Journal of Progressive Research in Mathematics(JPRM)
ISSN: 2395-0218

SCITECH
RESEARCH ORGANISATION|

Volume 8, Issue 1
Published online: June 30, 2016|

Journal of Progressive Research in Mathematics
www.scitecresearch.com/journals

On Invertible Weighted Composition Operator on Hardy Space $\mathbb{H}^2$.

Abood E. H. and Mohammed A. H.

Department of Mathematics, College of science, University of Baghdad, Jadirya, Baghdad, Iraq.

Abstract. In this paper we study the product of a weighted composition operator $\mathcal{W}_{f,\varphi}$ with the adjoint of a weighted composition operator $\mathcal{W}^*_{f,\psi}$ on the Hardy space $\mathbb{H}^2$. The order of the product give rise to different cases . We will try to completely describe when the operator $\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi}$ is invertible, isometric and unitary and when the operator $\mathcal{W}^*_{f,\psi}\mathcal{W}_{f,\varphi}$ is isometric and unitary.

1. Introduction

Let U denote the open unite disc in the complex plan, let $\mathbb{H}^\infty$ denote the collection of all holomorphic function on U and let $\mathbb{H}^2$ is consisting of all holomorphic self-map on U such that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ whose Maclaurin coefficients are square summable (i.e) $f(z) = \sum_{n=0}^{\infty}|a_n|^2 < \infty$. More precisely $f(z) = \sum_{n=0}^{\infty} a_n z^n$ if and only if $\|f\| = \sum_{n=0}^{\infty}|a_n|^2 < \infty$.The inner product inducing the $\mathbb{H}^2$ norm is given by $\langle f,g\rangle = \sum_{n=0}^{\infty} a_n \overline{b_n}$.

Given any holomorphic self-map φ on U, recall that the composition operator

is called the composition operator with symbol φ, is necessarily bounded. Let $f \in \mathbb{H}^\infty$, the operator $T_f: \mathbb{H}^2 \to \mathbb{H}^2$defined by

$$T_f(h(z)) = f(z)h(z), \quad for\ all\ z \in \text{U}, h \in \mathbb{H}^2$$

is called the Toeplitz operator with symbol f. Since $f \in \mathbb{H}^\infty$, then we call T_f a holomorphic Toeplitz operator. If T_f is a holomorphic Toeplitz operator,then the operator $T_f C_\varphi$ is bounded and has the form

$$T_f C_\varphi g = f(go\varphi) \qquad (g \in \mathbb{H}^2).$$

We call it the weighted composition operator with symbols f and φ [1] and [3], the linear operator

$$\mathcal{W}_{f,\varphi}\, g = f(go\varphi) \qquad (g \in \mathbb{H}^2).$$

We distinguish between the two symbols of weighted composition operator $\mathcal{W}_{f,\varphi}$, by calling f the multiplication symbol and φ composition symbol.

For given holomorphic self-maps f and φ of U, $\mathcal{W}_{f,\varphi}$ is bounded operator even if $f \notin \mathbb{H}^{\infty}$. To see a trivial example, consider $\varphi(z) = p$ where $p \in$ U and $f \in \mathbb{H}^{2}$,then for all $g \in \mathbb{H}^{2}$, we have

$$\| \mathcal{W}_{f,\varphi}\, g\|_2 = \|g(p)\|\|f\|_2 = \|f\|_2 |\langle g, K_p\rangle| \leq \|f\|_2 \|g\|_2 \|K_p\|_2.$$

In fact, if $f \in \mathbb{H}^{\infty}$,then $\mathcal{W}_{f,\varphi}$ is bounded operator on $\mathbb{H}^{2}$ with norm

$$\| \mathcal{W}_{f,\varphi} \| = \|T_f C_\varphi\| \leq \|f\|_\infty \|C_\varphi\| = \|f\|_\infty \sqrt{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}}.$$

2. Basic Concepts

We start this section, by giving the following results which are collect some properties of Toeplitz and composition operators.

Lemma (2.1):[4, 6] Let φbe a holomorphic self-map of U, then

(a) $C_\varphi T_f = T_{fo\varphi} C_{\varphi.}$

(b) $T_g T_f = T_{gf}$.

(c) $T_{f+\gamma g} = T_f + \gamma T_g$.

(d) $T_f^* = T_{\bar{f}}$.

