Journal of Progressive Research in Mathematics(JPRM)
ISSN: 2395-0218

SCITECH
RESEARCH ORGANISATION|

Volume 8, Issue 2
Published online: July 04, 2016|

Journal of Progressive Research in Mathematics
www.scitecresearch.com/journals

The Spectrum and The Numerical Range of $\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi}$ and $\mathcal{W}^*_{f,\psi}\mathcal{W}_{f,\varphi}$

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Abstract: In this paper we study the spectrum and the numerical range of weighted composition operator with the adjoint of weighted composition operator $\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi}$ and $\mathcal{W}^*_{f,\psi}\mathcal{W}_{f,\varphi}$ induced by linear –fractional self- maps φ *and* ψ of $\mathbb{U}$ on Hardy space $\mathbb{H}^2$.

1. Introduction

Let U denote the open unite disc in the complex plan ,let $\mathbb{H}^\infty$ denote the collection of all holomorphic function on U and let $\mathbb{H}^2$ is consisting of all holomorphic self-map on U such that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ whose Maclaurin coefficients are square summable (i.e)

$f(z) = \sum_{n=0}^{\infty} |a_n|^2 < \infty$. More precisely $f(z) = \sum_{n=0}^{\infty} a_n z^n$ if and only if $\|f\| = \sum_{n=0}^{\infty} |a_n|^2 < \infty$. The inner product inducing the $\mathbb{H}^2$ norm is given by $\langle f,g\rangle = \sum_{n=0}^{\infty} a_n \overline{b_n}$.

Given any holomorphic self-map φ on U, recall that the composition operator

Is called the composition operator with symbol φ, is necessarily bounded.Let

$f \in \mathbb{H}^\infty$,the operator $T_f : \mathbb{H}^2 \to \mathbb{H}^2$ defined by

$$T_f(h(z)) = f(z)h(z), \qquad for\ all\ z \in \mathrm{U}, h \in \mathbb{H}^2$$

is called the Toeplitz operator with symbol f.Since $f \in \mathbb{H}^\infty$,then we call T_f a holomorphic Toeplitz operator. If T_f is a holomorphic Toeplitz operator, then the operator $T_f C_\varphi$ is bounded and has the form

$$T_f C_\varphi g = f(go\varphi) \qquad (g \in \mathbb{H}^2).$$

We call it the weighted composition operator with symbols f and φ [1] and [3], the linear operator

$$\mathcal{W}_{f,\varphi}\, g = f(go\varphi) \qquad (g \in \mathbb{H}^2).$$

We distinguish between the two symbols of weighted composition operator $\mathcal{W}_{f,\varphi}$, by calling f the multiplication symbol and φ composition symbol.

For given holomorphic self-maps f and φ of U, $\mathcal{W}_{f,\varphi}$ is bounded operator even if $f \notin \mathbb{H}^{\infty}$. To see a trivial example, consider $\varphi(z) = p$ where $p \in$ U and $f \in \mathbb{H}^{\mathbf{2}}$, then for all $g \in \mathbb{H}^{\mathbf{2}}$, we have

$$\| \mathcal{W}_{f,\varphi}\, g\|_2 = \|g(p)\|\|f\|_2 = \|f\|_2|\langle g, K_p\rangle| \leq \|f\|_2\|g\|_2\|K_p\|_2.$$

In fact, if $f \in \mathbb{H}^{\infty}$,then $\mathcal{W}_{f,\varphi}$ is bounded operator on $\mathbb{H}^{\mathbf{2}}$ with norm

$$\| \mathcal{W}_{f,\varphi} \| = \|T_f C_\varphi\| \leq \|f\|_\infty \|C_\varphi\| = \|f\|_\infty \sqrt{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}}.$$

2. Basic Concepts

We start this section, by giving the following results which are collect some properties of Toeplitz and composition operators.

