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Reconstruction of a right-hand side of parabolic equation by radial basis functions method

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Abstract

The inverse problem of reconstructing the right-hand side (RHS) of a parabolic equation using the radial basis functions (RBF) method from a solution specified at internal points is investigated. In this paper, the RHS is unknown about time, and the method we use is the meshless method. Some numerical experiments are presented to illustrate the accuracy, stability and effectiveness.

Keywords: Inverse problem; radial basis functions; meshless method; parabolic equation; right-hand side.

MSC: 65M32; 65M70; 35R30.

1. Introduction

In this paper, we consider the problem of reconstructing the RHS of a parabolic equation from a solution specified at internal points, it has many important applications in physics and engineering.

This kind of inverse problem has been discussed in many papers [1-9]. In [1], the existence, uniqueness, regularity and the continuous dependence of the solution upon the data are demonstrated. In [2], the source term is state dependent. With the exception of [3] and [9], the unknown source has been sought either as a function of space or as a function of time. In [3], the source is defined by two unknown spatial functions with known time dependent, and the form of the functions about time is known. The unknown RHS in this paper is the same as that in [9], it is a function about space and time, and the dependence of the RHS on space is known, and on time is unknown.

There are various methods to solve this kind of inverse problem. such as finite difference method [5,8,9], boundary element method [4,6], variational method [7], fundamental solutions method [10], radial basis functions method [11-17], and so on. The method used in [9] is a special numerical method similar to the bordering method, it is dependent on difference scheme for space and time. In this paper, according to some ideas in [9], we use the meshless method based on the RBF method.

In meshless methods, a set of nodes are used instead of meshing the domain of the problem, because of this property, the meshless method is superior to the mesh dependent methods. The RBF method is a class of truly meshless method, it provides an interpolation formula for the approximation of the solution and its derivatives.

This paper is organized as follows. In section 2, we give an outline of the RBF method. In section 3, we solve the inverse problem using the meshless method based on the RBF method. Numerical experiments and discussions will be given in section 4.

2. The RBF method

The RBF is a univariate function, of which the independent variable is the distance between a fixed point x^* and any point x.

For given data $\{x_j, u_j\}, j = 1, 2, \cdots, N$. $x_1, x_2, \cdots, x_N$ are the distinct nodes, u_j is the value of function $u(x)$ at the node x_j for each $j = 1, 2, \cdots, N$. N is the number of distinct nodes. The interpolating function $u^h(x)$ can be written as

$$u^h(x) = \sum_{i=1}^{n} \lambda_i \phi(\| x - x_j \|) = \phi^T(x)\Lambda, \tag{2.1}$$

where λ_i is the unknown RBF coefficient, $\phi(\| x - x_j \|)$ is the RBF, and $\| x - x_j \|$ denotes the distance between x and x_j,

$$\phi(x) = [\phi(\| x - x_1 \|), \phi(\| x - x_2 \|), \cdots, \phi(\| x - x_N \|)]^T,$$

$$\Lambda = [\lambda_1, \lambda_2, \cdots, \lambda_N]^T.$$

In order to compute λ_i, substitute each $x_k (k = 1, 2, \cdots, N)$ for x in (2.1), then we have

$$\Phi\Lambda = U, \tag{2.2}$$

where

$$\Phi = \begin{bmatrix} \phi(\| x_1 - x_1 \|) & \phi(\| x_1 - x_2 \|) & \cdots & \phi(\| x_1 - x_N \|) \\ \phi(\| x_2 - x_1 \|) & \phi(\| x_2 - x_2 \|) & \cdots & \phi(\| x_2 - x_N \|) \\ \vdots & \vdots & & \vdots \\ \phi(\| x_N - x_1 \|) & \phi(\| x_N - x_2 \|) & \cdots & \phi(\| x_N - x_N \|) \end{bmatrix},$$

$$U = [u_1, u_2, \cdots, u_N]^T,$$

so we can get the coefficient

$$\Lambda = \Phi^{-1}U,$$

from (2.2).

