e_a, \{m_{e_a}\})\}$ of F_A which is denoted by $\tilde{m}/_A = \inf F_A$ is a soft element real number satisfying the following two conditions:*

1. $\tilde{\alpha}/_A \tilde{\geq} \tilde{m}/_A \;\; \forall \tilde{\alpha}/_A \tilde{\in} F_A$
2. *If* $\tilde{\alpha}/_A \tilde{\geq} \tilde{n}/_A \;\; \forall \tilde{\alpha}/_A \tilde{\in} F_A$, *then* $\tilde{m}/_A \tilde{\geq} \tilde{n}/_A$

These two conditions can be formulated in another equivalent form to give another equivalent definition for $\inf F_A$:

Proposition 4.3. *For a bounded from below soft subset F_A of soft element real numbers, $\tilde{m}/_A = \inf F_A$ iff*

1. *$\tilde{m}/_A$ is a lower bound for F_A.*
2. *$\forall \tilde{\varepsilon}/_A = \{(e_a, \{\varepsilon_{e_a}\})\} > \tilde{0}$ the soft element real number $\tilde{m}/_A \oplus \tilde{\varepsilon}/_A$ is not a lower bound for F_A*

Proof. Sraightforward ☐

Another equivalent form of the definition:

Proposition 4.4. *For a bounded from below soft subset F_A of soft element real number, $\tilde{m}/_A = \inf F_A$ iff*

1. *$\tilde{m}/_A$ is a lower bound for F_A.*
2. *$\forall \tilde{\varepsilon}/_A > \tilde{0} \; \exists$ a soft element real number $\tilde{\beta}/_A \tilde{\in} F_A$ such that $\tilde{\beta}/_A \tilde{<} \tilde{m}/_A \oplus \tilde{\varepsilon}/_A$ is not an upper bound for F_A*

Proof. Straightforward ☐

Definition 4.5. *For any soft set of soft elements real numbers F_A define $\ominus F_A$ by*

$$\ominus F_A = \{\tilde{-\alpha}/_A : \tilde{\alpha}/_A \tilde{\in} F_A\}.$$

Proposition 4.5. *Let $F_A \subset (R, E)$ be a soft subset of soft real numbers which is bounded. Then $\ominus F_A$ is also bounded and for which we have:*
(1) $\sup \ominus F_A = -\inf F_A$, *(2)*$\inf \ominus F_A = -\sup F_A$.

Proof. (1) Let $\inf F_A = \tilde{m}/_A$ then from the equivalent definition of infimum in proposition 2.3.4 we get:
(i) $\tilde{m}/_A \tilde{\leq} \tilde{\alpha}/_A \forall \tilde{\alpha}/_A \in F_A$, and
(ii) $\forall \tilde{\varepsilon}/_A > \tilde{0} \; \exists$ a soft real number $\tilde{\beta}/_A \tilde{\in} F_A$ such that $\tilde{\beta}/_A \tilde{<} \tilde{m}/_A \oplus \tilde{\varepsilon}/_A$
Multiplying by -1 we get:
(i) $\tilde{-\alpha}/_A \tilde{\leq} \tilde{-m}/_A \forall \tilde{-\alpha}/_A \in \ominus F_A$, and
(ii) $\forall \tilde{\varepsilon}/_A > \tilde{0} \; \exists$ a soft real number $\tilde{-\beta}/_A \tilde{\in} \ominus F_A$ such that $\tilde{-\beta}/_A \tilde{>} \tilde{-m}/_A \ominus \tilde{\varepsilon}/_A$
$\Rightarrow \sup \ominus F_A = -\tilde{m}/_A = -\inf F_A$.
(2) is similar to (1). ☐

5 Soft Sets, Soft real numbers and Countability

Definition 5.1. *A soft subset is called uncountable if it is not a union of countable set of soft elements, and it is countable if it is not uncountable .*

Proposition 5.1. *The union of two countable soft subsets is also countable.*

Proof. Let F_A, G_B be two countable soft subsets, where $F_A = \tilde{\cup}\{\tilde{\alpha}_n : n \in N\}$ $G_B = \tilde{\cup}\{\tilde{\beta}_n : n \in N\}$,then

$$F_A \tilde{\cup} G_B = \tilde{\cup}\{\gamma_n : n \in N\},$$

where

$$\tilde{\gamma}_n = \begin{cases} \tilde{\beta}_{\frac{n}{2}} & \text{if } n \text{ even} \\ \tilde{\alpha}_{\frac{n+1}{2}} & \text{if } n \text{ odd} \end{cases}$$

then $F_A \tilde{\cup} G_B$ is countable soft subset. □

Proposition 5.2. *Countable union of countable soft subsets is also countable.*

Proof. Straightforward □

Proposition 5.3. *Any soft subset (F, A) with support A is countable if A is countable and $F(e)$ is countable for each $e \in A$.*

