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Stability Analysis of Deterministic Cholera Model

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Abstract.

In this paper, we study two models models for the dynamics spread and transmission of cholera. For these models Lyapounov functions are used to show that when the basic reproduction number is less than or equal to one, the disease free equilibrium is globally stable , and when it is greater than one there is an endemic equilibrium which is also globally asymptotically stable.

Key words: Nonlinear epidemic model; Lyapounov function; asymptotic stability.

1 Introduction

Cholera is a severe diarrhoea disease caused by the bacterium Vibrio Cholerae. Transmission occurs to human when his food or water are contaminated, and also where he has a contact with cholera patient's faeces, vomit and corpse.

The purpose of this paper is to study the stability of cholera model of Wang and Modnak, according the following plan. In section 1 we present and study the model without controls.

In section 2, we show the global stability of disease free and endemic equilibria of Wang and Modnak [8].

2 The basic model

The basic model of cholera transmission can be written as a combined dynamical system $(S(t),I(t),R(t)\text{-}B(t))$, where $S(t)$, $I(t)$, $R(t)$ and $B(t)$ denote the susceptible, the infected recovered human and the environmental component respectively. Hence the model is given by :

$$\begin{cases} \dot{S} = \Lambda - \beta_e S \frac{B}{K+B} - \beta_h SI - \mu S \\ \dot{I} = \beta_e S \frac{B}{K+B} + \beta_h SI - (\gamma + \mu) I \\ \dot{B} = \xi I - \delta B \\ \dot{R} = \gamma I - \mu R \end{cases} \quad (1)$$

with these parameters :

with the following parameters :

Λ is the recruitment into the population ; β_e and β_h represent rates of ingesting vibrios from the contaminated water or through human to human interaction respectively ; μ denote the rate that an individual in the population died from reasons not related to the

disease ; γ is the rate that an infectious individual dies because of the disease, ξ is the rate of human contribution to $\mathrm{V}.$ cholerae, δ is the natural death of $\mathrm{V}.$ cholerae, K is the pathogen concentration that yield 50% chance of catching cholera see [8].

Proposition 1. Let $(S(t),I(t),R(t),B(t))$ *be the solution of system(1) with initial conditions* $(S0),I(0),R(0))$ *and the compact set :*

$$\Omega = \{(S,I,R) \in R_+^3; B \in R\}; S + I + R \leq \frac{\Lambda}{\mu}; B < \frac{\xi\Lambda}{\mu\delta}\} \quad (2)$$

Then , under the flow described by (1), Ω *is a positively set that attracts all solutions of* $\mathbb{R}_+^4$

Proof :Consider the following Lyapounov function

$$W(t) = (S(t) + I(t) + R(t))$$

Its time derivative satisfies :

$$\frac{dW(t)}{dt} = (\dot{S(t)} + \dot{I} + \dot{R}$$

$$= \Lambda - \mu W(t)$$

Hence, $\frac{dW(t)}{dt} \leq 0$ for $W(t) > \frac{\Lambda}{\mu}$, which implies that Ω is positively invariant set. Solving this differential equation one has that :

$$0 \leq W(t) \leq \frac{\Lambda}{\mu} + W(0)e^{-\mu t}$$

Where $\mathrm{W}(0)$ is the initial condition of $\mathrm{W(t)}$. Thus , as $t \to +\infty$ one has that

$0 < W(t) < \frac{\Lambda}{\mu}$.

In the same way one has $\frac{dB}{dt} = \xi I - \delta B \leq \frac{\xi\Lambda}{\mu} - \delta B \leq 0$ for $B(t) \geq \frac{\xi\Lambda}{\mu}$, this implies that :

$$0 \leq B(t) \leq \frac{\xi\Lambda}{\mu\delta} + B(0)e^{-\delta t}$$

at $t \to \infty$, $0 \leq B(t) \leq \frac{\xi\Lambda}{\mu\delta}$.

Then, one can conclude that Ω is an attractive set. This achieves the proof.