Proposition (2.2):[1] Let φ and ψ be two holomorphic self-map of U, then

1. $C_\varphi^n = C_{\varphi_n}$ for all positive integer n.

2. C_φ is the identity operator if and only if φ is the identity map.

3. $C_\varphi = C_\psi$ if and only if $\varphi=\psi$.

4. The composition operator cannot be zero operator.

For each $\alpha \in U$,the reproducing kernel at α ,defined by $K_\alpha(z) = \frac{1}{1-\bar{\alpha}z}$

It is easily seen for each $\alpha \in U$ and $f \in H^2$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ that

$$\langle f, K_\alpha\rangle = \sum_{n=0}^{\infty} a_n\, \alpha^n = f(\alpha).$$

When $\varphi(z) = (az+b)/cz+d)$ is linear-fractional self-map of U,Cowen in [2] establishes $C_\varphi^* = T_g C_\sigma T_h^*$,where the Cowen auxiliary functions g, σ and h are defined as follows:

$$g(z) = \frac{1}{-\bar{b}z+\bar{d}} \quad ,\sigma(z) = \frac{\bar{a}z-\bar{c}}{-\bar{b}z+\bar{d}} \qquad and \quad h(z) = cz+d \ .$$

If φ is linear fractional self-map U, then $W_{f,\varphi}^* = (T_f C_\varphi)^* = C_\varphi^* T_f^* = T_g C_\sigma T_h^*$.

Proposition (2.4):[5] Let each of $\varphi_1, \varphi_2, \dots \varphi_n$ be holomorphic self-mapsof Uand $f_1, f_2, \dots f_n \in \mathbb{H}^{\infty}$, then

$$\mathcal{W}_{f_1,\varphi_1} . \mathcal{W}_{f_2,\varphi_2} \dots . \mathcal{W}_{f_n,\varphi_n} = T_h C_\phi$$

Where$T_h = f_1 . (f_2 o\varphi_1) . (f_3 o\varphi_2 o\varphi_1) . \dots (f_2 o\varphi_{n-1} o\varphi_{n-2} o \dots . o\varphi_1)$ and

$C_\phi = \varphi_n o \varphi_{n-1} o \ldots . o \varphi_1$.

Corollary (2.5): Let φ be a holomorphic self-map of U and $f \in \mathbb{H}^\infty$ then

$$\mathcal{W}_{f,\varphi}^n = T_{f \ (f \ o\varphi \ )(f \ o\varphi_2)\ldots(f \ o\varphi_{n-1})} C_{\varphi_n}$$

The following lemma discuss the adjoint of weighted composition operator .

Lemma (2.6):[3] If the operator $\mathcal{W}_{f,\varphi}: \mathbb{H}^2 \to \mathbb{H}^2$ is bounded,then for each $\alpha \in U$

$$\mathcal{W}_{f,\varphi}^* K_\alpha = \overline{f(\alpha)} K_{\varphi(\alpha)}.$$

3- Invertible Weighted Composition Operator

In this section, we study the product of a weighted composition operator $\mathcal{W}_{f,\varphi}$ with the adjoint of a weighted composition operator $\mathcal{W}_{f,\psi}^*$ on the Hardy space $\mathbb{H}^2$. The order of the product give rise to different cases. We will try to completely describe when the operator $\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*$ is invertible, isometric and unitary and when the operator $\mathcal{W}_{f,\psi}^*\mathcal{W}_{f,\varphi}$ is isometric and unitary. First we try to obtain some properties of the operator $\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*$.

<u>Proposition (3.1):</u> Suppose φ *and* ψ be two holomorphic self-map of U and $f \in \mathbb{H}^\infty$, such that 0 is not a fixed point of U then $\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*$ is self-adjoint if and only if

$\psi(z) = \lambda\varphi(z)$, for all $z \in U$.