Lemma (2.1):[4, 6] Let φbe a holomorphic self-map of U, then

(a) $C_\varphi T_f = T_{fo\varphi} C_\varphi.$

(b) $T_g T_f = T_{gf}.$

(c) $T_{f+\gamma g} = T_f + \gamma T_g.$

(d) $T_f^* = T_{\bar{f}}.$

Proposition (2.2):[1] Let φ and ψ be two holomorphic self-map of U, then

1. $C_\varphi^n = C_{\varphi_n}$ for all positive integer n.
2. C_φ is the identity operator if and only if φ is the identity map.
3. C_φ= C$_\psi$ if and only if φ=ψ.
4. The composition operator cannot be zero operator.

For each $\alpha \in U$,the reproducing kernel at α ,defined by $K_\alpha(z) = \frac{1}{1-\bar{\alpha}z}$

It is easily seen for each $\alpha \in U$ and $f \in H^2$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ that

$$\langle f, K_\alpha\rangle = \sum_{n=0}^{\infty} a_n\, \alpha^n = f(\alpha).$$

When $\varphi(z) = (az + b)/cz + d)$ is linear-fractional self-map of U, Cowen in [2] establishes $C_\varphi^* = T_g C_\sigma T_h^*$,where the Cowen auxiliary functions g, σ and h are defined as follows: $g(z) = \frac{1}{-\bar{b}z+\bar{d}}$, $\sigma(z) = \frac{\bar{a}z-\bar{c}}{-\bar{b}z+\bar{d}}$ $\quad and \quad h(z) = cz + d$.

If φ is linear fractional self-map U, then $W_{f,\varphi}^* = (T_f C_\varphi)^* = C_\varphi^* T_f^* = T_g C_\sigma T_h^*.$

Proposition (2.4):[5] Let each of $\varphi_1, \varphi_2, \ldots \varphi_n$ be holomorphic self-maps of U and $f_1, f_2, \ldots f_n \in \mathbb{H}^{\infty}$**,** then

$$\mathcal{W}_{f_1,\varphi_1}.\mathcal{W}_{f_2,\varphi_2} \ldots .\mathcal{W}_{f_n,\varphi_n} = T_h C_\phi$$

Where$T_h = f_1.(f_2 o\varphi_1).(f_3 o\varphi_2 o\varphi_1). \dots (f_2 o\varphi_{n-1} o\varphi_{n-2} o \dots . o\varphi_1)$ and

$C_\phi = \varphi_n o\varphi_{n-1} o \dots . o\varphi_1$.

Corollary (2.5): Let φ be a holomorphic self-map of U and $f \in \mathbb{H}^\infty$ then

$$\mathcal{W}_{f,\varphi}^n = T_{f \ (f \ o\varphi \ )(f \ o\varphi_2)\dots(f \ o\varphi_{n-1})} C_{\varphi_n}$$

The following lemma discuss the adjoint of weighted composition operator.

Lemma (2.6):[3] If the operator $\mathcal{W}_{f,\varphi}: \mathbb{H}^2 \rightarrow \mathbb{H}^2$ is bounded,then for each $\alpha \in U$

$$\mathcal{W}_{f,\varphi}^* K_\alpha = \overline{f(\alpha)} K_{\varphi(\alpha)}.$$

3- The Spectrum and The Numerical Range of $\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*$ and $\mathcal{W}_{f,\psi}^*\mathcal{W}_{f,\varphi}$

In this section , we consider the operators $\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*$ *and* $\mathcal{W}_{f,\psi}^*\mathcal{W}_{f,\varphi}$ induced by linear –fractional self- maps φ *and* ψ of $\mathbb{U}$. We completely characterize the shape of the spectrum and numerical rang of these operators .

Recall that [6] ***the spectrum*** of an operator T, denoted by σ(T), is the set of all complex numbers λ for which T-λI is not invertible.

Recall that [6] ***the numerical range*** of an operator T on Hilbert space H is the set of complex numbers,

$$W(T) = \{\langle Tf, f\rangle : f \in \mathcal{H}, \|f\| = 1\}.$$

The following proposition collects some properties of the numerical range of an operator, for more details we refer the reader to [2].