Substituting (2.3) into (2.1),

$$u^h(x) = \phi^T(x)\Phi^{-1}U = \sum_{j=1}^{N} \psi_j(x) u_j,$$

where $\psi_j(x)$ is called the shape function, and

$$\Psi(x) = [\psi_1(x), \psi_2(x), \cdots, \psi_N(x)] = \phi^T(x)\Phi^{-1},$$

The well-known RBFs are listed in Table 1.

Table 1. Some well-known RBFs

	Definition
Gaussian (GA)	$\phi(r) = \exp(-cr^2)$
Hardy multiquadrics (MQ)	$\phi(r) = \sqrt{r^2 + c^2}$
Inverse multiquadrics (IMQ)	$\phi(r) = (\sqrt{r^2 + c^2})^{-1}$
Inverse quadric (IQ)	$\phi(r) = (r^2 + c^2)^{-1}$
Thin plate spline (TPS)	$\phi(r) = r^{2c} \log(r)$

3. The inverse problem and its numerical solution

In this section, we solve the inverse problem of reconstructing the RHS in a parabolic equation [18] by using the meshless method based on the RBF method.

The problem can be described as

$$u_t(x,t) = u_{xx}(x,t) + f(t), \quad 0 < x < l, 0 < t \le T, \tag{3.1}$$

with the initial and boundary conditions

$$u(x,0) = u_0(x), \quad 0 \le x \le l, \tag{3.2}$$

$$u(0,t) = 0, u(l,t) = 0, \quad 0 < t \le T. \tag{3.3}$$

If the function $f(x,t)$ is known, the equation (3.1) with conditions (3.2-3.3) is the direct problem. In this paper, we assume that the function $f(x,t)$ is unknown, and we solve the functions $u(x,t)$ and $f(x,t)$ with the additional observation of $u(x,t)$ at certain internal point $x_0 (0 < x_0 < l)$,

$$u(x_0, t) = E(t). \tag{3.4}$$

The mathematical model (3.1-3.4) arises in various areas of physics and engineering, such as hydrology, material sciences, heat transfer, transport problems, and so on. In the context of heat conduction, $u(x,t)$ represents temperature at the position x and time t, $f(x,t)$ is interpreted as a heat source, respectively. In practice, $f(x,t)$ is unknown in many cases, so it is required to reconstruct the unknown heat source by additional temperature measurements which are made at some single points, for example, finding a pollution source intensity and designing the final state in melting and freezing processes.

According to some transitions in [9], we assume the function $f(x,t)$ can be described as

$$f(x,t) = \eta(t)\xi(x), \tag{3.5}$$

where $\xi(x)$ is the known function, and satisfies the following restrictions:

(1) $\xi(x_0) \ne 0$,

(2) $\xi(x)$ is smooth enough,

(3) $\xi(x) = 0$ on the boundary of the computational domain.

Let

$$u(x,t) = \theta(t)\xi(x) + \omega(x,t), \tag{3.6}$$

where

$$\theta(t)=\int_0^t \eta(s)ds, \tag{3.7}$$

substituting (3.5-3.7) into (3.1), the equation (3.1) can be rewritten as

$$\omega_t(x,t)=u_{xx}(x,t)+\theta(t)\xi''(x) \qquad 0<x<l, 0<t\le T, \tag{3.8}$$

combining (3.4) with (3.6), we get

$$\theta(t)=\frac{E(t)-\omega(x_0,t)}{\xi(x_0)}, \tag{3.9}$$

substituting (3.9) into (3.8),

$$\omega_t(x,t)=u_{xx}(x,t)+\frac{E(t)-\omega(x_0,t)}{\xi(x_0)}\xi''(x) \qquad 0<x<l, 0<t\le T, \tag{3.10}$$

the initial and boundary conditions are

$$\omega(x,0)=u_0(x), \;\; 0\le x\le l, \tag{3.11}$$

$$\omega(0,t)=0, \omega(l,t)=0, \quad 0<t\le T. \tag{3.12}$$

On the basis of above descriptions, if we have the numerical solution $\hat{\omega}(x,t)$ of the equation (3.10), we can get the numerical solution $\hat{u}(x,t)$ and $\hat{f}(x,t)$ from (3.5-3.6).

