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Jordan (θ,θ)*- Derivation Pairs of Rings With Involution

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Abstract

Let R be a 6-torsion free ring with involution, θ is a mapping of R and let (d,g) : R → R be an additive mapping . In this paper we will give the relation between $(\theta,\theta)^*$-derivation pair and Jordan $(\theta,\theta)^*$-derivation pair. Also, we will prove that if (d,g) is a Jordan $(\theta,\theta)^*$-derivation pair , then d is a Jordan $(\theta,\theta)^*$-derivation.

Keywords: *- ring; involution; Jordan $(\theta,\theta)^*$-derivation pair; $(\theta,\theta)^*$-derivation pair; Jordan $(\theta,\theta)^*$-derivation.

1. Introduction

Let R be a 6-torsion free ring with involution and θ is a mapping of R. This paper consists of two sections. In section one, we recall some basic definitions and other concepts, which be used in our paper, we explain these concepts by examples and remarks . In section two, we introduce the concepts of $(\theta,\theta)^*$- derivation pair, Jordan $(\theta,\theta)^*$- derivation pair and we study the relation between them on R.

2.BASIC CONCEPTS

Definition 2.1:[1] A ring R is said to be n-torsion free where n≠0 is an integer if whenever na=0 with a∈R , then a = 0 .

Definition 2.2:[1] An additive mapping $x \to x^*$ on a ring R is called an involution if for all x,y ∈R , we have $(xy)^* = y^* x^*$ and $x^{**}x$. A ring equipped with an involution is called *- ring.

Definition 2.3:[1] Let R be a ring . An additive mapping d : R → R is called a derivation if : d(xy) = d(x)y + xd(y), for all x,y ∈R, and we say that d is a Jordan derivation if $d(x^2) = d(x)x + xd(x)$, for all x ∈R .

Example 2.4:[2] Let R = { $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$, a,b $\in$N, where N is the ring of integers } be a ring of 2x2 matrices with respect to usual addition and multiplication .

Let d : R $\rightarrow$ R , defined by d $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$, for all $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \in$R .

Then d is a derivation of R .

Remark 2.5:[2] Every derivation is a Jordan derivation, but the converse in general is not true .

Example 2.6:[2] Let R be a 2-torsion free ring and a$\in$R , such that xax = 0 , for all x $\in$R, but xay $\neq$0 , for some (x$\neq$y) $\in$R .

Define a map d : R $\rightarrow$ R , as follows : d(x)= ax. Then d is a Jordan derivation, but not a derivation .

Definition 2.7:[2] Let R be a *-ring . An additive mapping d : R $\rightarrow$ R is called a *-derivation if d(xy)= d(x)y^*+ xd(y) , for all x,y $\in$R, and we say that d is a Jordan *-derivation if d(x^2) = d(x)x^* + xd(x) , for all x $\in$R .

Definition 2.8:[3] Let R be a ring , additive mappings d,g : R $\rightarrow$ R is called a derivation pair and denoted by (d,g) if satisfy the system of equations:

d(xyx) = d(x)yx + xg(y)x + xyd(x) , for all x,y $\in$R .

g(xyx) = g(x)yx + xd(y)x + xyg(x) , for all x,y $\in$R .

And is called Jordan derivation pair if :

d(x^3) = d(x)x^2 + xg(x)x +x^2d(x) , for all x $\in$R .

g(x^3) = g(x)x^2 + xd(x)x +x^2g(x) , for all x $\in$R .

Example 2.9 :[3] Consider the ring:

R = { $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$, a,b $\in$S }. Where S be a ring. Define d,g : R $\rightarrow$ R , by :

d $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$, for all a,b $\in$S . And

g $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$, for all a,b $\in$S .

Then (d,g) is a derivation pair .

Remark 2.10:[3] Every derivation pair is a Jordan derivation pair , but the converse is in general not true .

Example 2.11:[3] Let R be a 2-torsion free non commutative ring , and a$\in$R such that xax =0 , for all x $\in$R , but xyax $\neq$0 , for some (x$\neq$y) $\in$R . Define an additive pair d,g : R $\rightarrow$ R , as follows : d(x)= ax , g(x) = xa

Then (d,g) is a Jordan derivation pair ,but not a derivation pair .