Proposition (3.1):[2],[4]

(1) $W(T)$ lies in the disc of center 0 and radius $\|T\|$.

(2) $\sigma_p(T) \subseteq W(T)$.

(3) $\sigma(T) \subseteq \overline{W(T)}$. ($\overline{W(T)}$ denotes the closure of $W(T)$).

(4) If T is the identity, then $W(T) = \{1\}$. More generally , if α, β ara complex numbers , then $W(\alpha T + \beta) = \alpha W(T) + \beta$.

(5) If T is normal operator, then $Conv\ \sigma(T) \subseteq \overline{W(T)}$.

($Conv\ \sigma(T)$ denotes the convex hull of $\sigma(T)$).

(6) $W(T)$ is convex set of C.

(7) If $\acute{T}$ is the compression of T to the closed subspace M , $W(\acute{T}) \subseteq W(T)$.

Lemma (3.2): Suppose that φ *and* ψ be two linear- fractional self-maps of $\mathbb{U}$ and $f \in \mathbb{H}^\infty$. then,$\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*$ is a weighted composition operator on Hardy space $\mathbb{H}^{\mathbf{2}}$.

Proof : Since each of φ *and* ψ is linear fractional , then put where $a, b, c, d, a_1, b_1, c_1$ $and\ d_1$ are complex constants , such that $cz + d \neq 0$ and $c_1 z + d_1 \neq 0$.

Thus , $C_\psi^* = T_g C_\sigma T_h^*$, where the Cowen auxiliary functions, g, σ $and\ h$ of ψ are defined as follows :

$$g(z) = \frac{1}{-\bar{b}z + \bar{d}} \quad , \sigma(z) = \frac{\bar{a}z - \bar{c}}{-\bar{b}z + \bar{d}} \qquad and \quad h(z) = cz + d$$

Note that ,

$$\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^* = T_f C_\varphi . \left(T_f C_\psi\right)^*$$

$$= T_f C_\varphi C_\psi{}^* T_f{}^*$$

$$= T_f C_\varphi T_g C_\sigma T_h^* T_f{}^*$$

$$= T_f C_\varphi T_{g.[\overline{hf} \circ \sigma]} C_\sigma$$

$$= T_{f.[(g.[\overline{hf} \circ \sigma])\varphi]} C_{\sigma \circ \varphi}$$

$$= T_k C_{\sigma \circ \varphi}$$

$$= \mathcal{W}_{k, \sigma \circ \varphi}$$

Where $k = f.[\left(g.\left[\overline{hf} \circ \sigma\right]\right)\varphi]$.

Since each of h, f $and\ g$ are in $\mathbb{H}^\infty$, then it is clear that $k \in \mathbb{H}^\infty$. In addition that

$$\sigma \circ \varphi = \frac{Az + B}{Cz + D} \quad ,$$

Where $A = \bar{a}a_1 - \bar{c}c_1$, $B = \bar{a}b_1 - \bar{c}d_1$, $C = \bar{d}b_1 - \bar{b}a_1$ and $D = \bar{d}d_1 - \bar{b}b_1$.

Lemma (3.3): Suppose that φ and ψ be two linear fractional self-maps of $\mathbb{U}$ and $f \in \mathbb{H}^\infty$, then $\mathcal{W}_{f,\psi}^* \mathcal{W}_{f,\varphi}$ is a weighted composition operator on Hardy space $\mathbb{H}^{\mathbf{2}}$.

Proof : Since ψ be linear fractional , then the Cowen auxiliary functions g, σ and h are defined as above . Thus,

$$\mathcal{W}_{f,\psi}^* \mathcal{W}_{f,\varphi} = \left(T_f C_\psi\right)^* T_f C_\varphi$$

$$= C_\psi{}^* T_f{}^* T_f C_\varphi$$

$$= T_g C_\sigma T_h^* T_f{}^* T_f C_\varphi$$

$$= T_g C_\sigma T_{\bar{h}|f|^2} C_\varphi$$

$$= T_{g(\bar{h}|f|^2 \circ \sigma)} C_{\varphi o \sigma}$$

$$= T_L C_{\varphi \circ \sigma}$$

$= \mathcal{W}_{L,\varphi\circ\sigma}$

Where $L = g(\bar{h}|f|^2 \circ \sigma)$.

