Journal of Progressive Research in Mathematics(JPRM)
ISSN: 2395-0218

SCITECH
RESEARCH ORGANISATION

Volume 7, Issue 2
Published online: April 20, 2016|

Journal of Progressive Research in Mathematics
www.scitecresearch.com/journals

Cauchy Problem for Fractional Ricatti Differential Equations Type with Alpha Order Caputo Fractional Derivatives

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Abstract

In this paper, we investigate solution of the fractional Ricatti differential equations (FRDEs) with alpha order Caputo fractional derivatives. In fact, FRDEs are analogous of the Ricatti ordinary differential equations. The multi power series method is used to obtain a useful formula that is implemented to find an explicit solution of Cauchy problem for FRDEs without solving any integral. This formula is explicit and easy to compute by using Maple software to get explicit solution. Also, it is shown that the proposed formula can be used to solve the Cauchy problem for Ricatti ordinary differential equations.

Keywords: Ricatti differential equation; Caputo fractional derivatives**;** multi power series method; Cauchy problems.

1. Introduction

The history of fractional calculus is back to 1695 when L'Hospital asked Leibniz about $\frac{d^{1/2}u(x)}{dx^{1/2}}$. Leibniz replied, "It will lead to a paradox." But he added prophetically, "From this apparent paradox, one day useful consequences will be drawn" [5]. In the past, only pure mathematician deal with fractional calculus because they are believed there is no applications to the fractional derivatives [14]. Based on the fact that a reasonable modeling of many physical phenomena having to depend on the time instant to gather with the prior time history, fractional calculus can be used successfully. Therefore, in recent year, fractional derivatives have been used in many phenomena in electromagnetic theory, fluid mechanics, viscoelasticity, circuit theory, control theory, biology, atmospheric physics, etc., [10] and [21]. However, there are two difficulties raised in the study of fractional derivatives, fractional derivatives cannot be expressed to a tangent direction as the classical first derivative. The second difficulty comes from complex integro-differential definitions which make the chain rule not valid for all type of fractional derivatives.

Fractional differential equations (FDEs) have been used to describe many real world problems such as damping laws, fluid mechanics, rheology, physics, mathematical biology, diffusion processes, electrochemistry, and so on. The solvability of a wide fractional differential equation types has been attracted many researchers [4], [6], [7], [11], [12], [16], [17], [19], [20], and [21]. Some theorems related to existence and uniqueness solutions of initial value problem for FDEs can be found in the books [18] and [10].

Any differential equation that is quadratic in the unknown function is called the Ricatti differential equation. This equation has many applications, especially in control theory. In recent year, the analytic or approximate solution of FRDEs has investigated by using many methods. For example, the decomposition method [15], the generalized Haar wavelet [13], Laplace-Adomian-Padé [8], the operator matrix of shifting Legendre polynomial [9], Legendre wavelet operational matrix [3], the generalized differential transform method [2]. Also, FRDEs may be considered as a good test example for investigating the accuracy and effectiveness of the numerical methods as shown in [1]. Motivated and inspired by the on-going research in this field, we will consider the following FRDEs type.

$$D^{\alpha}u(x)=A(x)+B(x)u(x)+C(x)u^{2}(x),\quad x\geq 0,\quad m-1<\alpha\leq m,\quad m\in N$$
$$u(0)=u_{0},u'(0)=u_{1},\ldots,u^{(m-1)}(0)=u_{m}, \tag{1}$$

Where $D^{\alpha}u(x)$ is Caputo derivative, [18] and [10], for any analytic function $u(x)$ which defined by

$$D^{\alpha}u(x)=\frac{1}{\Gamma(n-\alpha)}\int_{0}^{x}\frac{u^{(n)}(t)}{(x-t)^{\alpha-n+1}}dt,\quad n-1<\alpha<n,\ n\in Z^{+} \tag{2}$$

Clearly Caputo fractional derivative is linear operator i.e.

$$D^{\alpha}(a\,f(x)+b\,g(x))=D^{\alpha}a\,f(x)+D^{\alpha}a\,g(x),\ a,b\in R \tag{3}$$