2.1 Mathematical analysis

In this section , the model is analyzed in order to obtain the basic reproduction ratio, condition for the existence and uniqueness of non trivial equilibria

2.1.1 Basic reproduction number $\mathcal{R}_0$

The desease free equilibrium of the model is $(\frac{\Lambda}{\mu}, 0, 0, 0)$ Now, the basic reproduction Number $\mathcal{R}_0$ will be calculated by using the next generation matrix from Driessche and Woutmough , 2002 [9].

Let $\mathrm{X{=}(I,B,R,S)}$, system (1) becomes $\frac{dX}{dt} = \mathcal{F} - \mathcal{V}$, where $\mathcal{F} = \begin{pmatrix} \beta_e S\frac{B}{K+B} + \beta_h SI \\ 0 \\ 0 \\ 0 \end{pmatrix}$

and $\mathcal{V} = \begin{pmatrix} (\mu+\gamma)I \\ -\xi I + \delta B \\ -\gamma I + \mu R \\ -\Lambda + \beta_e S\frac{B}{K+B} + \beta_h SI + \mu S \end{pmatrix}$. The jacobian matrices of $\mathcal{F}$ $\mathcal{V}$ at the disease free equilibrium X_0 are respectively : $\mathcal{DF}(X_0) = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix}$. and $\mathcal{DV}(X_0) = \begin{pmatrix} V & 0 \\ J_1 & J_2 \end{pmatrix}$

where $F = \begin{pmatrix} \beta_h\frac{\Lambda}{\mu} & \beta_e\frac{\Lambda}{\mu} \\ 0 & 0 \end{pmatrix}$ and $V = \begin{pmatrix} \gamma+\mu & 0 \\ -\xi & \delta \end{pmatrix}$

$FV^{-1} \begin{pmatrix} \frac{\Lambda(K\beta_h\delta+\beta_e\xi)}{\mu\delta(\gamma+\mu)K} & \beta_e\frac{\Lambda}{\mu K\delta} \\ 0 & 0 \end{pmatrix}$ is the next generation matrix of system (1). The radius

FV^{-1} is

$$\rho(FV^{-1}) = \frac{\Lambda(K\beta_h\delta + \beta_e\xi)}{\mu\delta(\gamma+\mu)K}$$

Hence, the basic reproductive number is :

$$\mathcal{R}_0 = \frac{\Lambda(K\beta_h\delta + \beta_e\xi)}{\mu\delta(\gamma+\mu)K}$$

2.1.2 Stability of disease free equilibrium

Theorem 1. If $\mathcal{R}_0 < 1$, $\beta_e\Lambda - K\mu\delta = 0 \ et \ que \ \beta_h\Lambda - \mu^2 + \mu\xi = 0$

Proof Let be the following Lyapounov function $V(S,I,B,R) = (S - S^* ln(S)) + I + B + R$ and we have :

$$\dot{V}(S,I,B,R) = \dot{S}(1 - \frac{S^*}{S}) + \dot{I} + \dot{B} + \dot{R}$$

$$= (\Lambda - \beta_e S\frac{B}{K+B} - \beta_h SI - \mu S)(1 - \frac{S^*}{S}) + \beta_e S\frac{B}{K+B} + \beta_h SI - (\gamma+\mu)I + \xi I - \delta B + \gamma I - \mu R$$

$$= \mu(\frac{\Lambda}{\mu} - S)((1 - \frac{S^*}{S})) - (\beta_e S\frac{B}{K+B} - \beta_h SI)(1 - \frac{S^*}{S})) + \beta_e S\frac{B}{K+B} + \beta_h SI - (\gamma+\mu)I + \xi I - \delta B + \gamma I - \mu R$$

$$= \mu(S^* - S)((1 - \frac{S^*}{S})) + \beta_e S^*\frac{B}{K+B} - \beta_h S^* I - (\gamma+\mu)I + \xi I - \delta B + \gamma I - \mu R$$

$$\leq -\mu(\frac{(S-S^*)^2}{S} + (\beta\frac{S^*}{K} - \delta)B + (\beta_h S^* - \mu + \xi)I - \mu R$$

For $\beta\frac{S^*}{K} - \delta = 0$ and $\beta_h S^* - \mu + \xi = 0$ We have $\frac{dV}{dt} \leq 0$ Hence by LaSalle's principe , the disease free equilibrium is globally asymptotically stable in Ω.