Since each of $h, f\ and\ g$ are in $\mathbb{H}^\infty$, then it is clear that $L \in \mathbb{H}^\infty$. In addition that

$\varphi \circ \sigma = \frac{A_1 z + B_1}{C_1 z + D_1}$,

Where $A_1 = \bar{a}a_1 - \bar{b}b_1$, $B_1 = \bar{d}b_1 - \bar{c}a_1$, $C_1 = \bar{a}c_1 - \bar{b}d_1$ and $D_1 = \bar{d}d_1 - \bar{c}c_1$. ■

We begin this section with $\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*$ such that $\sigma \circ \varphi$ is a rational rotation of $\mathbb{U}$.

Remark (3.4): If $B = C = 0\ and\ |A| = |D|$, then it is clear that $\sigma \circ \varphi$ is rotation . Put $(\sigma \circ \varphi)(z) = \mu z$, $\mu = A/D = e^{i\theta}\ where\ \theta$ is rational number

Now, put $\mu = e^{2\pi i/n}$, where n is a positive integer number .

Suppose that j is a fixed integer number such that $0 \le j < n.$ If

$$\phi(z) = \sum_{k=0}^{\infty} a_k z^{kn+j} = z^j \sum_{k=0}^{\infty} a_k (z^n)^k = z^j \zeta(z^n)$$

where $\zeta(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathbb{H}^\infty$. Note that ,

$$C_{\sigma\circ\varphi}(\phi(z)) = \sum_{k=0}^{\infty} a_k ((\sigma\circ\varphi)(z))^{kn+j}$$

$$= \sum_{k=0}^{\infty} a_k \left(e^{2\pi i/n}\, z\right)^{kn+j}$$

$$= \sum_{k=0}^{\infty} a_k e^{2\pi i k} (e^{2\pi i/n})^j\, z^{kn+j}$$

$$= (e^{2\pi i/n})^j \sum_{k=0}^{\infty} a_k\, z^{kn+j}$$

$$= \mu^j \phi(z) \tag{1}$$

Clearly μ^j is the eigenvalue of $C_{\sigma\circ\varphi}$, for each $0 \le j < n$. Therefore, it is natural to look at the subspace H_j of functions that have such Maclaurin series. ■

Definition (3.5):[7] Let $n \ge 2$ and $0 \le j < n$, and let H_j be the set defined by

$$H_j = \{\phi : \phi(z) = z^j \zeta(z^n)\ , \zeta \in \mathbb{H}^2\}.$$

It is clear that H_j is a closed subspace of $\mathbb{H}^2$. Let P_j denote the orthogonal projection onto H_j .

Now, we compute the spectrum of the operator $\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*$ where $\sigma\circ\varphi$ a rational rotation of $\mathbb{U}$. First we need the following lemma .

Lemma (3.6):[7] Let $n>1$. Suppose $(z)=g(z^n)$ $where\ g\in\mathbb{H}^\infty$. If $0\leq l<n$, then $\sigma(T_\psi|H_l)=\overline{\psi(\mathbb{U})}$.

Theorem (3.7): Suppose that $\varphi\ and\ \psi$ be two automorphism of $\mathbb{U}$ and $f\in\mathbb{H}^\infty$. If σ is Cowen auxiliary function of ψ such that $\sigma\circ\varphi$ is a rational rotation of $\mathbb{U}$ and $k(z)=\zeta(z^n)$ where $\zeta\in\mathbb{H}^2$, then

$$\sigma(\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*)=\overline{\bigcup_{l=0}^{n-1}\mu^l k(\mathbb{U})}$$

where $\mu=e^{2\pi i/n}$, where n is a positive integer number .