For all $\alpha>0$ order Caputo derivative, one can have

$$D^{\alpha}C=0\ ,\ \alpha>0,\ C\ \text{, is a constant} \tag{4}$$

$$D^{\alpha}x^{n}=\begin{cases}0, & for\quad n\in N_{0}\,and\;\,n<[\alpha]\\ \dfrac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}x^{n-\alpha} & for\quad n\in N_{0}\,and\;\,n\geq[\alpha]\end{cases} \tag{5}$$

Where $[\alpha]$ is the smallest integer greater than or equal to α and $N_{0}=\{0\}\cup N$. Note that, when $\alpha\in N$, the Caputo differential derivative is coinciding with the classical differential derivative. For more details on fractional derivatives definitions and their properties, the reader may see the books [18], [10] and [21].

2. Main Result

In this section, we state and prove a new theorem to find a useful explicit formula to compute a solution of Cauchy problems for FRDEs without solving any integral. Now, we will state and prove the following lemma.

Lemma: Let $f(x)$ is a smooth functions for $x\geq 0$, and $u(x)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}U_{i,j}x^{i+\alpha j}$ then

(1) $$D^{\alpha}u(x)=\sum_{i=m}^{\infty}U_{i,0}\frac{\Gamma(i+1)}{\Gamma(i-\alpha+1)}x^{i-\alpha}+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}U_{i,j+1}\frac{\Gamma(i+\alpha j+\alpha+1)}{\Gamma(i+\alpha j+1)}x^{i+\alpha j},\quad m-1<\alpha<m$$

(2) $f(x)u(x)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(\sum_{s=0}^{i}\frac{f^{(s)}(0)}{s!}U_{i-s,j}\right)x^{i+\alpha j}$

(3) $f(x)u^2(x)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(\sum_{s=0}^{i}\sum_{r=0}^{i-s}\sum_{k=0}^{j}\frac{f^{(s)}(0)}{s!}U_{r,j-k}\,U_{i-s-r,k}\right)x^{i+\alpha j}$

Proof (1): Let us rewrite $u(x)$ as $u(x)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}U_{i,j}\ x^{i+\alpha j}=\sum_{i=0}^{\infty}U_{i,0}x^{i}+\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}U_{i,j}\ x^{i+\alpha j}$.

Then we apply D^{α} to have

$$D^{\alpha}u(x)=\sum_{i=m}^{\infty}U_{i,0}\frac{\Gamma(i+1)}{\Gamma(i-\alpha+1)}x^{i-\alpha}+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}U_{i,j+1}\frac{\Gamma(i+\alpha j+\alpha+1)}{\Gamma(i+\alpha j+1)}x^{i+\alpha j},\quad m-1<\alpha<m$$
.

Proof (2): By direct computation, one can have $f(x)u(x)=\left(\sum_{i=0}^{\infty}\frac{f^{(i)}(0)}{i!}x^{i}\right)\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}U_{i,j}x^{i+\alpha j}\right)$.

Assume $W_i=\sum_{j=0}^{\infty}U_{i,j}x^{\alpha j}$ we have,

$$f(x)u(x)=\left(\sum_{i=0}^{\infty}\frac{f^{(i)}(0)}{i!}x^{i}\right)\left(\sum_{i=0}^{\infty}W_i x^{i}\right)$$

$$=\sum_{i=0}^{\infty}\sum_{s=0}^{i}\frac{f^{(s)}(0)}{s!}W_{i-s}x^{i}$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(\sum_{s=0}^{i}\frac{f^{(s)}(0)}{s!}U_{i-s,j}\right)x^{i+\alpha j}$$

Proof (3): First, we compute $u^2(x)$

$$u^2(x)=\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}U_{i,j}x^{i+\alpha j}\right)\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}U_{i,j}x^{i+\alpha j}\right)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(\sum_{r=0}^{i}\sum_{s=0}^{j}U_{r,j-s}\,U_{i-r,s}\right)x^{i+\alpha j}$$