Proof : Let $\beta \in U$, then for each $z \in U$, we have

$$(\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*)^* K_\beta(z) = \mathcal{W}_{f,\psi}\mathcal{W}_{f,\varphi}^* K_\beta(z)$$
$$= T_f C_\psi\left(\overline{f(\beta)} K_{\varphi(\beta)}(z)\right)$$
$$= \overline{f(\beta)}\, f(z) K_{\varphi(\beta)}(\psi(z)) \quad .$$

On the other hand , for each $z \in U$, we have

$$\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^* K_\beta(z) = T_f C_\varphi\left(\overline{f(\beta)} K_{\psi(\beta)}(z)\right)$$
$$= \overline{f(\beta)}\, f(z) K_{\psi(\beta)}(\varphi(z)) \ .$$

Therefore, $\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*$ is self-adjoint if and only if for each $z \in U$

$$K_{\varphi(\beta)}(\psi(z)) = K_{\psi(\beta)}(\varphi(z))$$

Hence,

$$\frac{1}{1-\overline{\varphi(\beta)}\psi(z)} = \frac{1}{1-\overline{\psi(\beta)}\varphi(z)} \qquad (1)$$

In particular letting $\beta = 0$ in equation (3.1), we get

$\psi(z) = \lambda\varphi(z)$ *where* $\lambda = (\overline{\frac{\psi(0)}{\varphi(0)}})$ (note that $\varphi(0) \neq 0$) . ■

Recall that [2] an operator T is an isometry if $\|Tx\| = \|x\|$ for all x or equivalently $T^* T = I$.

Nordgren E.M [7] characterized the isometry composition operator as follows .

Theorem (3.2): A composition operator C_φ is an isometry if and only if φ is an inner function and $\varphi(0) = 0$.

Now , to characterize the inevitability of $\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi}$, we need the following results .

Lemma (3.3): Suppose that φ be a holomorphic self-map of U and $\in \mathbb{H}^\infty$. If $\mathcal{W}_{f,\varphi}$ is an isometry, then φ must be inner function and $\|f\| = 1$.

Proof : Let the operator $\mathcal{W}_{f,\varphi}$ is an isometry, then $\mathcal{W}^*_{f,\varphi}.\mathcal{W}_{f,\varphi} = I$. Thus for each $p \in U$, we have

$\|\mathcal{W}_{f,\varphi}K_p\| = \|K_p\|$, then $\|T_f C_\varphi K_p\| = \|K_p\|$.

This implies that $\|f(K_p \circ \varphi)\| = \|K_p\|$. Hence, by taking $p = 0$, then $K_0 = 1$

and thus $\|f(1 \circ \varphi)\| = \|1\|$ $, then$ $\|f\| = 1$

In addition that, if $g(z) = z$,then it is clear that $\|g\| = 1$ **.** Therefore

$\|\mathcal{W}_{f,\varphi}g\| = \|g\|$, and then $\|T_f C_\varphi g\| = \|g\|$.

Thus , $\|f(go\varphi)\| = \|g\|$, $then$ $\|f.\varphi\| = 1$.

Since $|\varphi(e^{it})| \le 1$ $a.e.$ $t \in [0,2\pi)$

and both $\|f\|$ and $\|f.\varphi\|$ are 1 .Then, by the integral representation of $\|f\|_{\mathbb{H}^2}$

$$\|f\|^2_{\mathbb{H}^2} = \frac{1}{2\pi}\int_0^{2\pi} |f(e^{it})|^2 dt \quad .$$

So that $|\varphi(e^{it})| = 1$ $a.e.$ $on\ U$, then φ is an inner function **.** ■

Gunatillake G. [5] studied the invertible weighted composition operator on Hardy space $\mathbb{H}^2$. He give the following result .

Theorem (3.4):[5] The operator $\mathcal{W}_{f,\varphi}$ on $\mathbb{H}^2$ is invertible if and only if f is both bounded and bounded away from zero on the unit disc and φ is an automorphism of the unit disc. The inverse operator is the weighted composition operator $\mathcal{W}^{-1}_{f,\varphi} = \mathcal{W}_{\frac{1}{(f\circ\varphi^{-1})},\varphi^{-1}}$.

We are ready to discuss the inevitability of the operator of the operator $\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi}$ **.**

Theorem (3.5): Suppose that φ and ψ be two holomorphic self-map of U and $f \in \mathbb{H}^\infty$. Then $\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi}$ is invertible if and only if each of $\mathcal{W}_{f,\varphi}$ and $\mathcal{W}_{f,\psi}$ is invertible operator.

Proof : Suppose that $\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi}$ is invertible, then the operator $\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi}$ is one-to-one and onto. Hence, $\mathcal{W}_{f,\varphi}$ is onto. Therefore it is clear that, φ is non- constant map.

Thus, $\mathcal{W}_{f,\varphi}$ is one-to-one . Hence $\mathcal{W}_{f,\varphi}$ is invertible.