Proof : Put $T_l=T_\psi|H_l$ for fixed$0\leq l<n$. If $\phi\in H_l$, then by (1) we have $C_{\sigma\circ\varphi}(\phi)=\mu^l\phi$, and so by lemma(3.2) we get

$$\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*(\phi)=T_kC_{\sigma\circ\varphi}(\phi)=\mu^l k\,.\,\phi$$

$$=\mu^l T_l(\phi)\in H_l \qquad (2)$$

Note that , it is easily seen that $H_0,H_1,\dots,H_{n-1}$ are pairwise orthogonal. Put

$\mathcal{W}_l=\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*|H_l$, so by (2) we have $\mathcal{W}_l=A^lT_l$. Thus ,

$$\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*=\mathcal{W}_0\oplus\mathcal{W}_1\oplus\ldots\oplus\mathcal{W}_{n-1} \qquad (3)$$

Hence, $\sigma(\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*)=\sigma(\mathcal{W}_0)\cup\sigma(\mathcal{W}_1)\cup\dots\cup\sigma(\mathcal{W}_{n-1})$.

Therefore, by spectral mapping theorem and lemma (3.5) we get

$\sigma(\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*)=\overline{\bigcup_{l=0}^{n-1}\mu^l k(\mathbb{U})}$. ■

Next, we compute the numerical rang of $\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*$ where $\sigma\circ\varphi$ is a rational rotation of $\mathbb{U}$. Before this , we establish the following lemma .

Lemma (3.8):[7] Let $\psi\in\mathbb{H}^\infty$. If H_j is an invariant subspace of T_ψ, then

$$W(T_\psi|H_j)=conv(\psi(\mathbb{U}))$$

Theorem (3.9): Suppose that $\varphi\ and\ \psi$ be two automorphism of $\mathbb{U}$ and $f\in\mathbb{H}^\infty$. If σ is Cowen auxiliary function of ψ such that $\sigma\circ\varphi$ is a rational rotation of $\mathbb{U}$ and $k(z)=\zeta(z^n)$ where $\zeta\in\mathbb{H}^2$, then

$$W(\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*)=conv\left(\bigcup_{l=0}^{n-1}\mu^l k(\mathbb{U})\right)$$

where $\mu=e^{2\pi i/n}$, where n is a positive integer number .

Proof: Put $T_l=T_\psi|H_l$ an $\mathcal{W}_l=\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*|H_l$ for fixed $0\leq l<n$. Thus , by (3)

we have that

$$W(\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^{*}) = conv\left(\bigcup_{l=0}^{n-1} W(\mathcal{W}_l\ )\right) \qquad (4)$$

If $\phi \in H_l$, then by (2) we have

$\langle \mathcal{W}_l(\phi), \phi\rangle = \langle \mu^l T_l(\phi), \phi\rangle = \mu^l \langle T_l(\phi), \phi\rangle$

It follows by lemma(3.8) that $W(\mathcal{W}_l) = \mu^l conv\big(k(U)\big)$. Thus , by (4) we have $W(\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^{*}) = conv\ (\bigcup_{l=0}^{n-1} \mu^l k(\mathbb{U})\ )$. ■

Recall that [ ] an operator T on a Hilbert space H is called convexiod if $conv\,\sigma(T) = \overline{W(T)}$. Note that by proposition (3.1) (5) we have every normal operator is convexiod.

Therefore , by theorem(3.7) and (3.9) we get the following consequence .

Corollary (3.10): Suppose that φ *and* ψ be two automorphism of $\mathbb{U}$ and $f \in \mathbb{H}^{\infty}$. If σ is Cowen auxiliary function of ψ such that $\sigma \circ \varphi$ is a rational rotation of $\mathbb{U}$ and $k(z) = \zeta(z^n)$ where $\zeta \in \mathbb{H}^2$, then$\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^{*}$is convexiod operator.

Remark (3.11): If A is a primitive n^{th} root of unity our previous results still holds in fact, if $A = e^{2\pi i m/n}$ where n *and* m are relatively prime and $0 \le m < n$.

Let $\lambda = e^{2\pi i m/n}$ and $\mu = e^{2\pi i/n}$. If P_k denotes $m_{jk}\,(mod\ n)$, then $\lambda^{jk} = \mu^{P_k}$.