Next, we solve the problem (3.10-3.12) using the meshless method based on RBF method.

According to (2.4), the approximate function $\hat{\omega}(x,t)$ of $\omega(x,t)$ at the time $t=t_m$ can be represented as

$$\hat{\omega}(x,t_m)=\sum_{j=1}^{N}\psi_j(x)\hat{\omega}(x_j,t_m), (m=1,2,\cdots,M, 0=t_1<t_2<\cdots<t_M=T)$$

where $\psi_j(x)$ is the shape function described in section 2.

Then

$$\hat{\omega}(x_0,t_m)=\sum_{j=1}^{N}\psi_j(x_0)\hat{\omega}(x_j,t_m),$$

$$\hat{\omega}_{xx}(x,t_m)=\sum_{j=1}^{N}\psi_j''(x)\hat{\omega}(x_j,t_m).$$

For the $\hat{\omega}_t(x,t_m)$, we apply one step forward difference formula to time,

$$\hat{\omega}_t(x,t_m)=\frac{\hat{\omega}(x,t_{m+1})-\hat{\omega}(x,t_m)}{\Delta t},$$

where $\Delta t=t_{m+1}-t_m, m=1,2,\cdots,M-1.$

So for $t=t_m$, the equation (3.10) can be rewritten as

$$\frac{\hat{\omega}(x,t_{m+1})-\hat{\omega}(x,t_m)}{\Delta t}=\sum_{j=1}^{N}\psi_j''(x)\hat{\omega}(x_j,t_m)+\frac{E(t_m)-\sum_{j=1}^{N}\psi_j(x_0)\hat{\omega}(x_j,t_m)}{\xi(x_0)}\xi''(x),$$

that is equivalent to

$$\hat{\omega}(x,t_{m+1}) = \hat{\omega}(x,t_m) + \Delta t[\sum_{j=1}^{N} \psi_j''(x)\hat{\omega}(x_j,t_m) + \frac{E(t_m) - \sum_{j=1}^{N} \psi_j(x_0)\hat{\omega}(x_j,t_m)}{\xi(x_0)} \xi''(x)],$$

by substituting each $x_k (k = 1, 2, \cdots, N)$ for x, we get

$$\hat{\omega}(x_k,t_{m+1}) = \hat{\omega}(x_k,t_m) + \Delta t[\sum_{j=1}^{N} \psi_j''(x_k)\hat{\omega}(x_j,t_m) + \frac{E(t_m) - \sum_{j=1}^{N} \psi_j(x_0)\hat{\omega}(x_j,t_m)}{\xi(x_0)} \xi''(x_k)],$$

Combining with the conditions (3.11-3.12), for the iterative system of equation (3.13),we can obtain the numerical solution $\hat{\omega}(x,t)$ for $k = 1, 2, \cdots, N$ and $t = 1, 2, \cdots, M$, using the iterative method, then we can have the numerical solutions

$$\hat{f}(x_k,t_m) = \hat{\eta}(t_m)\xi(x_k), \quad \hat{u}(x_k,t_m) = \hat{\theta}(t_m)\xi(x_k) + \hat{\omega}(x_k,t_m),$$

4. Numerical experiments and discussions

In this section, we give an example to test the accuracy, stability, and efficiency of the meshless method used in this work. In the experiments, we use Gaussian RBF.

Example. Consider the problem (3.1-3.5), with the conditions

$$u_0(x) = 0,\ E(t) = t,\ x_0 = 0.5.$$

and let $l = 1, T = 1$.

The exact solutions are

$$u(x,t) = t\sin(\pi x),\ f(x,t) = (1+\pi^2 t)\sin(\pi x),$$

with $\xi(x) = \sin(\pi x)$.

