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Double Reverse θ*- Centralizer of Rings With Involution

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Abstract.

Let R be a ring with involution and θ is a mapping of R .It was known that every double reverse θ^*-centralizer is a double Jordan θ^*- centralizer on R but the converse need not be true . In this paper we give conditions to the converse to be true .

Keywords:

*- Ring; semiprime *- ring; reverse *- centralizer; involution; double reverse θ^*- centralizer; double Jordan θ^*- centralizer .

1. Introduction

Let R be a ring with involution and θ is a mapping of R. This paper consists of two sections. In section one , we recall some basic definitions and other concepts, which be used in our paper , we explain these concepts by examples and remarks. In section two, we introduce the concepts of double reverse θ^*- centralizer , double Jordan θ^*- centralizer and we study the relation between them on R

2. BASIC CONCEPTS

Definition 2.1: [1] A ring R is called a prime ring if for any a,b $\in$R , aRb = {0} , implies that either a = 0 or b = 0 .

Definition 2.2: [1] A ring R is called a semiprime ring if for any a$\in$R , aRa = {0}, implies that a = 0 .

Remark 2.3 : [1] Every prime ring is a semiprime ring , but the converse in general is not true .

Example 2.4: [1] Let R = z_6 be a ring . To show that the ring R is semiprime ring , let a$\in$R such that aRa = 0, implies that a^2 = 0 , hence a = 0 , therefore R is semiprime ring . But R is not prime since 2 $\neq$0 and 3 $\neq$0 , implies that 2R3 = 0 .

Definition 2.5 :[1] A ring R is said to be n- torsion free where n $\neq$0 is an integer if whenever na = 0 with a$\in$R , then a = 0

Definition2.6 :[2] An additive mapping $x \rightarrow x^*$ on a ring R is called an involution if for all x,y$\in$R we have $(xy)^* = y^* x^*$ and $x^{**} = x$. A ring equipped with an involution is called *- ring .

Definition 2.7:[2] A left (right) reverse *- centralizer of *- ring R is an additive mapping T: R→R which satisfies T(xy) = T(y)x* (T(xy) = y*T(x)) for all x,y∈R . A reverse *- centralizer of a ring R is both left and right reverse *- centralizer .

Definition 2.8:[2] A left (right) Jordan *- centralizer of *- ring R is an additive mapping T:R→R which satisfies $T(x^2) = T(x)x^*$ ($T(x^2) = x^*T(x)$) for all x∈R . A Jordan *- centralizer of a ring R is both left and right Jordan *- centralizer .

Remark 2.9 :[3] Every left (right) reverse *- centralizer is a left (right) Jordan *- centralizer , but the converse in general is not true .

Example 2.10:[3] Let F be a field , and R be a ring of triangular matrices of the form

$$x = \begin{bmatrix} 0 & a & c & b \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ for all } a,b,c,d \in F,\ x \in R .$$

And the involution * define by

$$*\left(\begin{bmatrix} 0 & a & c & b \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & a & c & -b \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{for all } a,b,c, \in F.$$

And let T : R → R is an additive mapping defined as

$$T(x) = \begin{bmatrix} 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ for all } b \in F \text{ and } x \in R .$$

Then T is a Jordan * - centralizer but is not reverse * - centralizer .

Theorem 2.11:[4] Let R be a 2-torsion free semiprime *-ring , then every left (right) Jordan *- centralizer is a left (right) reverse *- centralizer .

Definition2.12:[3] Let R be a *- ring , and T , S : R →R be additive mappings , then a pair (T,S) is called a double reverse * - centralizer if T is a left reverse * - centralizer , S is a right reverse * - centralizer , and they satisfying the condition $x^*T(y)=S(x)y^*$, for all x , y ∈R .

Example 2.13:[3] Let F be a field , and let $M_2(F)$ be a ring of all matrices of order 2 over F , and the involution * on $M_2(F)$ defined by

$$*\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \quad \text{for all } a, b, c, d \in F .$$

Let T , S : $M_2(F) \rightarrow M_2(F)$ be additive mappings defined as

$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ b & d \end{bmatrix}$ for all a, b , c , d ∈F .

$S\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 & c \\ 0 & d \end{bmatrix}$ for all a, b , c , d ∈F .

Then (T,S) is a double reverse * - centralizer .