Suppose that $\phi \in H_j$, for some $0 \le j < n$. Then , it is easily see that

$C_{\sigma\circ\varphi}(\phi) = \mu^P \phi$ where $P \equiv m_j\,(mod\ n)$. Then ,

$\langle \mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^{*}(\phi), \phi\rangle = \langle T_k C_{\sigma\circ\varphi}(\phi), \phi\rangle = \mu^P \langle K\phi, \phi\rangle$.

Therefore , by lemma(3.8) we have

$W(\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^{*}|H_j) = \mu^P conv\ \big(k(\mathbb{U})\big)$.

But , $\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^{*} = \mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \ldots \oplus \mathcal{W}_{n-1}$this implies that

$W(\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^{*}) = conv\ \big(\mu^{P_0}k(\mathbb{U}) \cup \mu^{P_1}k(\mathbb{U}) \cup \ldots \cup \mu^{P_{n-1}}k(\mathbb{U})\big)$.

If $0 \le j_1 < j_2 < n$, then it is easily to see that $m_{j_1} \not\equiv m_{j_2}(mod\ n)$ and thus

$\{P_0, P_1, \ldots, P_{n-1}\} = \{0,1,2, \ldots, n-1\}$. ■

In what follows we study the numerical rang of the operator $(\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^{*})$ where $\sigma \circ \varphi$ is an irrational rotation of $\mathbb{U}$, i.e. $(\sigma \circ \varphi)(z) = Az$, $A = e^{2\pi i\theta}$

such that θ is irrational number .

Lemma (3.12): Suppose that φ *and* ψ be two automorphism of $\mathbb{U}$ and $f \in \mathbb{H}^{\infty}$. If σ is Cowen auxiliary function of ψ such that $\sigma \circ \varphi$ is an irrational rotation of $\mathbb{U}$ and

$k(z) = 1 + \widehat{K}_1 z + \widehat{K}_2 z^2 + \cdots$. Assume that n is a non-negative integer and m is a positive integer , Then $W(\mathcal{W}_{f,\varphi}.\mathcal{W}^*_{f,\psi})$ contains the ellipse with foci μ^n and μ^{n+m} whose major axis is $\sqrt{|\mu^n - \mu^{n+m}|^2 + |\widehat{K}_m|^2}$ and minor axis is$|\widehat{K}_m|$.

Proof : Let $Q = span\{e_1, e_2\}$ where $e_1(z) = z^n$ and $e_2(z) = z^{n+m}$. Then, we have

$$\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi}(e_1)(z) = T_k C_{\sigma\circ\varphi}(e_1)(z)$$

$$= k(z).e_1(\sigma \circ \varphi\varphi(z))$$

$$= (1 + \widehat{K}_1 z + \widehat{K}_2 z^2 + \cdots)A^n z^n$$

$$= \mu^n z^n + \mu^n \widehat{K}_1 z^{n+1} + \mu^n \widehat{K}_2 z^{n+2} + \cdots + \mu^n \widehat{K}_m z^{n+m} + \cdots$$

Similarly , we have

$$\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi}(e_2)(z) = \mu^{n+m} z^{n+m} + \mu^{n+m}\widehat{K}_1 z^{n+m+1} + \cdots$$

Thus, the matrix that represents $\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi}$ is

$$T = \begin{bmatrix} \mu^n & 0 \\ \mu^n \widehat{K}_m & \mu^{n+m} \end{bmatrix}$$

Therefore, $W(T)$ is the ellipse with fociμ^n and μ^{n+m} that described in (see[]). But $W(T) \subseteq W(\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi})$,as desired . ■

Now, we are ready to discuss the numerical rang of $\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi}$ when it is an isometry operator.

Recall that [] an operator T on Hilbert space $\mathcal{H}$ is said to be shift operator if T is an isometry and $(T^*)^n \to 0$ strongly . Gunatillake G. in [ ] described the numerical range of weighted composition operator $\mathcal{W}_{f,\varphi}$ when each of φ and ψ is inner function .