2.1.3 Existence of endemic equilibrium

System (1) has an endemic point $(\bar{S}, \bar{I}, \bar{B}, \bar{R})$ satisfying :

$$\left\{\begin{array}{c} \Lambda - \beta_e S\frac{B}{K+B} - \beta_h SI - \mu S = 0 \\ \beta_e S\frac{B}{K+B} + \beta_h SI - (\gamma+\mu)I = 0 \\ \xi I - \delta B = 0 \\ \gamma I - \mu R = 0 \end{array}\right. \quad (3)$$

and we obtain : $\bar{B} = \frac{\xi}{\delta}\bar{I}$; $\bar{S} = \frac{\Lambda}{\mu} - \frac{\gamma+\mu}{\mu}\bar{I}$ et $\bar{I}$ which is the solution of :

$$E\bar{I}^2 + F\bar{I} + G = 0 \tag{4}$$

with

$E = -\beta_h(\gamma+\mu)\xi$

$F = \beta_h\Lambda\xi - (\gamma+\mu)[\beta_e + \mu)\xi + \beta_h\delta K]$

and $G = \beta_e\Lambda\xi + \beta_h\Lambda K\delta - (\gamma+\mu)(\mu\delta K)$.

Then

- for

$$\mathcal{R}_0 = \frac{\Lambda(K\beta_h\delta + \beta_e\xi)}{\mu\delta(\gamma+\mu)K} > 1$$

G is positive and the fact that E is negative, one has $\bar{I}_1\bar{I}_2 = \frac{G}{E} < 0$ and the equation (4) has two solutions $\bar{I}_1 < 0$ et $\bar{I}_2 > 0$, one will take account only for one will take account only for $\bar{I}_2$ whom is in Ω

$$\bar{I}_2 = \frac{-F}{2E}$$

- For

$$\mathcal{R}_0 = \frac{\Lambda(K\beta_h\delta + \beta_e\xi)}{\mu\delta(\gamma+\mu)K} = 1$$

$\mathrm{G}{=}0$, equation(4) has $\bar{I}_1$ which is the disease free equilibrium and $\bar{I}_2 = \frac{-F}{E}$

- The case

$$\mathcal{R}_0 = \frac{\Lambda(K\beta_h\delta + \beta_e\xi)}{\mu\delta(\gamma+\mu)K} < 1$$

is not available, the two solutions are negatives and are not in Ω.

And one has the following result on stability of endemic equilibrium :

Theorem 2 *.If* $\mathcal{R}_0 > 1$ *the endemic equilibrium is globally asymptotically stable.*

$$V(S,I,B) = W_1(S - \bar{S}lnS) + W_2(I - \bar{I}lnI)$$

Proof: Define a Lyapounov function with W_1 and W_2 are positive constants to be chosen latter. We have :

$$\dot{V}(S,I,B) = W_1(1-\frac{\bar{S}}{S})\dot{S} + W_2(1-\frac{\bar{I}}{I})\dot{I}$$

taking account of system (2) :

$$\dot{V}(S,I,B) =_1 (1-\frac{\bar{S}}{S})(\beta_e\bar{S}\frac{\bar{B}}{K+\bar{B}} - \beta_e S\frac{B}{K+B} + \beta_h\bar{S}\bar{I} - \beta_h SI + \mu(\bar{S}-S))$$

$$+W_2(1-\frac{\bar{I}}{I})(\beta_e S\frac{B}{K+B} - \beta_e\bar{S}\frac{\bar{B}}{K+\bar{B}} + \beta_h SI - \beta_h\bar{S}\bar{I} - (\gamma+\mu)(I-\bar{I}))$$