Definition 2.12:[4] Let R be a *- ring , additive mappings d,g : R $\rightarrow$ R is called a *-derivation pair and denoted by (d,g) if satisfy the system of equations :

$d(xyx) = d(x)y^*x^* + xg(y)x^* + xyd(x)$, for all x,y $\in$R .

$g(xyx) = g(x)y^*x^* + xd(y)x^* + xyg(x)$, for all x,y $\in$R .

And is called Jordan *- derivation pair if :

$d(x^3) = d(x)x^{*2} + xg(x)x^* + x^2d(x)$, for all x $\in$R .

$g(x^3) = g(x)x^{*2} + xd(x)x^* + x^2g(x)$, for all x $\in$R .

3. JORDAN $(\theta,\theta)^*$- DERIVATION PAIR

Definition 3.1: Let R be a *- ring , θ is a mapping of R , additive mappings d,g : R $\rightarrow$ R is called a $(\theta, \theta)^*$- derivation pair and denoted by (d,g) if satisfy the system of equations :

$d(xyx) = d(x)\,\theta(y^*x^*) + \theta(x)g(y)\,\theta(x^*) + \theta(xy)d(x)$, for all x,y $\in$R . (1)

$g(xyx) = g(x)\,\theta(y^*x^*) + \theta(x)d(y)\,\theta(x^*) + \theta(xy)g(x)$, for all x,y $\in$R . (2)

And is called Jordan $(\theta, \theta)^*$- derivation pair if :

$d(x^3) = d(x)\,\theta(x^{*2}) + \theta(x)g(x)\,\theta(x^*) + \theta(x^2)d(x)$, for all x $\in$R . (3)

$g(x^3) = g(x)\,\theta(x^{*2}) + \theta(x)d(x)\,\theta(x^*) + \theta(x^2)g(x)$, for all x $\in$R . (4)

Remark 3.2: Let R be *- ring and θ is a mapping of R. Every $(\theta, \theta)^*$- derivation pair is a Jordan $(\theta, \theta)^*$- derivation pair, but the converse is in general not true.

Example 3.3: Let R be a 2-torsion free *- ring , with x^2 =0 and x^{*2} = 0 for all x $\in$R, and $xyx^* \neq xy^*x^*$ for some x,y $\in$R , we defined

d : R $\rightarrow$ R by $d(x) = x - x^*$ for all x $\in$R .

g : R $\rightarrow$ R by $g(x) = x^* - x$ for all x $\in$R .

And θ : R $\rightarrow$ R by $\theta(x) = x$ for all x $\in$R .

Then (d,g) is a Jordan $(\theta, \theta)^*$- derivation pair , but not $(\theta, \theta)^*$- derivation pair .

Lemma 3.4 : Let R be a 6-torsion free *- ring with an identity element , and let (d,g) be a Jordan $(\theta, \theta)^*$- derivation pair . Then d+g is a Jordan $(\theta, \theta)^*$- derivation , where θ is an automorphism of R .

Proof: Define an additive mapping $K : R \rightarrow R$, by $K(x) = d(x) + g(x)$,

for all $x \in R$. (5)

Then by using (3) , (4) and the above relation we get

$$K(x^3) = K(x)\,\theta(x^{*2}) + \theta(x)K(x)\,\theta(x^*) + \theta(x^2)K(x) \text{ , for all } x \in R . \tag{6}$$

Linearization the relation (6), we get

$$\begin{aligned}K(x^2y + yx^2 + xy^2 + y^2x + xyx + yxy) &= K(x)\,\theta(x^*y^*) + K(x)\,\theta(y^*x^*) + K(x)\,\theta(y^{*2}) + \\ &K(y)\,\theta(x^*y^*) + K(y)\,\theta(y^*x^*) + K(y)\,\theta(x^{*2}) + \theta(x)K(x)\,\theta(y^*) + \theta(x)K(y)\,\theta(x^*) + \theta(y)K(x)\,\theta(x^*) \\ &+ \theta(y)K(y)\,\theta(x^*) + \theta(x)K(y)\,\theta(y^*) + \theta(y)K(x)\,\theta(y^*) + \theta(xy)K(x) + \theta(yx)K(x) + \theta(y^2)K(x) \\ &+ \theta(xy)K(y) + \theta(yx)K(y) + \theta(x^2)K(y) \text{ , for all } x,y \in R .\end{aligned} \tag{7}$$