Now, by direct computation, one can have

$$f(x)u^2(x)=\left(\sum_{i=0}^{\infty}\frac{f^{(i)}(0)}{i!}x^{i}\right)\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(\sum_{r=0}^{i}\sum_{k=0}^{j}U_{r,j-k}\,U_{i-r,k}\right)x^{i+\alpha j}\right).$$

Assume $W_i=\sum_{j=0}^{\infty}\left(\sum_{r=0}^{i}\sum_{k=0}^{j}U_{r,j-k}\,U_{i-r,k}\right)x^{\alpha j}$ we have,

$$f(x)u^2(x)=\left(\sum_{i=0}^{\infty}\frac{f^{(i)}(0)}{i!}x^{i}\right)\left(\sum_{i=0}^{\infty}W_i x^{i}\right)=\sum_{i=0}^{\infty}\sum_{s=0}^{i}\frac{f^{(s)}(0)}{s!}\sum_{j=0}^{\infty}\left(\sum_{r=0}^{i-s}\sum_{k=0}^{j}U_{r,j-k}\,U_{i-s-r,k}\right)x^{i+\alpha j}$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(\sum_{s=0}^{i}\sum_{r=0}^{i-s}\sum_{k=0}^{j}\frac{f^{(s)}(0)}{s!}U_{r,j-k}\,U_{i-s-r,k}\right)x^{i+\alpha j}$$

Theorem: If $A(x), B(x)$ and $C(x)$ are smooth functions for $x \geq 0$, then the Cauchy problems for FRDEs type Eq.(1) has a solution $u(x)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}U_{i,j}x^{i+\alpha j}$ where, $U_{i,j}$ is the solution of the following forward linear algebraic system

$$U_{i,0}=u^{(i)}(0)=u_i,\ \forall i=0,1,2,\ldots,m-1, \tag{6}$$

$$U_{i,0}=0,\ i=m,m+1,m+2\ldots, \tag{7}$$

$$U_{i,1}=\frac{\Gamma(i+1)}{\Gamma(i+\alpha+1)}\left(\frac{A^{(i)}(0)}{i!}+\left(\sum_{s=0}^{i}\frac{B^{(s)}(0)}{s!}U_{i-s,0}\right)+\left(\sum_{s=0}^{i}\sum_{r=0}^{i-s}\frac{C^{(s)}(0)}{s!}U_{r,0}U_{i-s-r,0}\right)\right), \forall i=0,1,2,\ldots, \tag{8}$$

$$U_{i,j+1}=\frac{\Gamma(i+\alpha j+1)}{\Gamma(i+\alpha j+\alpha+1)}\left(\left(\sum_{s=0}^{i}\frac{B^{(s)}(0)}{s!}U_{i-s,j}\right)+\left(\sum_{s=0}^{i}\sum_{r=0}^{i-s}\sum_{k=0}^{j}\frac{C^{(s)}(0)}{s!}U_{r,j-k}U_{i-s-r,k}\right)\right), \quad \forall i=0,1,2,\ldots,\quad j=1,2,3,\ldots \tag{9}$$

Proof: Let $u(x)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}U_{i,j}x^{i+\alpha j}$, be a solution to Eq. (1), where $U_{i,j},\ i,j=0,1,2,\ldots$ are unknown coefficients. To find these coefficients one can substitute $u(x)$ in Eq. (1) and by using the above Lemma, we get

$$\sum_{i=m}^{\infty}U_{i,0}\frac{\Gamma(i+1)}{\Gamma(i-\alpha+1)}x^{i-\alpha}+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}U_{i,j+1}\frac{\Gamma(i+\alpha j+\alpha+1)}{\Gamma(i+\alpha j+1)}x^{i+\alpha j}=\sum_{i=0}^{\infty}\frac{A^{(i)}(0)}{i!}x^{i}$$
$$+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(\sum_{s=0}^{i}\frac{B^{(s)}(0)}{s!}U_{i-s,j}\right)x^{i+\alpha j}+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(\sum_{s=0}^{i}\sum_{r=0}^{i-s}\sum_{k=0}^{j}\frac{C^{(s)}(0)}{s!}U_{r,j-k}U_{i-s-r,k}\right)x^{i+\alpha j}.$$