Now, since each of $\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi}$ and $\mathcal{W}_{f,\varphi}$ is invertible, then we have $\mathcal{W}_{f,\psi}$ must be invertible operator.

The reverse induction follows immediately. ■

A straightforward consequence can obtained from theorem (3.4).

Corollary (3.6): Suppose that φ *and* ψ be two holomorphic self-map of U and $f \in \mathbb{H}^\infty$. Then $\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi}$ is invertible if and only if f is bounded and bounded away from zero on U and each of φ and ψ is an automorphism of U.

Corollary (3.7): Let φ *and* ψ be two holomorphic self-map of U and $f \in \mathbb{H}^\infty$. If $\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi}$ is invertible , then $(\mathcal{W}_{f,\varphi}.\mathcal{W}^*_{f,\psi})^{-1} = C^*_{\psi^{-1}}.\mathcal{W}_{[1/\overline{(f o \psi^{-1})}(f o \varphi^{-1})],\varphi^{-1}}$.

Proof : Since by theorem (3.1.3) we have

$\mathcal{W}^{-1}_{f,\varphi} = \mathcal{W}_{\frac{1}{(f o \varphi^{-1})},\varphi^{-1}}$ and $\mathcal{W}^{-1}_{f,\psi} = \mathcal{W}_{\frac{1}{(f o \psi^{-1})},\psi^{-1}}$. Then ,

$$(\mathcal{W}^*_{f,\psi})^{-1} = (\mathcal{W}^{-1}_{f,\psi})^* = (\mathcal{W}_{\frac{1}{(f o \psi^{-1})},\psi^{-1}})^* = (T_{\frac{1}{(f o \psi^{-1})}}C_{\psi^{-1}})^*$$

$$= C^*_{\psi^{-1}}T_{\frac{1}{\overline{(f o \psi^{-1})}}} \quad .$$

Hence , $\left(\mathcal{W}_{f,\varphi}.\mathcal{W}^*_{f,\psi}\right)^{-1} = (\mathcal{W}^*_{f,\psi})^{-1}(\mathcal{W}_{f,\varphi})^{-1}$

$$= (C^*_{\psi^{-1}}T_{\frac{1}{\overline{(f o \psi^{-1})}}}).(T_{\frac{1}{(f o \varphi^{-1})}}C_{\varphi^{-1}})$$

$$= C^*_{\psi^{-1}}\ T_{\frac{1}{\overline{(f o \psi^{-1})}(f o \varphi^{-1})}}\ C_{\varphi^{-1}}$$

$$= C^*_{\psi^{-1}}.\mathcal{W}_{\frac{1}{\overline{(f o \psi^{-1})}(f o \varphi^{-1})},\varphi^{-1}} \quad .■$$

In the following , we give the necessary and sufficient condition to the operator $\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi}$ to be isometry first we need the next lemma .

Lemma (3.8)[9]: If T is isometry operator and S is unitary operator , then $TS^*$is an isometry .

Theorem (3.9): Suppose that φ *and* ψ be two holomorphic self-maps of U and $f \in \mathbb{H}^\infty$ such that $\|f\|_{\mathbb{H}^\infty} = 1$. Then $\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi}$ is an isometry if and only if $\mathcal{W}_{f,\varphi}$ is an isometry and $\mathcal{W}_{f,\psi}$ is an unitary operator .

Proof : Suppose that $\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi}$ is an isometry , therefore

$(\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi})^*\ \mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi} = I$. Thus

$\mathcal{W}_{f,\psi}\mathcal{W}^*_{f,\varphi}\ \mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi} = I$ **.** Hence one can easily see that $\mathcal{W}_{f,\psi}$ is onto .

This it is clear that, ψ is non- constant map. Therefore by lemma (2.4.3) we have $\mathcal{W}_{f,\psi}$ is one-to-one.

Thus $\mathcal{W}_{f,\psi}$ invertible. Therefore by theorem (3.1.5) and corollary (3.1.6) ψ must be an automorphismof U. So that there exists $\eta \in \partial U$ *and* $p \in U$, that for each $z \in U$

$$\psi(z) = \eta\left(\frac{p-z}{1-\bar{p}z}\right), \quad where \psi(p) = 0 \quad .$$

But $\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^{*}$ is an isometry, then for every $p \in U$, we conclude that

$$\left\|\mathcal{W}_{f,\varphi}\, \mathcal{W}_{f,\psi}^{*} K_p\right\| = \left\|K_p\right\| \quad .$$

Thus , $\quad \left\|\mathcal{W}_{f,\varphi}\left(\, \overline{f(p)} K_{\psi(p)}\right)\right\| = \left\|K_p\right\| \;$.