Proof. If $\sharp(A) \leq \alpha_0$, $\sharp(F(e)) \leq \alpha_0 \;\; \forall e \in A$, then $\sharp(F, A) \leq \alpha_0 \times \alpha_0 = \alpha_0$ □

Lemma 5.1. *For any countable soft subset G_A of (X, E) there exist a countable collection of soft elements $\tilde{\alpha}_n = \{(e, \{x_{e_n}\}) : e \in A, n \in N\}$ such that $G_A = \tilde{\cup}_{n \in N}\{\tilde{\alpha}_n\}$.*

Proof. Let G_A be a countable soft subset of (X, E) i.e. $G(e)$ is a countable subset of X for every $e \in A$. Let $G(e) = \{g_n^e : n \in N\}$. Consider the soft elements $\tilde{\alpha}_n$, $\alpha_n(e) = g_n^e \; \forall e \in A, n \in N$. So,$\{\alpha_n : n \in N\}$ is a countable collection of soft elements and $\tilde{\cup}_{n \in N}\{\tilde{\alpha}_n\} = G_A$. □

Axiom 5.1. *Every nonempty soft set of soft real numbers that is bounded from above has a supremum.*

Since inf $F_A = -\sup(\ominus F_A)$ and F_A is bounded from below if and only if $\ominus F_A$ is bounded from above, it follows that every nonempty soft set of soft real numbers that is bounded from below has an infimum. The restriction to nonempty sets in the above axiom is necessary, since the empty set is bounded from above, but its supremum does not exist.

Theorem 5.1. *If $\tilde{\alpha} \tilde{\in} [R]_A^E$, then there exists $\tilde{n} \tilde{\in} [N]_A^E$ such that $\tilde{\alpha} \tilde{<} \tilde{n}$.*

Proof. Suppose, for contradiction, that there exists $\tilde{\alpha} \tilde{in} [R]_A^E$ such that $\tilde{\alpha} \tilde{>} \tilde{n}$ for every $\tilde{n} \tilde{\in} [N]_A^E$. Then $\tilde{\alpha}$ is an upper bound of $[N]_A^E$, so $[N]_A^E$ has a supremum $\tilde{r} = \sup[N]_A^E \tilde{\in} [R]_A^E$. Since $\tilde{n} \tilde{\leq} \tilde{r}$ for every $\tilde{n} \tilde{\in} [N]_A^E$, we have $\tilde{n} \ominus \tilde{1} \tilde{\leq} \tilde{r} \ominus \tilde{1}$ for every $\tilde{n} \tilde{\in} [N]_A^E$, which implies that $\tilde{n} \tilde{\leq} \tilde{r} \ominus \tilde{1}$ for every $\tilde{n} \tilde{\in} [N]_A^E$. But then $\tilde{r} \ominus \tilde{1}$ is an upper bound of $[N]_A^E$, which contradicts the assumption that $\tilde{r}$ is a least upper bound. □

The idea is to show that given any countable soft set of soft real numbers, there are additional soft real numbers in the "gaps" between them.

Theorem 5.2. *The soft set of soft elements real numbers is uncountable.*

Proof. Since the collection of all the soft points real numbers has the same cardinality as $E \times R$ for any set of parameters R and since $\#(R) = C_0 > \alpha_0$, where C_0 is the cardinality of the continous interval and α_0 is the cardinality of the natural numbers. So, $\#(E \times R) \geq C_0 > \alpha_0$. Also, any soft element real number is an arbitrary union of soft points which implies that the cardinality of the set of soft real numbers is greater than α_0 □

6 Properties of Supremum and Infimum in Soft Real Numbers

In this section, some pasic properties of Supremum and Infimum in Soft Real Numbers are given in details.

Proposition 6.1. *Suppose that F_A, G_B are nonempty soft sets of soft elements real numbers such that $\tilde{\alpha}\tilde{\leq}\tilde{\beta}$ for all $\tilde{\alpha}\tilde{\in}F_A$ and $\tilde{\beta}\tilde{\in}G_B$. Then $\sup F_A\tilde{\leq}\inf G_B$.*

Proof. Fix $\tilde{\beta}\tilde{\in}G_B$. Since $\tilde{\alpha}\tilde{\leq}\tilde{\beta}$ for all $\tilde{\alpha}\tilde{\in}F_A$, it follows that $\tilde{\beta}$ is an upper bound of F_A, so $\sup F_A$ is finite and $\sup F_A\tilde{\leq}\tilde{\beta}$. Hence, $\sup F_A$ is a lower bound of G_B, so $\inf G_B$ is finite and $\sup F_A\tilde{\leq}\inf G_B$. □