Replace x by –x in the relation (7) we get

$$\begin{aligned}K(x^2y + yx^2 - xy^2 - y^2x + xyx - yxy) &= K(x)\,\theta(x^*y^*) + K(x)\,\theta(y^*x^*) - K(x)\,\theta(y^{*2}) - K(y)\,\theta(x^*y^*) \\ &- K(y)\,\theta(y^*x^*) + K(y)\,\theta(x^{*2}) + \theta(x)K(x)\,\theta(y^*) + \theta(x)K(y)\,\theta(x^*) + \theta(y)K(x)\,\theta(x^*) - \\ &\theta(y)K(y)\,\theta(x^*) - \theta(x)K(y)\,\theta(y^*) - \theta(y)K(x)\,\theta(y^*) + \theta(xy)K(x) + \theta(yx)K(x) - \theta(y^2)K(x) - \\ &\theta(xy)K(y) - \theta(yx)K(y) + \theta(x^2)K(y) \text{ , for all } x,y \in R .\end{aligned} \tag{8}$$

According to the relation (7) , (8) we obtain

$$\begin{aligned}K(x^2y + yx^2 + xyx) &= K(x)\,\theta(x^*y^*) + K(x)\,\theta(y^*x^*) + K(y)\,\theta(x^{*2}) + \theta(x)K(x)\,\theta(y^*) + \\ &\theta(y)K(x)\,\theta(x^*) + \theta(x)K(y)\,\theta(x^*) + \theta(xy)K(x) + \theta(yx)K(x) + \theta(x^2)K(y) ,\end{aligned}$$

for all $x,y \in R$. (9)

Setting x=y=1 in (9) , and since R is a 6-torsion free , we get $K(1) = 0$

Now replace y by 1 in the relation (9) we get

$$K(x^2) = K(x)\,\theta(x^*) + \theta(x)K(x) \text{ , for all } x \in R . \tag{10}$$

Hence , d + g is a Jordan $(\theta, \theta)^*$- derivation

Lemma 3.5 : Let R be a 6-torsion free *- ring with an identity element, and let (d,g) be a Jordan $(\theta, \theta)^*$- derivation pair then

$$(d - g)(x) = a\,\theta(x^*) + \theta(x)\,a \text{ , for all } x \in R . \tag{11}$$

$$(g - d)(y) = b\,\theta(y^*) + \theta(y)\,b \text{ , for all } y \in R . \tag{12}$$

Where a = d(1) , and b = g(1) , θ is an automorphism of R .

Proof: From the relation (3) one obtains (see how the relation (9) was obtained from (7))

$d(x^2y + yx^2 + xyx) = d(x)\,\theta(x^*y^*) + d(x)\,\theta(y^*x^*) + d(y)\,\theta(x^{*2}) + \theta(x)g(x)\,\theta(y^*)$ +

$\theta(y)g(x)\,\theta(x^*) + \theta(x)g(y)\,\theta(x^*) + \theta(xy)d(x) + \theta(yx)d(x) + \theta(x^2)d(y)$,

for all $x,y \in R$. (13)

Setting x =1 , in the relation (13) we get

$d(y) = 2(a\,\theta(y^*) + \theta(y)\,a) + (b\,\theta(y^*) + \theta(y)\,b) + g(y)$, for all $y \in R$. (14)

Similar , we can show

$g(y) = 2(b\,\theta(y^*) + \theta(y)\,b) + (a\,\theta(y^*) + \theta(y)\,a) + d(y)$, for all $y \in R$. (15)

Comparing the relation (14) and (15) , we arrive at (11) , (12) .

Theorem 3.6 : Let R be a 6-torsion free *- ring with an identity element , and let (d,g) is a Jordan $(\theta, \theta)^*$- derivation pair then (d,g) is a $(\theta, \theta)^*$- derivation pair , where θ is an automorphism of R .