Hence , $\quad \left\|T_f C_\varphi\left(\, \overline{f(p)} K_0\right)\right\| = \left\|K_p\right\| \;$.

Then , $\quad \left\|\overline{f(p)} T_f C_\varphi(K_0)\right\| = \left\|K_p\right\| \;$.

Therefore , $\quad \left\|\overline{f(p)}\, f(K_0 \circ \varphi)\right\| = \left\|K_p\right\| \;$.

But $\;(K_0 \circ \varphi = \; 1 \circ \varphi \; = 1)\quad , \quad \left\|\overline{f(p)}\, f\right\| = \left\|K_p\right\| \;$.

Hence , $\left|\overline{f(p)}\right| \| f \| = \left\|K_p\right\| \quad$.

Then , $\quad \left|\langle f, K_p\rangle\right| = \left\|K_p\right\| = \|f\| \left\|K_p\right\| \quad$.

Thus, by Cauchy –Schwartz inequality , we have

$$f(z) = \alpha K_p(z) = \frac{\alpha}{1-\bar{p}z} \qquad for\ some\ \ \alpha \in \mathbb{C}$$

But $\| f \| = 1$, then it easily see that $f(z) = r\frac{K_p}{\|K_p\|}$where$|r| = 1 \;\; and\ \psi(p) = 0$

Hence by theorem (2.9) we have $\mathcal{W}_{f,\psi}$ is unitary operator .

Conversely , if $\mathcal{W}_{f,\varphi}$ is an isometry and $\mathcal{W}_{f,\psi}$ is unitary , then

$$\mathcal{W}_{f,\varphi}^{*}\mathcal{W}_{f,\varphi} = \mathcal{W}_{f,\psi}^{*}\mathcal{W}_{f,\psi} = \mathcal{W}_{f,\psi}\mathcal{W}_{f,\psi}^{*} = I \ (2)$$

Hence from (3.2) we have

$$(\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^{*})^{*}\, \mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^{*} = \mathcal{W}_{f,\psi}\, \mathcal{W}_{f,\varphi}^{*}\, \mathcal{W}_{f,\varphi}\, \mathcal{W}_{f,\psi}^{*} = I \ .$$

Therefore $\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^{*}$ is an isometry ,as desired . ■

Corollary (3.10): Suppose $\varphi\ and\ \psi$ be two holomorphic self-map of U and $f \in \mathbb{H}^{\infty}$such that$\|f\|_{\mathbb{H}^{\infty}} = 1$. Then $\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^{*}$ is unitary if and only if each of $\mathcal{W}_{f,\varphi}$ and $\mathcal{W}_{f,\psi}$ is an unitary operator .

Proof : Suppose that $\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^{*}$ is an unitary operator , then it is isometry. Therefore by theorem (3.9) we have $\mathcal{W}_{f,\psi}$ is unitary operator . But since $\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^{*}$ is unitary, then $\mathcal{W}_{f,\psi}\mathcal{W}_{f,\varphi}^{*}$ is also unitary , thus by theorem (3.9) we have $\mathcal{W}_{f,\varphi}$ is unitary operator .

The converse is clear .

Now , the corollary (3.9) and theorem (2.9) we get the following consequence .

Corollary (3.11): Suppose φ *and* ψ be two holomorphic self-map of U and $f \in \mathbb{H}^{\infty}$such that$\|f\|_{\mathbb{H}^{\infty}} = 1$. Then $\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^{*}$ is unitary if and only if each of φ *and* ψ is an automorphism of U and $f(z) = r\frac{K_p}{\|K_p\|}$ such that $p \in U$ where $|r| = 1$ and

$$\varphi\,(p) = \psi(p) = 0 \;\;.$$

We are in a position to examine when $\mathcal{W}_{f,\psi}^{*}\mathcal{W}_{f,\varphi}$ dose admit characterization analogous to the operator $\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^{*}$, we first record result regarding norm .