$$= W_1(1-\frac{\bar{S}}{S})(\beta_e\bar{S}\frac{\bar{B}}{K+\bar{B}} - \beta_e S\frac{\bar{B}}{K+\bar{B}} + \beta_e S\frac{\bar{B}}{K+\bar{B}} - \beta_e S\frac{B}{K+B} + \beta_h\bar{S}\bar{I} + \beta_h S\bar{I} - \beta_h S\bar{I} - \beta_h SI + \mu(\bar{S}-S))$$

$$+W_2(1-\frac{\bar{I}}{I})(\beta_e S\frac{B}{K+B} - \beta_e S\frac{\bar{B}}{K+\bar{B}} + \beta_e S\frac{\bar{B}}{K+\bar{B}} - \beta_e\bar{S}\frac{\bar{B}}{K+\bar{B}}$$

$$+\beta_h SI - \beta_h\bar{S}I + \beta_h\bar{S}I - \beta_h\bar{S}\bar{I} - (\gamma+\mu)(I-\bar{I}))$$

$$= -W_1\frac{1}{S}(\mu + \beta_e\bar{S}\frac{\bar{B}}{K+\bar{B}} + \beta_h\bar{I})(S-\bar{S})^2 + \beta_e(S-\bar{S})(\frac{\bar{B}}{K+\bar{B}} - \frac{B}{K+B}) - W_1\beta_h(S-\bar{S})(I-\bar{I})$$

$$+W_2\beta_h(S-\bar{S})(I-\bar{I}) + W_2(\frac{I-\bar{I}}{I})\beta_e S(\frac{B}{K+B} - \frac{\bar{B}}{K+\bar{B}}) - W_2\beta_e\frac{\bar{B}}{K+\bar{B}}(\frac{I-\bar{I}}{I})(S-\bar{S})$$

$$+W_2(\beta\bar{S} - (\gamma+\mu))\frac{(I-\bar{I})^2}{I}$$

$$\leq -W_1\frac{1}{S}(\mu + \beta_e\bar{S}\frac{\bar{B}}{K+\bar{B}} + \beta_h\bar{I})(S-\bar{S})^2 + W_1\beta_e(S-\bar{S})(\frac{\bar{B}}{K+\bar{B}} - \frac{B}{K+B})$$

$$+(W_2 - W_1)\beta_h(S-\bar{S})(I-\bar{I}) + W_2(\frac{I-\bar{I}}{I})\beta_e S(\frac{B}{K+B} - \frac{\bar{B}}{K+\bar{B}}) - W_2\beta_e\frac{\bar{B}}{K+\bar{B}}(\frac{I-\bar{I}}{I})(S-\bar{S})$$

$$+W_2(\beta\bar{S} - (\gamma+\mu))\frac{(I-\bar{I})^2}{I}$$

for $W_1 = W_2 = 1$ and the fact that $\beta\bar{S} - (\gamma+\mu) = 0$ one has

$$\frac{dV}{dt} \leq -\frac{1}{S}(\mu + \beta_e\bar{S}\frac{\bar{B}}{K+\bar{B}} + \beta_h\bar{I})(S-\bar{S})^2 + \beta_e\bar{S}\frac{B}{K+B}$$

$$-(\frac{\bar{I}}{I})\beta_e S(\frac{B}{K+B} - \frac{\bar{B}}{K+\bar{B}}) - \beta_e\frac{\bar{B}}{K+\bar{B}}(\frac{\bar{I}}{I})\bar{S}$$

$$\leq -\frac{1}{S}(\mu + \beta_e\bar{S}\frac{\bar{B}}{K+\bar{B}} + \beta_h\bar{I})(S-\bar{S})^2 - \beta_e\frac{\bar{B}}{K+\bar{B}}(\frac{\bar{I}}{I})\bar{S} < 0$$

Hence by LaSalle's principe ,the endemic equilibrium is globally asymptotically stable in Ω.