Proof : Putting 1 for y in the relation (13) we get

$3d(x^2) = 2A(x) + B(x) + a\,\theta(x^{*2}) + \theta(x^2)a + \theta(x)b\,\theta(x^*)$, for all $x \in R$. (16)

Where $A(x) = d(x)\,\theta(x^*) + \theta(x)d(x)$ and $B(x) = g(x)\,\theta(x^*) + \theta(x)g(x)$.

Using Lemma 3.5 , and relation (16) we get

$3d(x^2) = 2A(x) + B(x) + (d-g)(x^2) + \theta(x)b\,\theta(x^*)$, for all $x \in R$. (17)

Hence ,

$d(x^2) + (d+g)(x^2) = 2A(x) + B(x) + \theta(x)b\,\theta(x^*)$, for all $x \in R$. (18)

By using Lemma 3.4 , we get

$d(x^2) + (d+g)(x)\,\theta(x^*) + \theta(x)\,(d+g)(x) = 2A(x) + B(x) + \theta(x)b\,\theta(x^*)$, for all $x \in R$.

Then from above relation we obtain

$d(x^2) = A(x) + \theta(x)b\,\theta(x^*)$, for all $x \in R$. (19)

Linearization the relation (19) we obtain

$d(xy+yx)= d(x)\,\theta(y^*) + \theta(x)d(y) + d(y)\,\theta(x^*) + \theta(y)\,d(x) + \theta(x)b\,\theta(y^*) + \theta(y)b\,\theta(x^*)$, for all $x,y \in R$. (20)

Replace y by xy + yx , in the relation (20) we get

$d(x^2y + yx^2) + 2\,d(xyx) = 2(d(x)\,\theta(y^*x^*) + \theta(x)d(y)\,\theta(x^*) + \theta(xy)d(x) + \theta(xy)b\,\theta(x^*) + \theta(x)b\,\theta(y^*x^*)) + d(x)\,\theta(x^*y^*) + \theta(x)d(x)\,\theta(y^*) + \theta(x^2)d(y) + \theta(x^2)\,b\,\theta(y^*) + d(y)\,\theta(x^{*2}) + \theta(y)d(x)\,\theta(x^*) + \theta(y)b\,\theta(x^{*2}) + \theta(yx)d(x) + \theta(x)b\,\theta(x^*y^*) + \theta(yx)\,b\,\theta(x^*)$,

for all $x,y \in R$. (21)

Replace x by x^2 in the relation (20) and using (19) we get

$d(x^2y + yx^2)= d(x)\,\theta(x^*y^*) + \theta(x)d(x)\,\theta(y^*) + \theta(x^2)d(y) + \theta(x^2)\,b\,\theta(y^*) + d(y)\,\theta(x^{*2}) + \theta(y)d(x)\,\theta(x^*) + \theta(y)b\,\theta(x^{*2}) + \theta(yx)d(x) + \theta(x)b\,\theta(x^*y^*) + \theta(yx)\,b\,\theta(x^*)$,

for all $x,y \in R$. (22)

Comparing the relations (21) , (22) we get

$d(xyx) = d(x)\,\theta(y^*x^*) + \theta(x)d(y)\,\theta(x^*) + \theta(xy)d(x) + \theta(xy)b\,\theta(x^*) + \theta(x)b\,\theta(y^*x^*)$,

for all $x,y \in R$. (23)

Using Lemma 3.5 , then we gets

$d(xyx) = d(x)\,\theta(y^*x^*) + \theta(x)g(y)\,\theta(x^*) + \theta(xy)d(x) + \theta(xy)b\,\theta(x^*) + \theta(x)b\,\theta(y^*x^*) + \theta(x)a\,\theta(y^*x^*) + \theta(xy)a\,\theta(x^*)$, for all $x,y \in R$. (24)

Since $(a\,\theta(y^*) + \theta(y)\,a) + (b\,\theta(y^*) + \theta(y)\,b) = 0$ (see the relations (14) , (15)) then from relation (24) we obtains

$d(xyx) = d(x)\,\theta(y^*x^*) + \theta(x)g(y)\,\theta(x^*) + \theta(xy)d(x)$, for all $x,y \in R$.

and ,

$g(xyx) = g(x)\,\theta(y^*x^*) + \theta(x)d(y)\,\theta(x^*) + \theta(xy)g(x)$, for all $x,y \in R$.