Assume $U_{i,0}=0,\ i=m,m+1,m+2,\ldots$, so one cane have

$$\sum_{i=0}^{\infty}U_{i,1}\frac{\Gamma(i+\alpha+1)}{\Gamma(i+1)}x^{i}+\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}U_{i,j+1}\frac{\Gamma(i+\alpha j+\alpha+1)}{\Gamma(i+\alpha j+1)}x^{i+\alpha j}=\sum_{i=0}^{\infty}\frac{A^{(i)}(0)}{i!}x^{i}$$
$$+\sum_{i=0}^{\infty}\left(\sum_{s=0}^{i}\frac{B^{(s)}(0)}{s!}U_{i-s,0}\right)x^{i}+\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\left(\sum_{s=0}^{i}\frac{B^{(s)}(0)}{s!}U_{i-s,j}\right)x^{i+\alpha j}+\sum_{i=0}^{\infty}\left(\sum_{s=0}^{i}\sum_{r=0}^{i-s}\frac{C^{(s)}(0)}{s!}U_{r,0}U_{i-s-r,0}\right)x^{i}$$
$$+\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\left(\sum_{s=0}^{i}\sum_{r=0}^{i-s}\sum_{k=0}^{j}\frac{C^{(s)}(0)}{s!}U_{r,j-k}U_{i-s-r,k}\right)x^{i+\alpha j}$$

$$U_{i,1}=\frac{\Gamma(i+1)}{\Gamma(i+\alpha+1)}\left(\frac{A^{(i)}(0)}{i!}+\left(\sum_{s=0}^{i}\frac{B^{(s)}(0)}{s!}U_{i-s,0}\right)+\left(\sum_{s=0}^{i}\sum_{r=0}^{i-s}\frac{C^{(s)}(0)}{s!}U_{r,0}U_{i-s-r,0}\right)\right), \forall i=0,1,2,\ldots$$

$$U_{i,j+1}=\frac{\Gamma(i+\alpha j+1)}{\Gamma(i+\alpha j+\alpha+1)}\left(\left(\sum_{s=0}^{i}\frac{B^{(s)}(0)}{s!}U_{i-s,j}\right)+\left(\sum_{s=0}^{i}\sum_{r=0}^{i-s}\sum_{k=0}^{j}\frac{C^{(s)}(0)}{s!}U_{r,j-k}U_{i-s-r,k}\right)\right), \quad \forall i=0,1,2,\ldots,\quad j=1,2,3,\ldots$$

3. Illustrated examples

In this section, we used the formula $u(x)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}U_{i,j}x^{i+\alpha j}$ in the main theorem to find an analytic or approximate solution for any value of α, where α is the order of FRDEs. For any integer number, M, we compute the value of the coefficients $U_{i,j}$, $\forall i,j=0,1,2,\ldots,M$ by substituting the value of $U_{i,0}$ $\forall i=0,1,2,\ldots,M$ in Eq. (8) to compute $U_{i,1}$ $\forall i=0,1,2,\ldots,M$. Then, we use Eq. (9), to compute the rest of coefficients value $U_{i,j}$,$i,j=0,1,2,\ldots,M$. This procedure is carried out to have $u(x)=\sum_{i=0}^{M}\sum_{j=0}^{M}U_{i,j}x^{i+\alpha j}$. In fact, this procedure is not entailed any integration or any complex manipulations. So that it is a very easy method to find the solution of the Cauchy problems for FRDEs type Eq. (1). The accuracy and effectiveness of the method will depend on the number of M. The forward linear algebraic system Eq. (6) - Eq. (9) is easy to compute by using Maple software or by setting a computer code to get explicit an approximate solution to Eq. (1). Several examples are adopted to illustrate the advantage of the proposed Method.