Lemma (3.13): Suppose that φ be a holomorphic self-map of U and $\in \mathbb{H}^\infty$. If $\mathcal{W}_{f,\varphi}$ is an isometry, then φ must be inner function and $\|f\| = 1$.

Proof : Let the operator $\mathcal{W}_{f,\varphi}$ is an isometry, then $\mathcal{W}^*_{f,\varphi}.\mathcal{W}_{f,\varphi} = I$. Thus for each $p \in U$, we have

$\|\mathcal{W}_{f,\varphi}K_p\| = \|K_p\|$, then $\|T_f C_\varphi K_p\| = \|K_p\|$.

This implies that $\|f(K_p \circ \varphi)\| = \|K_p\|$. Hence , by taking $p = 0$, then $K_0 = 1$

and thus $\|f(1 \circ \varphi)\| = \|1\|$, $then$ $\|f\| = 1$

In addition that, if $g(z) = z$, then it is clear that$\|g\| = 1$ **.** Therefore

$\|\mathcal{W}_{f,\varphi}g\| = \|g\|$, and then $\|T_f C_\varphi g\| = \|g\|$.

Thus , $\|f(go\varphi)\| = \|g\|$, $then$ $\|f.\varphi\| = 1$.

Since $\left|\varphi(e^{it})\right| \leq 1 \quad a.e. \quad t \in [0,2\pi)$

and both $\|f\|$ and $\|f.\varphi\|$ are 1 .Then, by the integral representation of $\|f\|_{\mathbb{H}^2}$

$$\|f\|_{\mathbb{H}^2}^2 = \frac{1}{2\pi}\int_0^{2\pi} \left|f(e^{it})\right|^2 dt \quad .$$

So that $\left|\varphi\left(e^{it}\right)\right| = 1 \quad a.e. \quad on\ U$, then φ is an inner function **.** ■

Theorem (3.14):[] Let φ be an inner function that fixes the origin .If ψ is also a non-constant inner function , then $\mathcal{W}_{\psi,\varphi}$ is a shift operator such that $W(\mathcal{W}_{\psi,\varphi}) = \mathbb{U}$.

Corollary (3.15): Suppose that $\varphi\ and\ \psi$ be two linear- fractional self-maps of $\mathbb{U}$ and $f \in \mathbb{H}^\infty$ such that each of $\varphi\ and\ \sigma$ fixes the origin, then $\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*$ is a shift operator and $W(\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*) = \mathbb{U}$.

Proof : Since $\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*$ is an isometry operator , then by lemma (3.2) and lemma (3.13) $\sigma \circ \varphi$ is an inner function and $\|k\|_\infty = 1$. But by our assumption that each of $\varphi\ and\ \sigma$ fixes the origin , then $\sigma \circ \varphi$ is also. In addition that since$\|k\|_\infty = 1$, then we have $|k(0)| < 1$, hence k is a non-constant inner function. Therefore by theorem (3.2.13) $\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*$ is shift operator such that $W(\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*) = \mathbb{U}$. ■

Next , we study the numerical rang of $\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*$ when it is an isometry that is not unitary .

Theorem (3.16):[] Let $\mathcal{W}_{\psi,\varphi}$ be an isometry on $\mathbb{H}^2$. If $\mathcal{W}_{\psi,\varphi}$ is not a unitary operator, then $W(\mathcal{W}_{\psi,\varphi}) = \mathbb{U} \cup \sigma_p(\mathcal{W}_{\psi,\varphi})$.

Corollary (3.17): Let $\varphi\ and\ \psi$ be two linear- fractional self-maps of U and $f \in \mathbb{H}^\infty$ such that either $\varphi\ or\ \psi$ is not automorphism of $\mathbb{U}$. If $\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*$ is an isometry, then $W(\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*) = \mathbb{U} \cup \sigma_p\left(\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*\right)$.