Definition 2.14 [3]: Let R be * - ring , and T , S : R →R be additive mappings , then a pair (T,S) is called a double Jordan * - centralizer if T is a left Jordan * - centralizer , S is a right Jordan * - centralizer , and they satisfying the condition $x^*T(x)=S(x)x^*$, for all x ∈R .

Remark 2.15 [3]: Every double reverse * - centralizer is a double Jordan * - centralizer, but the converse in general is not true .

Example2.16 [3]: Let R and the involution * be as in the Example (2.10) , and

T , S : R →R be additive mappings defined as

$T(x) = \begin{bmatrix} 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $S(x) = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ for all a , b ∈F and x ∈R .

Then (T,S) is a double Jordan * - centralizer , but is not a double reverse * - centralizer .

Definition 2.17[5]:A left (right) reverse θ^*-centralizer of *-ring R is an additive mapping T : R → R which satisfies $T(xy) = T(y)\theta(x^*)$ ($T(xy) = \theta(y^*)T(x)$) for all x , y ∈R , where θ is a mapping of R . A reverse θ^*-centralizer of *-ring R is both left and right θ^*-centralizer .

Definition 2.18[5]: A left (right) Jordan θ^*-centralizer of *-ring R is an additive mapping T : R → R which satisfies $T(x^2) = T(x)\theta(x^*)$ ($T(x^2) = \theta(x^*)T(x)$) for all x ∈R , where θ is a mapping of R . A Jordan θ^*-centralizer of *-ring R is both left and right Jordan θ^*-centralizer .

3. DOUBLE REVERSE θ*- CENTRALIZER

Definition 3.1: Let R be a *-ring , and T , S : R →R be additive mappings , then a pair (T,S) is called a double reverse θ^* - centralizer if T is a left reverse θ^*-centralizer, S is a right reverse θ^*-centralizer, and they satisfying the condition $\theta(x^*)T(y) = S(x)\,\theta(y^*)$, for all x,y ∈ R , where θ is a mapping of R .

Example 3.2: Let F be a field , and let $M_2(F)$ be a ring of all matrices of order 2 over F , and the involution * on $M_2(F)$ defined by

$*\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ for all a, b , c , d ∈F .

Let T , S : $M_2(F) \to M_2(F)$ be additive mappings defined as

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ b & d \end{bmatrix} \quad \text{for all } a, b, c, d \in F.$$

$$S\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 & c \\ 0 & d \end{bmatrix} \quad \text{for all } a, b, c, d \in F.$$

And let θ : $M_2(F) \to M_2(F)$ defined by

$$\theta\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{for all } a, b, c, d \in F.$$

Then (T,S) is a double reverse θ^* - centralizer .

Definition 3.3: Let R be a *-ring , and T , S : R →R be additive mappings , then a pair (T,S) is called a double Jordan θ^* - centralizer if T is a left Jordan θ^*-centralizer, S is a right Jordan θ^*-centralizer, and they satisfying the condition $\theta(x^*)T(x) = S(x)\theta(x^*)$, for all x ∈ R , where θ is a mapping of R .

Remark 3.4: Let R be *-ring and θ is a mapping of R .Every double reverse θ^* - centralizer is a double Jordan θ^* - centralizer, but the converse in general is not true .

Example 3.5: Let F be a field , and R be a ring of triangular matrices of the form .

$$x = \begin{bmatrix} 0 & a & c & b \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{for all } a,b,c,d \in F,\ x \in R .$$

And the involution * define by

$$*\left(\begin{bmatrix} 0 & a & c & b \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & a & c & -b \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{for all } a,b,c \in F \text{, and}$$

T , S : R →R be additive mappings defined as

$$T(x) = \begin{bmatrix} 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } S(x) = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ for all } a, b \in F \text{ and } x \in R.$$

And let $\theta: R \to R$ defined by

$$\theta(x) = \begin{bmatrix} 0 & a & c & b \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{for all } a,b,c, \in F \text{ and } x \in R.$$

Then (T,S) is a double Jordan θ^* - centralizer , but is not a double reverse θ^* - centralizer .

The converse of Remark (3.4) is true under certain conditions , as in the following theorems .

Theorem 3.6: Let R be a 2-torsion free semiprime *- ring , then every double Jordan θ^* - centralizer is a double reverse θ^* - centralizer , where θ is an automorphism of R .