Example 1. Consider the following nonlinear FRDE

$$D^{\alpha}u(x)=1+2u(x)-u^2(x),\quad x\ge 0,\quad 0<\alpha\le 1,\ u(0)=0 \tag{10}$$

Where the exact solution for $\alpha=1$ is

$$u(x)=1+\sqrt{2}\tanh\left(\sqrt{2t}+\frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}-1}\right)\right) \tag{11}$$

Using $M=4$ and solve Eq. (6) - Eq. (9) for any value of $0<\alpha\le 1$, we find a solution of FRDE Eq. (10) as

$$\begin{aligned}u(x)=&\frac{x^{\alpha}}{\Gamma(\alpha+1)}++\frac{2x^{2\alpha}}{\Gamma(1+2\alpha)}-\frac{\left(\Gamma(1+2\alpha)-4\Gamma(\alpha+1)^2\right)x^{3\alpha}}{\Gamma(\alpha+1)^2\Gamma(1+3\alpha)}\\&-\frac{\sqrt{\pi}(1+4\alpha)\left(\Gamma(1+2\alpha)^2-4\Gamma(\alpha+1)^2\Gamma(1+2\alpha)+2\Gamma(1+3\alpha)\Gamma(\alpha+1)\right)16^{-a}x^{4\alpha}}{\Gamma\left(\frac{3}{2}+2\alpha\right)\Gamma(\alpha+1)^2\Gamma(1+2\alpha)^2}+\cdots\end{aligned}$$

If $M=8$ and $\alpha=1$ then the solution of Eq. (10) is

$$u(x)=x+x^2+\frac{1}{3}x^3-\frac{1}{3}x^4-\frac{7}{15}x^5-\frac{7}{45}x^6+\frac{53}{315}x^7+\frac{71}{315}x^8-\cdots$$

This result is agreement with the exact result in Eq. (11). Also If $M=6$ and $\alpha=\frac{1}{2}$ then the solution of Eq. (10) is

$$\begin{aligned}u(x)=&\frac{2\sqrt{x}}{\sqrt{\pi}}+2x+\frac{16}{3}\frac{(\pi-1)x^{3/2}}{\pi^{3/2}}+\frac{(\pi-4)x^2}{\pi}-\frac{32}{45}\frac{\left(3\pi^2+44\pi-32\right)x^{5/2}}{\pi^{5/2}}\\&-\frac{1}{36}\frac{\left(333\pi^2+284\pi-512\right)x^3}{\pi^2}+\cdots\\&=1.128379167\sqrt{x}+2.x+2.051213127x^{3/2}-0.2732395433x^2-5.521879267x^{5/2}\\&-10.32009894x^3-\cdots\end{aligned}$$

This result is agreement with the result in, example 6 [1], example 3.2 [15].

Example 2. Consider the following nonlinear FRDE

$$D^{\alpha}u(x)=1-u^{2}(x),\quad x\geq 0,\quad 0<\alpha\leq 1,\ u(0)=0 \tag{12}$$

Where the exact solution for $\alpha=1$ is

$$u(x)=\frac{e^{2x}-1}{e^{2x}+1} \tag{13}$$

Using $M=8$ and solve Eq. (6) - Eq. (9) for any value of $0<\alpha\leq 1$, we find a solution of FRDE Eq. (12) as

$$u(x)=\frac{x^{\alpha}}{\Gamma(\alpha+1)}-\frac{\Gamma(1+2\alpha)x^{3\alpha}}{\Gamma(1+3\alpha)\Gamma(\alpha+1)^{2}}+\frac{2\Gamma(1+4\alpha)\Gamma(1+2\alpha)x^{5\alpha}}{\Gamma(1+5\alpha)\Gamma(\alpha+1)^{3}\Gamma(1+3\alpha)}$$
$$-\frac{2(64)^{\alpha}\Gamma\left(\frac{3}{2}+3\alpha\right)\Gamma(1+2\alpha)\left(\Gamma(1+2\alpha)\Gamma(1+5\alpha)+4\Gamma(1+4\alpha)\Gamma(1+3\alpha)\right)x^{7\alpha}}{\sqrt{\pi}(1+6\alpha)\Gamma(1+3\alpha)\Gamma(1+7\alpha)\Gamma(\alpha+1)^{4}\Gamma(1+5\alpha)}+\cdots$$