3 The model with controls

We present, in this section, the model of Wang and Modnak [8] which is the extension of model (1)by adding vaccination , treatment and water sanitation.The model (1) becomes :

$$\begin{cases} \dot{S} = \Lambda - \beta_e S \frac{B}{K+B} - \beta_h SI - \mu S - vS \\ \dot{I} = \beta_e S \frac{B}{K+B} + \beta_h SI - (\gamma + \mu) I - aI \\ \dot{B} = \xi I - \delta B - wB \\ \dot{R} = \gamma I - \mu R + vS + aI \end{cases} \tag{5}$$

with v is the rate of vaccination, a the rate of therapeutic treatment applied to infected people vibrios and w is the rate of the death of vibrios by leading sanitary water.

Like in section 1 the next set

$$\Sigma = \{(S, I, R) \in R_+^3; B \in R\}; S + I + R \leq \frac{\Lambda}{\mu}; B < \frac{\xi\Lambda}{\mu(\delta + w)}\} \tag{6}$$

is a positively set that attracts all solutions of $\mathbb{R}_+^4$ Denote by $\overset{\circ}{\Sigma}$ the interior of Σ

One has the disease free equilibrium : $(s^*, I^*, B, R^*) = (\frac{\Lambda}{\mu+v}, 0, 0, 0)$ and the basic reproduction number :

$$\mathcal{R}_0 = \frac{\Lambda(K\beta_h(\delta + w) + \beta_e \xi)}{\mu(\delta + w)(\mu + v)(\gamma + \mu + a)K}$$

which gives us the following results.

3.1 Stability of disease free equilibrium

Theorem 3. if $\mathcal{R}_0 < 1$ *the disease free equilibrium is globally asymptotically stable*

Proof : Let be the following Lyapounov function V :

$$V(S, E, I) = I(t)$$

We have :

$$\frac{dV}{dt} = \beta_e S \frac{B}{K+B} + \beta_h SI - (\gamma+\mu)I - aI$$

$$= (\mu(\delta+w)(\mu+v)(\gamma+\mu+a))(\frac{\beta_h S}{\mu(\delta+w)(\mu+v)(\gamma+\mu+a)} - \frac{(\gamma+\mu+a)}{\mu(\delta+w)(\mu+v)(\gamma+\mu+a)K})I$$
$$+\beta_e S \frac{B}{K+B}$$
$$\leq (\mu(\delta+w)(\mu+v)(\gamma+\mu+a))(\frac{\Lambda(K\beta_h(\delta+\omega)+\beta_e\xi)S}{\mu(\delta+w)(\mu+v)(\gamma+\mu+a)} - \frac{1}{\mu(\delta+w)(\mu+v)})I$$
$$+\beta_e S \frac{B}{K+B}$$
$$\leq (\mu(\delta+w)(\mu+v)(\gamma+\mu+a))(\mathcal{R}_0 S(t) - \frac{1}{\mu(\delta+w)(\mu+v)})I$$
$$+\beta_e S \frac{B}{K+B}$$

We see that

$\frac{dV}{dt} \leq 0$ for $\mathcal{R}_0 < \frac{1}{\mu(\delta+w)(\mu+v)} < 1$ and $B=0$

If $\mathcal{R}_0 < 1$ then $\frac{dV}{dt} = 0 \leftrightarrow I(t)=0$

If $\mathcal{R}_0 = 1$ then $\frac{dV}{dt} = 0 \leftrightarrow \mathcal{R}_0 = \frac{1}{\mu(\delta+w)(\mu+v)}$

Hence by LaSalle 's principe the disease free equilibrium is stable in $\sum$ □

If $\mathcal{R}_0 > 1$ the disease free equilibrium is instable by theorem 3. Moreover, the behavior of the local dynamics near the disease free equilibrium implies that the system (5) is uniformly persistent in $\sum$; namely there exists a constant $C > 0$ such that :

Lim $\inf_{t\to\infty} S(t) > C$ lim $\inf_{t\to\infty} S(t) > C$, lim $\inf_{t\to\infty} S(t) > C$

The uniform persistence together with boundedness of $\sum$ is equivalent to the existence of a compact set $D \in \overset{\circ}{\sum}$, which is absorbing for (5).