Definition 6.1. *If $F_A\tilde{\subset}(R,E)$ and $\tilde{r}\tilde{\in}[R]^E_A$, then we define*

$$\tilde{r} \odot F_A = \{\tilde{l}\tilde{\in}[R]^E_A : \tilde{l} = \tilde{r} \odot \tilde{\gamma} \textit{ for every } \tilde{\gamma}\tilde{\in}F_A\}.$$

Proposition 6.2. *If $\tilde{r}\tilde{\geq}\tilde{0}$, then*

$$\sup(\tilde{r} \odot F_A) = \tilde{r}\sup F_A, \ \inf(\tilde{r} \odot F_A) = \tilde{r}\inf F_A.$$

If $\tilde{r}\tilde{<}\tilde{0}$, then

$$sup(\tilde{r} \odot F_A) = \tilde{r}\inf F_A, \ \inf(\tilde{r} \odot F_A) = \tilde{r}\sup F_A.$$

Proof. The result is obvious if $\tilde{r} = \tilde{0}$. If $\tilde{r}\tilde{>}\tilde{0}$, then $(\tilde{r} \odot \tilde{\gamma})\tilde{\leq}\tilde{\alpha}$ if and only if $\tilde{\gamma}\tilde{\leq}\tilde{\alpha}/\tilde{r}$, which shows that $\tilde{\alpha}$ is an upper bound of $\tilde{r} \odot F_A$ if and only if $\tilde{\alpha} = \tilde{r}$ is an upper bound of F_A, so $\sup(\tilde{r} \odot F_A) = \tilde{r}\sup F_A$. If $\tilde{r}\tilde{<}\tilde{0}$, then $(\tilde{r} \odot \tilde{\gamma})\tilde{\leq}\tilde{\alpha}$ if and only if $\tilde{\gamma}\tilde{\geq}\tilde{\alpha}/\tilde{r}$, so $\tilde{\alpha}$ is an upper bound of $\tilde{r} \odot F_A$ if and only if $\tilde{\alpha} = \tilde{r}$ is a lower bound of F_A, so

$$sup(\tilde{r} \odot F_A) = \tilde{r}\inf F_A.$$

The remaining results follow similarly. □

Definition 6.2. *If $F_A, G_B\tilde{\subset}(R,E)$, then we define*

$$F_A \oplus G_B = \{\tilde{r}\tilde{\in}[R]^E_{\leq E} : \tilde{r} = \tilde{\alpha} \oplus \tilde{\beta} \textit{ for some } \tilde{\alpha}\tilde{\in}F_A, \tilde{\beta}\tilde{\in}G_B\},$$

$$F_A \ominus G_B = \{\tilde{l}\tilde{\in}[R]^E_{\leq E} : \tilde{l} = \tilde{\alpha} \ominus \tilde{\beta} \textit{ for some } \tilde{\alpha}\tilde{\in}F_A, \tilde{\beta}\tilde{\in}G_B\},$$

Proposition 6.3. *If F_A, G_B are nonempty soft real sets, then*

$$\sup(F_A \oplus G_B) = \sup F_A \oplus \sup G_B,\ \inf(F_A \oplus G_B) = \inf F_A \oplus \inf G_B,$$

$$\sup(F_A \ominus G_B) = \sup F_A \ominus \inf G_B,\ \inf(F_A \ominus G_B) = \inf F_A \ominus \sup G_B.$$

Proof. The soft set $F_A \oplus G_B$ is bounded from above if and only if F_A and G_B are bounded from above, so $\sup(F_A \oplus G_B)$ exists if and only if both $\sup F_A$ and $\sup G_B$ exist. In that case, if $\tilde{\alpha}\tilde{\in}F_A$ and $\tilde{\beta}\tilde{\in}G_B$, then

$$\tilde{\alpha} \oplus \tilde{\beta} \tilde{\leq} \sup F_A \oplus \sup G_B,$$

so $\sup F_A \oplus \sup G_B$ is an upper bound of $F_A \oplus G_B$, and therefore

$$\sup(F_A \oplus G_B) \tilde{\leq} \sup F_A \oplus \sup G_B.$$

To get the inequality in the opposite direction, suppose that $\tilde{\epsilon}\tilde{>}\tilde{0}$. Then there exist $\tilde{\alpha}\tilde{\in}F_A$ and $\tilde{\beta}\tilde{\in}G_B$ such that

$$\tilde{\alpha}\tilde{>} \sup F_A \ominus \frac{\tilde{\epsilon}}{2},\ \ \tilde{\beta}\tilde{>} \sup G_B \ominus \frac{\tilde{\epsilon}}{2}.$$