If $M=8$ and $\alpha=1$ then the solution of Eq. (12) is

$$u(x)=x-\frac{1}{3}x^{3}+\frac{2}{15}x^{5}-\frac{17}{315}x^{7}+\cdots$$

This result is agreement with the exact result in Eq. (13). Also If $M=8$ and $\alpha=\frac{1}{2}$ then the solution of Eq. (12) is

$$u(x)=\frac{2\sqrt{x}}{\sqrt{\pi}}-\frac{16}{3}\frac{x^{3/2}}{\pi^{3/2}}+\frac{1024}{45}\frac{x^{5/2}}{\pi^{5/2}}-\frac{8192}{75}\frac{x^{7/2}}{\pi^{7/2}}+\cdots$$
$$=1.128379167\sqrt{x}-0.9577979845x^{3/2}+1.300806688x^{5/2}-1.987486218x^{7/2}+\cdots$$

This result is agreement with the result in, example 2 [8].

Example 3. Consider the following nonlinear FRDE

$$D^{\alpha}u(x)=x^{2}+2u(x)+u^{2}(x),\quad x\geq 0,\quad 0<\alpha\leq 1,\ u(0)=0 \tag{14}$$

Using $M=4$ and solve Eq. (6) - Eq. (9) for any value of $0<\alpha\leq 1$, we find a solution of FRDE Eq. (14) as

$$u(x)=\frac{2x^{2+\alpha}}{\Gamma(3+\alpha)}+\frac{4x^{2+2\alpha}}{\Gamma(3+2\alpha)}+\frac{8x^{2+3\alpha}}{\Gamma(3+3\alpha)}+\frac{4\Gamma(5+2\alpha)x^{4+3\alpha}}{\Gamma(5+3\alpha)\Gamma(3+\alpha)^{2}}+\frac{16x^{2+4\alpha}}{\Gamma(3+4\alpha)}$$
$$+\left(\frac{8\Gamma(5+2\alpha)\Gamma(5+3\alpha)}{\Gamma(5+3\alpha)\Gamma(5+4\alpha)\Gamma(3+\alpha)^{2}}+\frac{16\Gamma(5+3\alpha)}{\Gamma(3+\alpha)\Gamma(5+4\alpha)\Gamma(3+2\alpha)}\right)x^{4+4\alpha}+\cdots$$

If $M=6$ and $\alpha=\frac{1}{4}$ then the solution of Eq. (14) is

$$u(x)=0.7845423289x^{9/4}+1.203604445x^{5/2}+1.808731848x^{11/4}+0.4089298840x^{19/4}+2.666666667x^{3}$$
$$+1.776863848x^{5}+3.862362235x^{13/4}+5.089518943x^{21/4}+0.3860668648x^{29/4}+5.502191747x^{7/2}$$
$$+12.01899082x^{11/2}+2.712447261x^{15/2}+\cdots$$

This result is agreement with the result in, example 1 [2].

If $M=6$ and $\alpha=\frac{1}{2}$ then the solution of Eq. (14) is

$$u(x)=0.6018022226x^{5/2}+0.6666666667x^{3}+0.6877739684x^{7/2}+0.1509625991x^{11/2}$$
$$+0.6666666667x^{4}+0.4415553048x^{6}+0.6113546389x^{9/2}+0.8293160657x^{13/2}$$
$$+0.06141305036x^{17/2}+0.5333333333x^{5}+1.254210816x^{7}+0.2812575859x^{9}+\cdots$$

This result is agreement with the result in, example 2 [2].

If $M=6$ and $\alpha=\frac{3}{4}$ then the solution of Eq. (14) is

$$u(x)=0.4521829621x^{11/4}+0.3438869842x^{7/2}+0.2271977785x^{17/4}+0.05094746950x^{25/4}$$
$$+0.1333333333x^{5}+0.09465289534x^{7}+0.07063849782x^{23/4}+0.1091095254x^{31/4}$$
$$+0.008269955518x^{39/4}+0.03420165810x^{13/2}+0.09834685897x^{17/2}+0.02330746635x^{21/2}+\cdots$$

This result is agreement with the result in, example 3 [2].