Theorem (3.12): Suppose φ *and* ψ be two holomorphic self-map of U and $f \in \mathbb{H}^{\infty}$ such that $\|f\|_{\mathbb{H}^{\infty}} = |f(0)|^2 = 1$. Then $\left\|\mathcal{W}_{f,\psi}^{*}\mathcal{W}_{f,\varphi}\right\| = 1$ if and only if

$\psi(0) = \varphi(0) = 0$.

Proof : If $\left\|\mathcal{W}_{f,\psi}^{*}\mathcal{W}_{f,\varphi}\right\| = 1$, then for each $\alpha, z \in U$ we get that

$\mathcal{W}_{f,\psi}^{*}\mathcal{W}_{f,\varphi}\, K_{\alpha}(z) = \mathcal{W}_{f,\psi}^{*}(f(z)\, K_{\alpha}(\varphi(z))$.

Thus by letting $\alpha = 0$ *and* $z = 0$, yields

$$\begin{aligned} \mathcal{W}_{f,\psi}^{*}\mathcal{W}_{f,\varphi}\, K_{\alpha}(z) &= \mathcal{W}_{f,\psi}^{*}\mathcal{W}_{f,\varphi}\, K_0(0) \\ &= \mathcal{W}_{f,\psi}^{*}(f(0)\, K_0 \circ \varphi(0)) \\ &= f(0)\mathcal{W}_{f,\psi}^{*}\,(K_0) \\ &= f(0)\overline{f(0)}\, K_{\psi(0)} \\ &= |f(0)|^2 K_{\psi(0)} \\ &= K_{\psi(0)}\,. \end{aligned}$$

Hence , we have

$$\left\|K_{\psi(0)}\right\| \le \left\|\mathcal{W}_{f,\psi}^{*}\mathcal{W}_{f,\varphi}\right\| = 1 \qquad (3.3)$$

Thus ,

$$\left\|K_{\psi(0)}\right\|^2 = \frac{1}{1 - |\psi(0)|^2} \le 1$$

which implies that $\psi(0) = 0$. But we know that ,

$$\left\|\mathcal{W}_{f,\psi}^{*}\mathcal{W}_{f,\varphi}\right\| = \left\|\mathcal{W}_{f,\varphi}^{*}\mathcal{W}_{f,\psi}\right\| = 1 \;\;.$$

Therefore , similarly we obtain that $\varphi(0) = 0$, as desired .

Conversely , assume that $\varphi(0) = \psi(0) = 0$. Thus ,

$$\begin{aligned} \left\|\mathcal{W}_{f,\psi}^{*}\mathcal{W}_{f,\varphi}\right\| &\le \left\|\mathcal{W}_{f,\psi}\right\|\left\|\mathcal{W}_{f,\varphi}\right\| \\ &\le \|f\|_{\mathbb{H}^{\infty}}^2 \left\|C_{\psi}\right\|\left\|C_{\varphi}\right\| \end{aligned}$$

$$\leq \|f\|_{\mathbb{H}^\infty}^2 \sqrt{\frac{1+|\varphi(0)||1+\psi(0)|}{1-|\varphi(0)||1-\psi(0)|}}$$

And the hypothesis$\varphi(0) = \psi(0) = 0$ and $\|f\| = 1$ implies that

$\|\mathcal{W}_{f,\psi}^* \mathcal{W}_{f,\varphi}\| \leq 1$. Moreover , from (3) we have

$$\|\mathcal{W}_{f,\psi}^* \mathcal{W}_{f,\varphi}\| \geq \|K_{\psi(0)}\| = 1 \ .$$

Hence , $\|\mathcal{W}_{f,\psi}^* \mathcal{W}_{f,\varphi}\| = 1$.■

Corollary (3.13): Suppose φ *and* ψ be two holomorphic self-map of U and $f \in \mathbb{H}^\infty$such that $\|f\|_{\mathbb{H}^\infty} = |f(0)|^2 = 1$. If $\mathcal{W}_{f,\psi}^* \mathcal{W}_{f,\varphi}$ is an isometry, then $\psi(0) = \varphi(0) = 0$.

Proof : If $\mathcal{W}_{f,\psi}^* \mathcal{W}_{f,\varphi}$ is an isometry, then its norm is one .Thus by theorem(3.1.12) we conclude that $\psi(0) = \varphi(0) = 0$. ■

Now, consider the case $\mathcal{W}_{f,\psi}^* \mathcal{W}_{f,\varphi}$ is an isometry . We will require some preliminary results.