Firstly, in order to illustrate the numerical accuracy of the method in this work, we plot the error functions $f(x,t) - \hat{f}(x,t)$ and $u(x,t) - \hat{u}(x,t)$ in Figure 1.

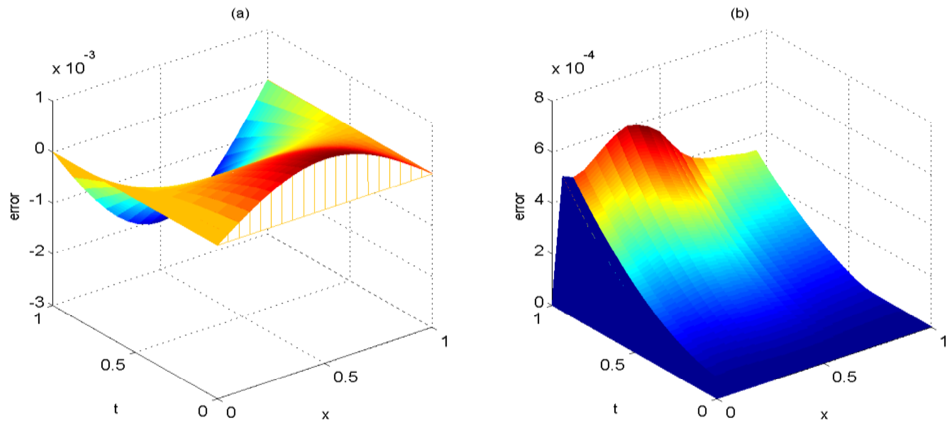

Fig 1: error function (a): $f(x,t) - \hat{f}(x,t)$ **,(b):** $u(x,t) - \hat{u}(x,t)$.

For purpose of observing the effect of approximation more clearly, we plot the exact and numerical solution of the functions $f(x,t)$ and $u(x,t)$ at $x = 0.1$ in Figure 2.

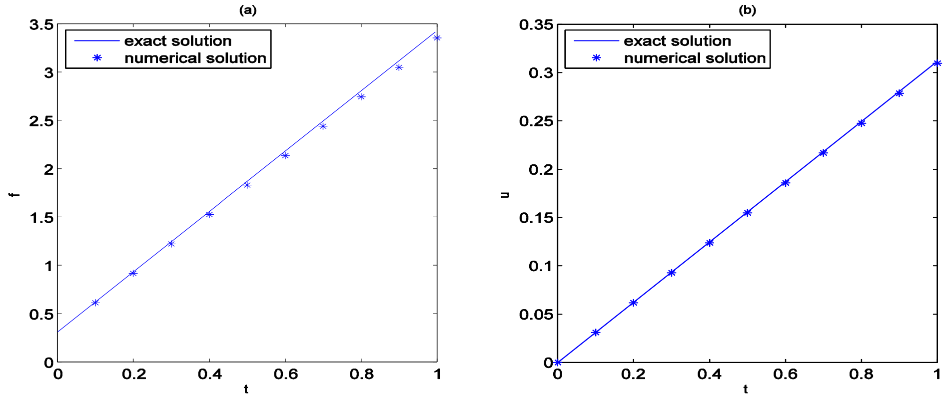

Fig 2: the exact and numerical solutions of (a): $f(x,t)$,(b): $u(x,t)$ at $x=0.1$.

From the above two figures, we see that the effect of the approximation using the meshless method is well.

Secondly, in order to test the stability of the numerical solution, we give small perturbation on the overspecified data $E(t)$, and the artificial error is defined as

$$E_\gamma(t) = E(t)(1+\gamma),$$

where γ represents the noisy parameter.

The results are shown in Figures 3-4. In Figure 3, we plot the error function $f(x,t)-\hat{f}(x,t)$ with $\gamma=0.001$ and $\gamma=0.01$, respectively. In Figure 4, we plot exact and numerical solution of the function $f(x,t)$ and $u(x,t)$ at $x=0.1$ with $\gamma=0.01$.