3.2 Stability of endemic equilibrium

The fourth equation of (5) is not appear in the third other. One can reduce the system (5) to :

$$\begin{cases} \dot{S} = \Lambda - \beta_e S \frac{B}{K+B} - \beta_h SI - \mu S - vS \\ \dot{I} = \beta_e S \frac{B}{K+B} + \beta_h SI - (\gamma + \mu)I - aI \\ \dot{B} = \xi I - \delta B - wB \end{cases} \quad (7)$$

which gives the following endemic equilibrium : $\bar{S} = \frac{\Lambda - (\gamma+\mu+a)\bar{I}}{\mu+v}$; $\bar{B} = \frac{\xi \bar{I}}{\delta+w}$ et $\bar{I}$

$$E\bar{I}^2 + F\bar{I} + G = 0 \quad (8)$$

with

$$E = -\beta_h(\gamma + \mu + a)\xi\,;$$
$$F = \beta_h \Lambda \xi - (\gamma + \mu + a)[\beta_e + \mu + v)\xi + \beta_h(\delta + w)K]\,;$$

and $G = \beta_e \Lambda \xi + \beta_h \Lambda K(\delta + w) - (\gamma + \mu + a)(\mu(\delta + w)K).$

and like in system (1), $\bar{I}$ is unique in Σ. One has :

Theorem 4. *Here we use the technical method of Muldowney [5], see also* $[1,\ 2,3,4,\ 7]$
Assume that :

- *There exist a compact absorbing set and* $K \in D$ -

- *The system has an unique equilibrium point* $\bar{x}$ *in* D

- $\bar{x}$ *is locally stable*

- *the system satisfies the Poincare Bendixon property*

- *each periodic orbit of (5) in* D *is orbitally asymptotically stable.*

Then the endemic equilibrium is globally asymptotically stable

The proof of this theorem is based on monotone dynamical systems, as developed in[5] and needs some following properties :

Theorem 5. *There exist a compact absorbing set* $K \in D$

Definition 1. *The system (5) is called competitive if there exists a diagonal matrix* H *with entries* ± 1 *such that each off-diagonal entry of* $H(\frac{\partial f}{\partial x})(\bar{x})H$ *is nonpositive in D,*

where $(\frac{\partial f}{\partial x})(\bar{x})$ *is the Jacobian matrix of (5)*

Definition 2. *Let be a three dimensional matrix* $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ *one define the second compound matrix by* $A^{[2]} = \begin{pmatrix} a_{11}+a_{22} & a_{23} & -a_{13} \\ a_{32} & a_{11}+a_{33} & a_{12} \\ -a_{31} & a_{21} & a_{22}+a_{33} \end{pmatrix}$

Theorem 6. IF $\mathcal{R}_0 > 1$ the endemic equilibrium is locally asymptotically stable

Proof : the proof of this theorem is the application of the following Arino, Mcluskey lemma :

Lemma 1. If $tr(\frac{\partial f}{\partial x})(\bar{x} < 0$, $det((\frac{\partial f}{\partial x})(\bar{x}) < 0$ *and* $det((\frac{\partial f^{[2]}}{\partial x})(\bar{x} < 0)$ *are negative , then its engenvalues have real part negative.*

Proof We have :

$$\frac{\partial f}{\partial x}(\bar{x}) = \begin{pmatrix} -(\frac{\beta_e \bar{B}}{K+\bar{B}} + \beta_h \bar{I} + \mu + v) & -\beta_h \bar{S} & -\frac{\beta_e K \bar{S}}{(K+\bar{B})^2} \\ \frac{\beta_e \bar{B}}{K+\bar{B}} + \beta_h \bar{I} & \beta_h \bar{S} - (\gamma + \mu + a) & \frac{\beta_e K \bar{S}}{(K+\bar{B})^2} \\ 0 & \xi & -(\delta + w) \end{pmatrix}.$$