Proposition (3.14)[9]: Let S and T be contractive operators on a Hilbert space . If S^*T is an isometry , then T is an isometry and we have $T = SS^*T$.

Lemma (3.15)[9]: Suppose φ *and* ψ are holomorphic self-maps of U such that φ is non-constant and $C_\varphi = C_\psi T$ for some $T \in B(\mathbb{H}^2)$. Thus there is a holomorphic self -map α of U such that $T = C_\alpha$ *and* $\varphi = \alpha \circ \psi$.

Corollary (3.16):

Suppose φ *and* ψ are holomorphic self-maps of U such that $f \in \mathbb{H}^\infty/\{0\}$. If φ is non-constant map and $\mathcal{W}_{f,\varphi} = \mathcal{W}_{f,\psi} S$ for some $S \in B(\mathbb{H}^2)$. Then there is a holomorphic self -map α of U such that $S = C_\alpha$ *and* $\varphi = \alpha \circ \psi$.

Proof : It follows from $\mathcal{W}_{f,\varphi} = \mathcal{W}_{f,\psi} S$ that for each $z \in U$, $g \in \mathbb{H}^2$

$f(z) C_\varphi g(z) = f(z) C_\psi S\, g(z)$. Hence , $C_\varphi = C_\psi S$ **.** Hence the consequence follows immediately by lemma(3.15). ■

We are now in a position to analyze $\mathcal{W}_{f,\psi}^* \mathcal{W}_{f,\varphi}$ in the case where the product is isometry.

Theorem (3.17): Suppose φ *and* ψ be two holomorphic self-map of U and $f \in \mathbb{H}^\infty$ such that $\|f\|_{\mathbb{H}^\infty} = |f(0)|^2 = 1$. If $\mathcal{W}_{f,\psi}^* \mathcal{W}_{f,\varphi}$ is an isometry, each of φ *and* ψ is an inner function with $\psi(0) = \varphi(0) = 0$ and $\varphi = \alpha \circ \psi$ where $\alpha: U \to U$ is inner with $\alpha(0) = 0$.

Proof : Suppose $\mathcal{W}_{f,\psi}^* \mathcal{W}_{f,\varphi}$ is an isometry . By corollary (3.1.13) we have $\psi(0) = \varphi(0) = 0$. This implies that ,

$$\|\mathcal{W}_{f,\varphi}\| \leq \|f\|_{\mathbb{H}^\infty} \|C_\varphi\| \leq \|f\|_{\mathbb{H}^\infty} \sqrt{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}} = 1 \ .$$

Similarly $\|\mathcal{W}_{f,\psi}\| \leq 1$, therefore each of $\mathcal{W}_{f,\psi}$ *and* $\mathcal{W}_{f,\varphi}$ is contractive on $\mathbb{H}^2$. Now, applying corollary (3.1.16) with $S = \mathcal{W}_{f,\psi}$ and $T = \mathcal{W}_{f,\varphi}$, we get that $\mathcal{W}_{f,\varphi}$ is isometry and $\mathcal{W}_{f,\varphi} =$

$\mathcal{W}_{f,\psi}\mathcal{W}_{f,\psi}^{*}\mathcal{W}_{f,\varphi}$. Therefore , by lemma (3.3) we get that φ is an inner function . Thus it is clear that φ is non-constant.

Now , by corollary (3.16) there exists a holomorphic self-map α of U such that

$C_{\alpha} = \mathcal{W}_{f,\psi}^{*}\mathcal{W}_{f,\varphi}$ and $\varphi = \alpha \circ \psi$.

Now , C_{α} is an isometry , then by theorem (3.2) we have α is inner function such that $\alpha(0) = 0$. Since each of φ and α is inner function , then ψ is also .