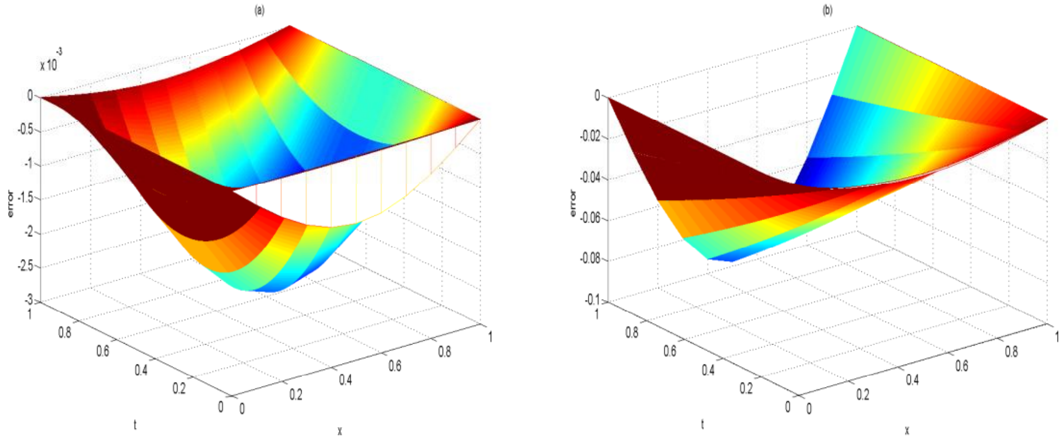

Fig 3: error function $f(x,t)-\hat{f}(x,t)$ (a):. $\gamma=0.001$, (b):. $\gamma=0.01$.

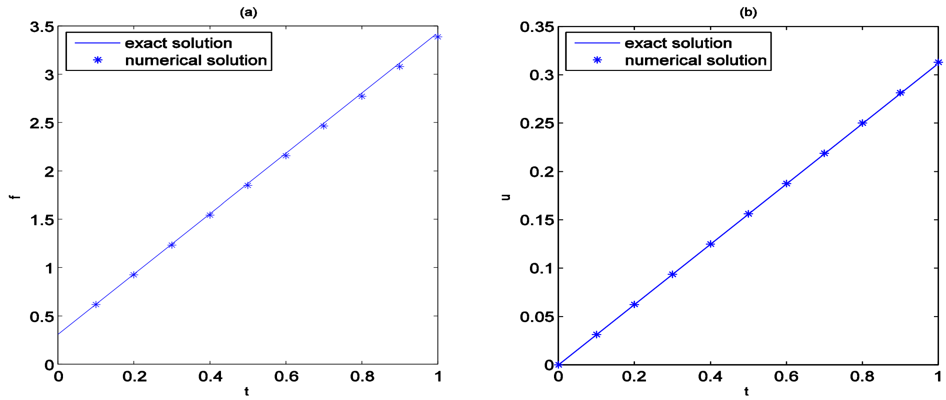

Fig 4: the exact and numerical solutions of (a): $f(x,t)$,(b): $u(x,t)$ at $x=0.1$ with $\gamma=0.01$.

From these two figures, we can see that with the increase of the noisy data, the error function $f(x,t)-\hat{f}(x,t)$ increases as well, even so, there is no obvious oscillation about the numerical solution, so the effect of approximation is stable as a whole.

Thirdly, in order to test the effectiveness of the numerical solution, we make a comparison between the method (RBF) in this paper and the finite difference method (FDM), we plot the exact and numerical solution of $f(x,t)$ and $u(x,t)$ in three different cases: $\gamma=0$, $\gamma=0.001$ and $\gamma=0.01$, in Figures 5-7, respectively.