Then we get (d,g) is a $(\theta, \theta)^*$- derivation pair .

Lemma 3.7 : Let R be a 2-torsion free *- ring with an identity element , and let (d,g) be a Jordan $(\theta, \theta)^*$- derivation pair, such that d(1)=g(1) then d(x)=g(x) for all $x \in R$, where θ is an automorphism of R .

Proof : Define the mapping $f : R \to R$ by $f(x) = d(x) - g(x)$, for all $x \in R$.

Then by using (3) , (4) we get

$f(x^3) = f(x)\,\theta(x^{*2}) - \theta(x)f(x)\,\theta(x^*) + \theta(x^2)f(x)$, for all $x \in R$. (25)

Linearization the relation (25) we get

$$f(x^2y + yx^2 + xy^2 + y^2x + xyx + yxy) = f(x)\,\theta(x^*y^*) + f(x)\,\theta(y^*x^*) + f(x)\,\theta(y^{*2}) + f(y)\,\theta(x^*y^*) + f(y)\,\theta(y^*x^*) + f(y)\,\theta(x^{*2}) - \theta(x)f(x)\,\theta(y^*) - \theta(x)f(y)\,\theta(x^*) - \theta(y)f(x)\,\theta(x^*) - \theta(y)f(y)\,\theta(x^*) - \theta(x)f(y)\,\theta(y^*) - \theta(y)f(x)\,\theta(y^*) + \theta(xy)f(x) + \theta(yx)f(x) + \theta(y^2)f(x) + \theta(xy)f(y) + \theta(yx)f(y) + \theta(x^2)f(y)$$

, for all $x,y \in R$. (26)

From the relation (26) one obtains (see how the relation (9) was obtained from (7))

$$f(x^2y + yx^2 + xyx) = f(x)\,\theta(x^*y^*) + f(x)\,\theta(y^*x^*) + f(y)\,\theta(x^{*2}) - \theta(x)f(x)\,\theta(y^*) - \theta(y)f(x)\,\theta(x^*) - \theta(x)f(y)\,\theta(x^*) + \theta(xy)f(x) + \theta(yx)f(x) + \theta(x^2)f(y)\,,$$

for all $x,y \in R$. (27)

Replace x by 1 in the above relation we get

$2\,f(y) = f(1)\,\theta(y^*) + \theta(y)f(1)$, for all $y \in R$. (28)

Since $f(1)=0$, then from (28) we get , $d(x)=g(x)$, for all $x \in R$.

Lemma 3.8 : Let R be a 2-torsion free *- ring with an identity element , and let

$d : R \to R$ be an additive mapping satisfies

$d(xyx) = d(x)\,\theta(y^*x^*) + \theta(x)d(y)\,\theta(x^*) + \theta(xy)d(x)$, for all $x,y \in R$. (29)

Then d is a Jordan $(\theta, \theta)^*$- derivation , where θ is an automorphism of R .

Proof : Setting $y=x=1$ in (29) we get , $d(1)=0$.

Replace y by 1 in (29) therefore we obtain

$d(x^2) = d(x)\,\theta(x^*) + \theta(x)d(x)$, for all $x \in R$.

Then d is a Jordan $(\theta, \theta)^*$- derivation .

Theorem 3.9 : Let R be a 6-torsion free *- ring with an identity element , and let (d,g) be a Jordan $(\theta, \theta)^*$- derivation pair such that $d(1)=g(1)$, then d is a Jordan $(\theta, \theta)^*$- derivation , where θ is an automorphism of R .

Proof : By using Theorem 3.6 , we get (d,g) is a $(\theta, \theta)^*$- derivation pair , and by Lemma 3.7 , we get d satisfy the relation (29) , hence by using Lemma 3.8 , we get d is a Jordan $(\theta, \theta)^*$- derivation .

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