Proof : Assume that $\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*$ is isometry operator, then by lemma(3.2) and lemma (3.13), $\|k\|_\infty = 1$. Since either $\varphi\ or\ \psi$ is not automorphism of $\mathbb{U}$, then $\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*$ is not unitary, Therefore, by theorem(3.15) we have $W(\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*) = \mathbb{U} \cup \sigma_p\left(\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*\right)$.

■

The following result describe the spectra of unitary weighted composition operator.

Theorem (3.16):[] Suppose $\mathcal{W}_{\psi,\varphi}$ is unitary. If φ is elliptic with fixed point p, then $|\acute{\varphi}(p)| = 1$

and the spectrum of $\mathcal{W}_{\psi,\varphi}$ is the closure of the set $\{\psi(p)\acute{\varphi}(p)^n : n = 0,1,2,\ldots\}$.

If φ is parabolic or hyperbolic, then the spectrum is the unit circle.

Theorem (3.19): Suppose that $\varphi\ and\ \psi$ be two automorphism self-maps of $\mathbb{U}$ and $f \in \mathbb{H}^\infty$. If $K(z) = \frac{r\,K_p(z)}{\|K_p\|}$ $where\ |r| = 1\ such\ that\ \varphi(p) = \psi(p) = 0$,then we have the following statements :

1- If $\sigma \circ \varphi$is hyperbolic or parabolic, then $W(\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi}) = \mathbb{U}$.

2- If $\sigma \circ \varphi$is elliptic with fixed point p $\in \mathbb{U}$ .Then, if $(\sigma \circ \varphi)'(p)$ is a root of unity, we have $W(\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi}) = \mathbb{U}$. Otherwise, $W(\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi})$ is a polygon with vertices $k(p), k(p).(\sigma \circ \varphi)'(p), \dots, k(p)\,.((\sigma \circ \varphi)'(p))^{n-1}$.

Proof: Since each of $\varphi\ and\ \psi$ is automorphism with above hypotheses, therefore $\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi}$is unitary operator .Therefore,$\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi}$ is convexiod, thus

$$conv\ \sigma(\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi}) = \overline{W(\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi})} \qquad (5)$$

Therefore we can get the following cases :

(1) If $\sigma \circ \varphi$ is hyperbolic then by theorem(3.18) we have $W(\mathcal{W}_{f,\varphi}\cdot^*_{f,\psi}) = \mathbb{U}$. On the other hand , if $\sigma \circ \varphi$ is parabolic, then by theorem(3.18) we have $W(\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi}) = \mathbb{U}$.

(2) If $\sigma \circ \varphi$ is elliptic with fixed point p $\in \mathbb{U}$, then by theorem(3.18) we have $\left|(\psi^{-1} o \varphi)'(p)\right| = 1$ and

$\sigma\left(\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi}\right) = \{k(p)\,.((\sigma \circ \varphi)'(p))^n : n = 0,1,2, \dots\}$. Thus two cases which are appeared
a. If $(\sigma \circ \varphi)'(p)$ is a root of unity.

Therefore it is clear that

$$\sigma\left(\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi}\right) = \{\ k(p), k(p).(\sigma \circ \varphi)'(p), \dots, k(p)\,.((\sigma \circ \varphi)'(p))^{n-1}\}$$

Hence it is clear that by (5) we have $W(\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi})$ is a polygon with vertices $k(p), k(p).(\sigma \circ \varphi)'(p), \dots, k(p)\,.((\sigma \circ \varphi)'(p))^{n-1}$.

b. If $(\sigma \circ \varphi)'(p)$ is not a root of unity.

Since$\|k\|_\infty = 1$,then $|k(p)| = 1$, therefore we have by [] that the set $\{k(p)\,.(\sigma \circ \varphi)'(p))^n : n = 0,1,2, \dots\}$ is dense in $\sigma\left(\mathcal{W}_{f,\varphi}.\mathcal{W}^*_{f,\psi}\right)$. But $(\sigma \circ \varphi)'(p)$ is not a root of unity , then $\sigma\left(\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi}\right) = \partial\mathbb{U}$. This implies by (5) that $W(\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi}) = \mathbb{U}$.

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