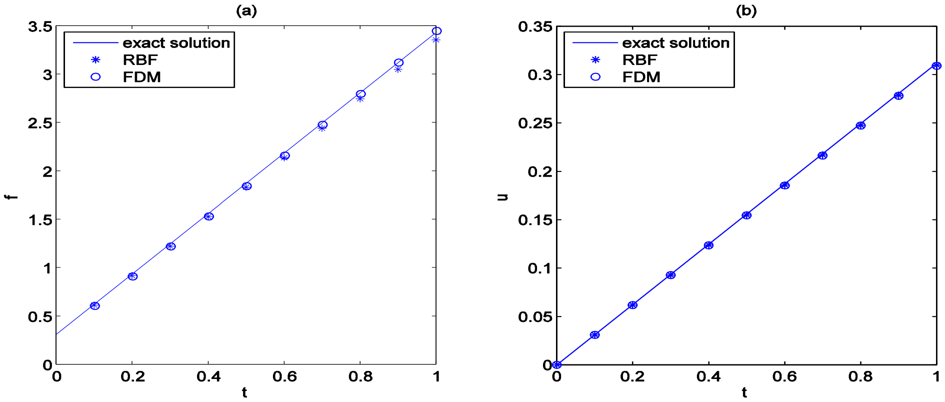

Fig 5: the exact and numerical solutions of (a): $f(x,t)$,(b): $u(x,t)$ at $x=0.1$ with $\gamma=0$.

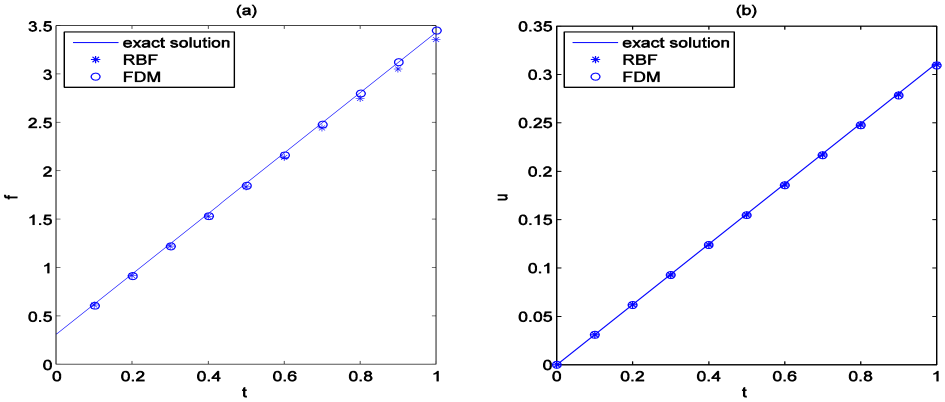

Fig 6: the exact and numerical solutions of (a): $f(x,t)$,(b): $u(x,t)$ at $x=0.1$ with $\gamma=0.001$.

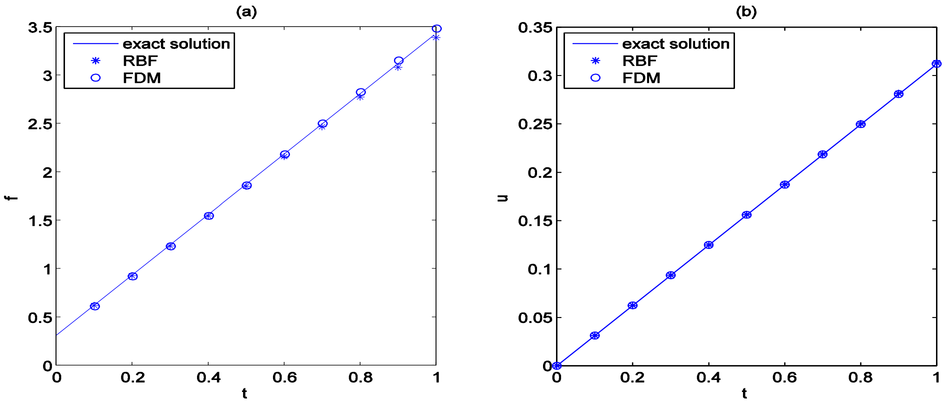

Fig 7: the exact and numerical solutions of (a): $f(x,t)$,(b): $u(x,t)$ at $x=0.1$ with $\gamma=0.01$.