It follows that

$$\tilde{\alpha} \oplus \tilde{\beta} \tilde{>} \sup F_A \oplus \sup G_B \ominus \tilde{\epsilon}$$

for every $\tilde{\epsilon}\tilde{>}\tilde{0}$, which implies that

$$\sup(F_A \oplus G_B) \tilde{\geq} \sup F_A \oplus \sup G_B.$$

Thus,

$$\sup(F_A \oplus G_B) = \sup F_A \oplus \sup G_B.$$

It follows from this result and proposition 2.3.5 that

$$\sup(F_A \ominus G_B) = \sup F_A \oplus \sup(\ominus G_B) = \sup F_A \ominus \inf G_B.$$

The proof of the results for $\inf(F_A \oplus G_B)$ and $\inf(F_A \ominus G_B)$ is similar, or we can apply the results for the supremum to $\ominus F_A$ and $\ominus G_B$. □

As an example of the proposition 2.5.4 , we have the following:

Example 6.1. *Let $\{\tilde{r_n}/_A\}$ and $\{\tilde{l_n}/_A\}$, $n \in N$ be a sequence of soft element real numbers. Then*
(1) $\sup(\tilde{r_n}/_A \oplus \tilde{l_n}/_A) \tilde{\leq} \sup \tilde{r_n}/_A \oplus \sup \tilde{l_n}/_A$,
(2) $\inf \tilde{r_n}/_A \oplus \inf \tilde{l_n}/_A \tilde{\leq} \inf(\tilde{r_n}/_A \oplus \tilde{l_n}/_A)$.

Proof. In fact, let $\tilde{m}/_A = \sup \tilde{r_n}/_A, \tilde{n}/_A = \sup \tilde{l_n}/_A$ and $\tilde{\varepsilon}/_A = \sup(\tilde{r_n}/_A \oplus \tilde{l_n}/_A)$. It is then required to show that:

$$\tilde{\varepsilon}/_A \tilde{\leq} \tilde{m}/_A \oplus \tilde{n}/_A.$$

From the definition we get;

$$\tilde{r_n}/_A \tilde{\leq} \tilde{m}/_A\ \forall n \text{ and } \tilde{l_n}/_A \tilde{\leq} \tilde{n}/_A\ \forall n$$

Thus,

$$\tilde{r_n}/_A \oplus \tilde{l_n}/_A \tilde{\leq} \tilde{m}/_A \oplus \tilde{n}/_A \ \forall n \in N$$

Hence $\tilde{m}/_A \oplus \tilde{n}/_A$ is an upper bound for the soft real numbers $(\tilde{r_n}/_A \oplus \tilde{l_n}/_A)$ and consequently is greater than or equal to the least upper bound $\tilde{\varepsilon}/_A$.
The proof of (2) is similarly. □

Proposition 6.4. *Let F_A and G_B be bounded soft subsets of soft elements real numbers such that $F_A \tilde{\subseteq} G_B$, then we get;*

$$\inf G_B \tilde{\leq} \inf F_A \tilde{\leq} \sup F_A \tilde{\leq} \sup G_B$$

Proof. Let $\inf G_B = \tilde{m}$ then from the definition we get;

$$\tilde{m} \tilde{\leq} \tilde{\alpha} \quad \forall \tilde{\alpha} \tilde{\in} G_B$$

Consequently,

$$\tilde{m} \tilde{\leq} \tilde{\alpha} \quad \forall \tilde{\alpha} \tilde{\in} F_A$$

Hence $\tilde{m}$ is a lower bound for the soft subset F_A. It follows then that $\tilde{m} \tilde{\leq} \inf F_A$
Clearly,

$$\inf F_A \tilde{\leq} \tilde{\alpha} \tilde{\leq} \sup F_A \ \ \forall \tilde{\alpha} \tilde{\in} F_A$$

. Finally, let $\sup G_B = \tilde{r}$ then

$$\tilde{r} \tilde{\geq} \tilde{\alpha} \quad \forall \tilde{\alpha} \tilde{\in} G_B$$

Therefore,

$$\tilde{r} \tilde{\geq} \tilde{\alpha} \quad \forall \tilde{\alpha} \tilde{\in} F_A$$

Hence $\tilde{r}$ is an upper bound for the soft subset F_A and then $\tilde{r} \tilde{\geq} \sup F_A$.

□

7 conclusion

Operations on soft elements real numbers is given. Also, some properties of countable and uncountable soft sets is introduced. Finally, least upper bound and greatest lower bound and some properties of supremum and infimum in soft real analysis is studied.

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