Example 4. Consider the following nonlinear FRDE type

$$D^{\alpha}u(x)=1-u^{2}(x),\quad x\geq 0,\;\; 1<\alpha\leq 2,\;\; u(0)=\sqrt{2},u'(0)=-1 \tag{15}$$

If $M=3$ and $\alpha=2$ then the solution of Eq. (15) is

$$u(x)=\sqrt{2}-x+\frac{1}{2}x^{2}+\frac{1}{3}\sqrt{2}x^{3}-\frac{1}{12}x^{4}-\frac{1}{12}\sqrt{2}x^{4}-\frac{1}{60}x^{5}+\frac{1}{36}\sqrt{2}x^{6}-\frac{1}{252}x^{7}+\frac{1}{360}x^{6}$$
$$-\frac{1}{90}\sqrt{2}x^{7}-\frac{17}{3360}x^{8}+\frac{5}{3024}\sqrt{2}x^{9}+\cdots$$

If $M=8$ and $\alpha=\frac{3}{2}$ then the solution of Eq. (15) is

$$u(x)=1.414213562-1.x+0.7522527778x^{3/2}+0.8510768648x^{5/2}-0.1719434921x^{7/2}$$
$$-0.4714045206x^{3}-0.1250000000x^{4}+0.2121320343x^{5}-0.02500000000x^{6}$$
$$+0.08797191783x^{9/2}-0.1558708833x^{11/2}-0.08436962655x^{13/2}+0.04040835286x^{15/2}+\cdots$$

This result is agreement with the result in, example 3.3 [15].

Example 5. Consider the following nonlinear FRDE type

$$D^{\alpha}u(x)=\sin(x)+\cos(2x)u(x)+\tan(x)u^{2}(x),\quad x\geq 0,\;\; 1<\alpha\leq 2,\;\; u(0)=1,u'(0)=1 \tag{16}$$

If $M=3$ and $\alpha=2$ then the solution of Eq. (16) is

$$u(x)=1+x+\frac{1}{2}x^{3}-\frac{1}{60}x^{5}+\frac{1}{30}x^{6}-\frac{1}{2520}x^{7}+\frac{1}{672}x^{8}+\frac{1591}{362880}x^{9}+\cdots$$

If $M=3$ and $\alpha=\frac{3}{2}$ then the solution of Eq. (15) is

$$u(x)=1+x+\frac{8}{5}\frac{x^{5/2}}{\sqrt{\pi}}-\frac{32}{189}\frac{x^{9/2}}{\sqrt{\pi}}+\frac{1}{8}x^4+\frac{7}{40}x^5-\frac{1}{144}x^6+\frac{64}{3465}\frac{x^{11/2}}{\sqrt{\pi}}+\frac{2176}{45045}\frac{x^{13/2}}{\sqrt{\pi}}$$
$$+\frac{4096}{45045}\frac{\left(\frac{247}{720}+\frac{64}{25\pi}\right)x^{15/2}}{\sqrt{\pi}}+\cdots$$

If $M=3$ and $\alpha=\sqrt{\pi}$ then the solution of Eq. (16) is

$$u(x)=1+x+0.6604508265x^{\sqrt{\pi}+1}-0.06113973675x^{3+\sqrt{\pi}}+0.05330353742x^{1+2\sqrt{\pi}}$$
$$+0.07252965997x^{2+2\sqrt{\pi}}-0.002447974756x^{3+2\sqrt{\pi}}+0.002282238135x^{1+3\sqrt{\pi}}$$
$$+0.005812050616x^{2+3\sqrt{\pi}}+0.01477733722x^{3+3\sqrt{\pi}}+\cdots$$

.

4. Conclusion

In this paper, we introduce a useful formula to compute a solution of the Cauchy problem for FRDEs type Eq. (1). This formula is not entailed any integration or any complex manipulations even if the FRDEs is content high nonlinearity terms as in the example (5). Also, it is explicit and easy to compute by using Maple software or by setting a computer code to get an explicit solution to this problem. It should be emphasized that this formula can be used to find a solution of high order Ricatti ordinary differential equations. The accuracy and effectiveness of the considering method depend on the number of terms as shown in illustrated examples. For more accuracy and effectiveness, we advise to use Padé approximation as (Khan at el. 2013) and (Momani at el. 2006).

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