Proof: Using Theorem (2.11) , we have T is a left reverse θ^* - centralizer, S is a right reverse θ^* - centralizer. We are done if we can show that $\theta(x^*)T(y) = S(x)\,\theta(y^*)$, for all $x,y \in R$, by hypotheses

$\theta(x^*)T(x) = S(x)\,\theta(x^*)$, for all $x \in R$ (1)

If we substitute x+y for x in (1) , we get

$\theta(x^*)T(y) + \theta(y^*)T(x) = S(x)\,\theta(y^*) + S(y)\,\theta(x^*)$, for all $x,y \in R$. (2)

Replace yz by y in (2) we arrive at

$\theta(x^*)T(z)\,\theta(y^*) + \theta(z^*)\,\theta(y^*)\,T(x) = S(x)\,\theta(z^*)\,\theta(y^*) + \theta(z^*)\,S(y)\,\theta(x^*)$,

for all $x,y,z \in R$. (3)

Yields

$(\theta(x^*)T(z) - S(x)\,\theta(z^*))\,\theta(y^*) = \theta(z^*)\,(S(y)\,\theta(x^*) - \theta(y^*)T(x))$,

for all $x,y,z \in R$. (4)

Putting z = x in (4) , we get

$\theta(x^*)\,(S(y)\,\theta(x^*) - \theta(y^*)T(x)) = 0$, for all $x,y \in R$. (5)

Replace y by yz in (5) , we get

$\theta(x^*)\,\theta(z^*)\,(S(y)\,\theta(x^*) - \theta(y^*)T(x)) = 0$, for all $x,y,z \in R$. (6)

We have , therefore

$\theta(x^*)\,R\,(S(y)\,\theta(x^*) - \theta(y^*)T(x)) = 0$, for all $x,y \in R$. (7)

If we substitute x + w for x in (7) , we get

$(\theta(x^*) + \theta(w^*))\,R\,(S(y)\,\theta(x^*) + S(y)\,\theta(w^*) - \theta(y^*)T(x) - \theta(y^*)T(w)) = 0$,

for all $x,y,w \in R$.

Implies that

$\theta(w^*)$ R $(S(y)\,\theta(w^*) - \theta(y^*)T(w)) + \theta(w^*)$ R $(S(y)\,\theta(x^*) - \theta(y^*)T(x)) +$

$\theta(x^*)$ R $(S(y)\,\theta(x^*) - \theta(y^*)T(x)) + \theta(x^*)$ R $(S(y)\,\theta(w^*) - \theta(y^*)T(w)) = 0$,

for all $x,y,w \in R$. (8)

Yields

$\theta(x^*)$ R $(S(y)\,\theta(w^*) - \theta(y^*)T(w)) + \theta(w^*)$ R $(S(y)\,\theta(x^*) - \theta(y^*)T(x)) = 0$,

for all $x,y,w \in R$. (9)

Implies that

$\theta(x^*)$ R $(S(y)\,\theta(w^*) - \theta(y^*)T(w)) = - \theta(w^*)$ R $(S(y)\,\theta(x^*) - \theta(y^*)T(x))$,

for all $x,y,w \in R$. (10)

Right multiplication (10) by R $\theta(x^*)$ R $(S(y)\,\theta(w^*) - \theta(y^*)T(w))$, we obtain

$\theta(x^*)$ R $(S(y)\,\theta(w^*) - \theta(y^*)T(w))$ R $\theta(x^*)$ R $(S(y)\,\theta(w^*) - \theta(y^*)T(w)) = - \theta(w^*)$ R $(S(y)\,\theta(x^*) - \theta(y^*)T(x))$ R $\theta(x^*)$ R $(S(y)\,\theta(w^*) - \theta(y^*)T(w))$, for all $x,y,w \in R$.

Using relation (7) , we obtain

$\theta(x^*)$ R $(S(y)\,\theta(w^*) - \theta(y^*)T(w))$ R $\theta(x^*)$ R $(S(y)\,\theta(w^*) - \theta(y^*)T(w)) = 0$,

for all $x,y,w \in R$. (11)

By the semiprimeness of R , we get

$\theta(x^*)$ R $(S(y)\,\theta(w^*) - \theta(y^*)T(w)) = 0$, for all $x,y,w \in R$. (12)

Replace $\theta(x^*)$ by $(S(y)\,\theta(w^*) - \theta(y^*)T(w))$ in (12) , we get

$(S(y)\,\theta(w^*) - \theta(y^*)T(w))$ R $(S(y)\,\theta(w^*) - \theta(y^*)T(w)) = 0$, for all $x,y,w \in R$. (13)

Again by the semiprimeness of R , we get

$\theta(y^*)T(w) = S(y)\,\theta(w^*)$, for all $y,w \in R$.