Conversely , if each of φ *and* ψ is inner function such that $\varphi(0) = \psi(0) = 0$

$\varphi = \alpha \circ \psi$ where $\alpha: U \to U$ is inner with $\alpha(0) = 0$. Using the identity $C_{\alpha} = \mathcal{W}_{f,\psi}^{*}\mathcal{W}_{f,\varphi}$, we obtain by theorem (3.2) that C_{α} is an isometry , as desired .■

Now, we are ready to use the isometric characterization to describe precisely when $\mathcal{W}_{f,\psi}^{*}\mathcal{W}_{f,\varphi}$ is a unitary operator**.**

Corollary (3.18): Suppose φ *and* ψ be two holomorphic self-map of U and $f \in \mathbb{H}^{\infty}$such that$\|f\|_{\mathbb{H}^{\infty}} = |f(0)|^2 = 1$. Then $\mathcal{W}_{f,\psi}^{*}\mathcal{W}_{f,\varphi}$ is unitary if and only if each of φ *and* ψ is an inner function with $\psi(0) = \varphi(0) = 0$ and there exists inner function α with $\alpha(0) = 0$ such that $\varphi = \alpha \circ \psi$.

Proof : Suppose $\mathcal{W}_{f,\psi}^{*}\mathcal{W}_{f,\varphi}$ is unitary , then by theorem (3.17) both φ *and* ψ is an inner function with $\psi(0) = \varphi(0) = 0$ and there exists inner function α with $\alpha(0) = 0$ such that $\varphi = \alpha \circ \psi$.

As in theorem (3.17) we have $\mathcal{W}_{f,\psi}^{*}\mathcal{W}_{f,\varphi} = C_{\alpha}$, and so C_{α} is unitary. This implies $\alpha(z) = \lambda z$ for some λ with $|\lambda| = 1$. Therefore $\varphi(z) = \lambda\psi(z)$. The reverse induction is clear. ■

Now , we are ready to recover the inevitability of the operator $\mathcal{W}_{f,\psi}^{*}\mathcal{W}_{f,\varphi}$. We need the following lemma.

Lemma (3.19)[10]: Suppose φ be univalent , holomorphic self-map of U. Then C_{φ} has closed range on $\mathbb{H}^{\mathbf{2}}$ if and only if φ is an automorphism of U .

Theorem (3.20): Suppose φ *and* ψ be two holomorphic self-map of U such that ψ is univalent and $f \in \mathbb{H}^{\mathbf{2}}$which is bounded and bounded away from zero . Then $\mathcal{W}_{f,\psi}^{*}\mathcal{W}_{f,\varphi}$ is invertible if and only if each of φ *and* ψ are automorphism of U.

Proof : Suppose that $\mathcal{W}_{f,\psi}^{*}\mathcal{W}_{f,\varphi}$ is invertible, then $\mathcal{W}_{f,\psi}^{*} = C_{\psi}^{*}T_{f}^{*}$ is onto. Therefore, it is clear that C_{ψ}^{*} is onto . This implies that C_{ψ} is bounded from below and so the range of C_{ψ} is closed. Thus by lemma (3.19) we have ψ is an automorphism . Therefore by applying theorem (3.4) we have that $\mathcal{W}_{f,\psi}$ is invertible operator. Hence $\mathcal{W}_{f,\psi}^{*}$ is invertible and then $\mathcal{W}_{f,\varphi}$ is invertible.

Therefore again by theorem (3.4) that φ is an automorphism.

The converse is follows immediately by theorem (3.4). ■

References

[1] Abood E.H., " The composition operator on Hardy space H^2", Ph.D. Thesis, University of Baghdad, (2003).

[2] Berberian, S.K., Introduction to Hilbert Space, Sec. Ed., Chelesa publishing Com., New York, N.Y., 1976.

[3] CowenC. C. and Ko E.,"Hermitian weighted composition operator on H^2 "Trans.Amer. Math. Soc., 362(2010), 5771-5801.

[4] Deddnes J. A. , "Analytic Toeplitz and Composition Operators", Con. J. Math., vol(5), 859-865, (1972).

[5] Gunatillake G.,"invertible weighted composition operator ",J. Funct. Anal., 261(2011),

831-860.

[6] Halmos P. R.,"A Hilbert space problem book", Sprinrer- Verlag, NewYork,(1974).

[7] Nordgren, E. A., Composition operator, Can. J. Math. 20(1968), 442-449.

[8] Shapiro J.H., "Composition Operators and Classical Function Theory", Springer-Verlage, New York, (1993).

[9] Clifford, J. H. , Le, T. and Wiggins, A. ,Invertible composition operators : The product of a composition operators with adjoint of a composition operators,

[10] Akeroyd, J. R. and Ghatage, P. G.,Closed range composition operators on A^2, Illinois J. Math. 52 (2008), 533-549 .