One has

$$tr(\frac{\partial f}{\partial x})(\bar{x}) = -(\frac{\beta_e \bar{B}}{K+\bar{B}} + \beta_h \bar{I} + \mu + v) - \beta_h(\frac{(\gamma+\mu+a)\bar{I}}{\mu+v}) - (\gamma+\mu+a) - (\delta+w) + \beta_h \frac{\Lambda}{\mu+v} < 0$$

$$det(M) = -(\delta+w)[\beta_h \Lambda + (\gamma+\mu+a)(\frac{\beta_e \bar{B}}{K+B} + (\mu+v) - \beta_h(\gamma+\mu+a)\frac{K}{(K+\bar{B})^2} < 0$$

$\frac{\partial f^{[2]}}{\partial x}(\bar{x})$ is the second additive compound of the jacobian matrix $\frac{\partial f}{\partial x})(\bar{x})$ is $\frac{\partial f^{[2]}}{\partial x})(\bar{x})$

$$\begin{pmatrix} -(\frac{\beta_e \bar{B}}{K+\bar{B}} + 2\mu + a + \gamma + v) + \beta_h(\bar{S} - \bar{I}) & \frac{\beta_e K \bar{S}}{(K+\bar{B})^2} & \frac{\beta_e K \bar{S}}{(K+\bar{B})^2} \\ \xi & -(\frac{\beta_e \bar{B}}{K+\bar{B}} + \beta_h \bar{I} + \mu + v + \delta + w) & -\beta_h \bar{S} \\ 0 & \frac{\beta_e \bar{B}}{K+\bar{B}} + \beta_h \bar{I} & \beta_h \bar{S} - (\gamma + \mu + a + \delta + w) \end{pmatrix}$$

. et $det(\frac{\partial f^{[2]}}{\partial x})(\bar{x}) = -\beta_h \bar{I}[(\gamma+\mu+a+\delta+w)(\frac{\beta_e \bar{B}}{K+\bar{B}+\mu+v+\delta+w)} - \frac{\beta_e \bar{B}}{(K+\bar{B})^2}] + \beta_h \bar{S}[(\gamma+\mu+a+\delta+w)$
$(\frac{\beta_e \bar{B}}{k+\bar{B}} + \mu+v+\delta+w) - \xi \frac{\beta_e K \bar{S}}{(K+\bar{B})}^2] - (\gamma+\mu+a+\delta+w)(\frac{\beta_e \bar{B}}{K+\bar{B}} + 2\mu+a+\gamma+v)(\frac{\beta \bar{B}}{K+\bar{B}} + \mu+v+\delta+w)$

Theorem 7. *Pour* $\mathcal{R}_0 > 1$ *differential system (1) est competitive*

Proof : Consider a three dimensional matrix $H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

One has $H\frac{\partial f}{\partial x}H = \begin{pmatrix} -(\frac{\beta_e B}{K+B} + \beta_h I + \mu + v) & -\beta_h S & -\frac{\beta_e KS}{(K+B)^2} \\ -\frac{\beta_e B}{K+B} + \beta_h I & \beta_h S - (\gamma + \mu + a) & -\frac{\beta_e KS}{(K+B)^2} \\ 0 & -\xi & -(\delta + w) \end{pmatrix}$.

For $\beta_h S - (\gamma + \mu + a) < 0)$ the system (5) is competitive and possesses the Poincaré - Bendixon property.

Theorem 8. *Any periodic orbits of the system (5) is asymptotically orbitally stable.*

The proof is based on the following Muldowney's theorem :

Theorem 9. *A sufficient condition for a periodic orbit* $\Gamma = \{\phi(t) : 0 < t \leq \nu\}$

$$\dot{y} = A^{[2]}(t)y \tag{9}$$

is asymptotically stable, where $A(t) = \frac{\partial f}{\partial x}$ is the second additive compound matrix of the jacobian et $A^{[2]}$ $de\ A(t)$