Lemma 3.7: Let R be *- ring with an identity element . Then (T,S) is a double Jordan θ^* - centralizer if and only if T and S are of the form $T(x) = a\,\theta(x^*)$ and $S(x) = \theta(x^*)\,a$ for some fixed element $a \in R$, where θ is an automorphism of R .

Proof: Let (T,S) be a double Jordan θ^* - centralizer , then

$T(x^2) = T(x)\,\theta(x^*)$ for all $x \in R$. (1)

$S(x^2) = \theta(x^*)\,S(x)$ for all $x \in R$. (2)

$\theta(x^*)T(x) = S(x)\,\theta(x^*)$ for all $x \in R$ (3)

Replace x by x + 1 in (1) , we get

$T(x) = a\,\theta(x^*)$ for all $x \in R$. Where $a = T(1)$

Also , replace x by x + 1 in (2) , we get

$S(x) = \theta(x^*)\,b$ for all $x \in R$. Where $b = S(1)$

Now , setting x = 1 in (3) , we obtain a = b .

Thus , $T(x) = a\,\theta(x^*)$ and $S(x) = \theta(x^*)\,a$, for all $x \in R$.

To show the converse , assume that $T(x) = a\,\theta(x^*)$ and $S(x) = \theta(x^*)\,a$, then

$T(x^2) = a\,(\theta(x^*))^2 = T(x)\,\theta(x^*)$ for all $x \in R$.

Hence, T is a left Jordan θ^* - centralizer .

In similar way, we can show that S is a right Jordan θ^* - centralizer where $S(x) = \theta(x^*)\,a$, for all $x \in R$. Since $\theta(x^*)T(x) = \theta(x^*)\,a\,\theta(x^*) = S(x)\,\theta(x^*)$, therefore

(T,S) is a double Jordan θ^* - centralizer .

Lemma 3.8: Let R be *- ring with an identity element , then $T : R \to R$ is a left (right) reverse θ^* - centralizer if and only if T is of the form $T(x) = a\,\theta(x^*)$ ($T(x) = \theta(x^*)\,a$) for some fixed element $a \in R$, where θ is an automorphism of R.

Proof: Let T be a left reverse θ^* - centralizer, then

$T(xy) = T(y)\,\theta(x^*)$, for all $x,y \in R$.

Replace y by 1 we get

$T(x) = a\,\theta(x^*)$, for all $x \in R$. Where $a = T(1)$

If $T(xy) = \theta(y^*)T(x)$, for all $x,y \in R$

Replace x by 1 we get

$T(y) = \theta(y^*)\,a$, for all $y \in R$. Where $a = T(1)$

To show the converse , assume $T(x) = a\,\theta(x^*)$ for all $x \in R$. Then

$T(xy) = a\,\theta((xy)^*) = a\,\theta(y^* x^*) = a\,\theta(y^*)\,\theta(x^*) = T(y)\,\theta(x^*)$, for all $x,y \in R$.

Hence T is a left reverse θ^* - centralizer , similar we can show T is right reverse

θ^* - centralizer if $T(x) = a\,\theta(x^*)$, for all $x \in R$.

Theorem 3.9: Let R be *- ring with an identity element , then every double Jordan θ^* - centralizer is a double reverse θ^* - centralizer , where θ is an automorphism of R .

Proof: Let (T,S) be a double Jordan θ^* - centralizer , then from Lemma (3.7) , we get T and S are of the form $T(x) = a\,\theta(x^*)$ and $S(x) = \theta(x^*)\,a$ for some fixed element $a \in R$, and from Lemma (3.8), we get T is a left reverse θ^* - centralizer, S is a right reverse θ^* - centralizer , and $\theta(x^*)T(y) = \theta(x^*)\,a\,\theta(y^*) = S(x)\,\theta(y^*)$, therefore (T,S) is a double reverse θ^* - centralizer

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