From these three figures, we can see that under the condition of the same nodes, the approximation achieved by using RBF is better than using FDM, especially in the case that there exists noisy data.

At last, in order to further illustrate the accuracy, stability and effectiveness of the method, we define the root mean square error (RE) of the functions $f(x,t)$ and $u(x,t)$ as follows

$$RE(f)=\sqrt{\frac{\sum_{i=1}^{M}\sum_{j=1}^{N}(f(x_i,t_j)-\hat{f}(x_i,t_j))^2}{MN}},$$

where $f(x_i,t_j)$ and $\hat{f}(x_i,t_j)$ are the exact and numerical solution at the node x_i,t_j, respectively, M and N are the number of nodes about x and t, in order to make a comparison with the finite difference method, we use the uniform nodal arrangement. The definition of $RE(u)$ is same to $RE(f)$. The results in different cases are given in Table 2.

Table 2. The errors for different $\Delta x, \Delta t$ and noisy parameter

			RE(f)		***RE(u)***	
Δx	Δt	γ	*RBF*	*FDM*	*RBF*	*FDM*
0.1	0.001	0	8.9636×10^{-3}	8.0577×10^{-2}	1.6295×10^{-4}	0.0802×10^{-15}
0.1	0.0001	0	3.3836×10^{-3}	1.1873×10^{-1}	1.6303×10^{-4}	1.2946×10^{-14}
0.05	0.0005	0	2.6363×10^{-3}	1.1760×10^{-2}	2.0125×10^{-4}	4.0137×10^{-15}
0.05	0.0001	0	6.3887×10^{-4}	2.6768×10^{-2}	20.129×10^{-4}	6.7091×10^{-15}
0.1	0.001	0.001	4.7862×10^{-3}	8.4609×10^{-2}	4.8746×10^{-4}	3.8935×10^{-4}
0.1	0.0001	0.001	1.3263×10^{-3}	1.2306×10^{-1}	4.8743×10^{-4}	3.8926×10^{-4}
0.05	0.0005	0.001	2.9984×10^{-3}	1.4907×10^{-2}	5.8765×10^{-4}	3.9846×10^{-4}
0.05	0.0001	0.001	4.9774×10^{-3}	3.1079×10^{-2}	5.8765×10^{-4}	3.9842×10^{-4}
0.1	0.001	0.01	3.5674×10^{-2}	1.2241×10^{-1}	3.9731×10^{-3}	3.8935×10^{-3}
0.1	0.0001	0.01	4.1048×10^{-2}	1.6257×10^{-1}	3.9723×10^{-3}	3.8926×10^{-3}
0.05	0.0005	0.01	4.3398×10^{-2}	5.3300×10^{-2}	4.1703×10^{-3}	3.9846×10^{-3}
0.05	0.0001	0.01	4.5846×10^{-2}	7.1334×10^{-2}	4.1699×10^{-3}	3.9842×10^{-3}

From Table 2, we can get the approximation effect of the method FDM is feasible when there is no noisy data, but when the noisy parameter is increased, the effect of the approximation become worse suddenly. In the practical problems, the existence of noisy data is an unbeatable truth, from the data in Table 2, we obtain that the method RBF is better than FDM for the approximation effect of $f(x,t)$ under any case, meanwhile, for the approximation effect of $u(x,t)$, the method RBF is not worse than FDM when there exist noisy data. As a result, the RBF method is accurate, stable and effective.

5. Conclusion

In this paper, we use the meshless method based on the RBF method to solve the inverse problem of reconstructing the right-hand side in parabolic equation. From the experiments, we get that the numerical solutions are close to the exact solutions, with the noisy data γ increasing, the results have a corresponding change, however, there is no obvious oscillation. So the method in this work is accurate, stable and efficient.

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