The system (9) is called the second compound system of the orbit $\phi(t)$

Proof Let be Y=(X,Y,Z), la matrice $A^{[2]}$ est

$$\begin{pmatrix} -(\frac{\beta_e B}{K+B} + 2\mu + a + \gamma + v) + \beta_h(S - I) & \frac{\beta_e KS}{(K+B)^2} & \frac{\beta_e KS}{(K+B)^2} \\ \xi & -(\frac{\beta_e B}{K+B} + \beta_h I + \mu + v + \delta + w) & -\beta_h S \\ 0 & \frac{\beta_e B}{K+B} + \beta_h I & \beta_h S - (\gamma + \mu + a + \delta + w) \end{pmatrix}$$

system (7) becomes :

$$\begin{cases} \dot{X} = (-(\frac{\beta_e B}{K+B} + 2\mu + a + \gamma + v) + \beta_h(S - I))X + \frac{\beta_e KS}{(K+B)^2}Y + \frac{\beta_e KS}{(K+B)^2}Z \\ \dot{Y} = \xi X - (\frac{\beta_e B}{K+B} + \beta_h I + \mu + v + \delta + w)Y - \beta_h SZ \\ \dot{Z} = \frac{\beta_e B}{K+B} + \beta_h IY + (\beta_h S - (\gamma + \mu + a + \delta + w))Z \end{cases} \tag{10}$$

Let be

$$V(t) = \sup\{|X|, \frac{I}{B}(|Y| + |Z|)\}$$

By direct calculations, we can obtain the following inequalities :

$$D_+(|X|) \leq (-(\frac{\beta_e B}{K+B} + 2\mu + a + \gamma + v) + \beta_h(S-I))|X| + \frac{\beta_e KS}{(K+B)^2}\frac{I}{B}[\frac{B}{I}(|Y|+|Z|]$$

$$D_+(|Y|) \leq \xi|X| - (\frac{\beta_e B}{K+B} + \beta_h I + \mu + v + \delta + w)|Y| - \beta_h S|Z|$$

$$D_+(|Z|) \leq (\frac{\beta_e B}{K+B} + \beta_h I)|Y| + (\beta_h S - (\gamma + \mu + a + \delta + w))|Z|$$

We deduce :

$$D_+[|Y|+|Z|] \leq \xi|X| - (\mu+\delta+w)[|Y|+|Z|] - v|Z| - (\gamma+a)|Z|$$

Thus

$$D_+(\frac{I}{B}(|Y|+|Z|) = \frac{I}{B}D_+(|Y|) + D_+(|Z|) + (\frac{\dot{I}}{I} - \frac{\dot{B}}{B})(|Y|+|Z|)$$

and

$$D_+(\frac{I}{B}(|Y|+|Z|) \leq \frac{I}{B}(|Y|+|Z|)(\frac{\dot{I}}{I} - \frac{\dot{B}}{B} - \mu - \delta - w) + \frac{I}{B}\xi|X|$$

which implies this inequality :

$$D_+(V(t) = \sup\{g_1(t), g_2(t)\}V(t) \tag{11}$$

where :

$$\begin{aligned} g_1(t) &= -(\frac{\beta_e B}{K+B} + 2\mu + a + \gamma + v) + \beta_h(S-I) + \frac{\beta_e KS}{I(K+B)^2} \\ &\leq -(\frac{\beta_e B}{K+B} + 2\mu + a + \gamma + v) + \frac{\dot{I}}{I} \\ &\leq \frac{\dot{I}}{I} - \mu \\ g_2(t) &= (\frac{\dot{I}}{I} - \frac{\dot{B}}{B} - \mu - \delta - w) + \frac{I}{B}\xi \end{aligned}$$

and :

$$g_2(t) \leq \frac{\dot{I}}{I} - \mu$$

which gives us :

$$D_+(V(t)) \leq \sup\{g_1(t), g_2(t)) \leq \frac{\dot{I}}{I} - \mu$$

By application of Gronwall inquality, one has :

$$V(t) \leq V(0)I(t)exp(-\mu t) \leq \frac{\Lambda}{\mu}V(0)exp(-\mu t)$$

this implies tha $\mathrm{V(t)} \to 0$ when $\mathrm{t} \to \infty$, a result the second compound differential system (7) is asymptotically stable. Thus the periodic solution $(\mathrm{S(t)},\mathrm{I(t)},\mathrm{R(t)})$ asymptotically stable with asymptotically phase.

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