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Ec-CLS-modules

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Abstract.

In this not we consider generalization of the notion of y-closed submodule and CLS-module called y-ec-closed submodule and an Ec-CLS-module respectively. And we study the properties of this kind of module. Also we study the direct sum of an Ec-CLS-module.

Keywords: y-closed submodule; Extending and CLS-modules.

Introduction:

Throughout this paper R will be a commutative ring with identity, and all modules will be unitary left R–modules. A proper submodule N of an R–module M is called an essential submodule in M, if for every nonzero submodule K of M has nonzero intersection with N [1].A submodule N of M is called closed in M, if it has no proper essential extension in M [1]. A submodule N is called ec-closed submodule if N contains essentially a cyclic submodule, i.e. there exists n∈N such that $<n>\subseteq_e N$ [2]. A submodule N is called y-closed submodule of a module M, if $\frac{M}{N}$ is nonsingular [1]. An R-module M is called anc (CS-module), if every submodule of M is essential in a direct summand of M. A Tercan introduced the following concept: An R-module M is called a CLS-module, if every y-closed submodule of M is a direct summand of M [3].

In this paper, we introduce the concept of y-ec-closed submodule of a module M, and we defined a CLS-module that every y-ec-closed submodule is a direct summand.

In section one, we defined y-ec-closed submodules and give some properties of these submodules.

In section two, we give the definition of Ec-CLS-module, we also give their properties. We prove that any direct summand of an Ec-CLS-module is an Ec-CLS-module.

In section three, we study the direct sum of an Ec-CLS-module. It is shown that if $M = M_1 \oplus M_2$, where M_1 and M_2 are an Ec-CLS-modules such that $annM_1+annM_2 = R$, then M is an Ec-CLS-module.

1. y-ec-closed submodule.

Definition (1.1): Let N be an ec-submodule of M, N is called y-ec-closed submodule of M, if $\frac{M}{N}$ is nonsingular.

Remarks and examples (1.2):

1. For any uniform R-module M is y-ec-closed submodule of M.

2. The module Z as Z-module contains only <0> and Z which are y-ec-closed submodules of Z.

3. Every y-ec-closed submodule of M is y-closed submodule.

4. Every y-ec-closed submodule of M is ec-closed submodule of M and, then it is closed. The converse is not true, for example; M = Z_6 as Z-module and $\{\bar{0}, \bar{2}, \bar{4}\}$ is ec-closed (closed) submodule of Z_6, but $\frac{Z6}{\{0,2,4\}} = \{\bar{0}, \bar{3}\}$ is not y-ec-closed.

Proposition (1.3): Let M be a nonsingular R-module, and let N be an ec-submodule of M. Then N is y-ec-closed in M, if and only if N is ec-closed submodule.

Proof: The necessity is given by Remark 1.2(4). Conversely, suppose that M is nonsingular R-module, and N is ec-closed submodule of M. If $Z(\frac{M}{N}) = \frac{A}{N}$, where A is a submodule of M with N⊆A. Hence $N \subseteq_e A$ by [1, prop.1.21, p.32], but N is ec-closed, then N is closed. Therefore, N = A and there exists n∈N such that $nR \subseteq_e N$. Thus $Z(\frac{M}{N}) = 0$, and hence N is y-ec-closed submodule of M.

Examples (1.4):

1. Every nonsingular simple R-module M is y-ec-closed of M.

2. Every nonsingular uniform R-module M has only two y-ec-closed submodule<0> and M.

Remark (1.5): Let R be an integral domain and let N be an ec-submodule of R-module M. If $\frac{M}{N}$ is torsion free, then N is y-ec-closed of M.

Proof: Since R is an integral domain and $\frac{M}{N}$ is torsion free, then $0 = T(\frac{M}{N}) = Z(\frac{M}{N})$ by

[1, p.31]. Then $\frac{M}{N}$ is nonsingular and hence N is an y-ec-closed submodule.

Proposition (1.6): Let M be an R-module and A, B be ec-submodule of M such that A⊆B then:

1. If A is an y-ec-closed of M, then A is an y-ec-closed of B.

2. If $\frac{B}{A}$ is y-ec-closed of $\frac{M}{A}$, then B is y-ec-closed submodule of M.

3. If A is y-ec-closed of M, then $\frac{M}{B}$ is singular if and only if $B \subseteq_e M$.

4. If A is y-ec-closed in B, and B isy-ec-closed of M, then A is y-ec-closed submodule of M.

Proof:

1. It is clear.

2. Let $\frac{B}{A}$ be y-ec-closed of $\frac{M}{A}$, then there exists $\frac{<x>}{A} \subseteq_e \frac{B}{A}$, where x∈B and $\frac{\frac{M}{A}}{\frac{B}{A}}$ is nonsingular. But $\frac{\frac{M}{A}}{\frac{B}{A}} \cong \frac{M}{B}$ by third isomorphism theorem), then B is y-closed and since $<x> \subseteq_e B$, thus B is y-ec-closed of M. By the same we can prove the converse.

3. By Remark 1.2(3) and [4, prop.2.1.18, p.27].

4. It is clear by Remark 1.2(4) and [4, prop.2.1.10, p.24].

Proposition (1.7): Let M be an R-module and A, B be an ec-submodules of M, then A is y-ec-closed submodule of A+B if and only if A∩B is y-ec-closed submodule of B.

Proof: Assume that A is y-ec-closed submodule of A+B, then A is y-closed submodule of A+B by Remark 1.2(3). Therefore A∩B is y-closed of B, by [4, prop.2.16, p.22]. But A is y-ec-closed, then there exists x∈Asuch that $<x> \subseteq_e A$, $<x> \cap (A \cap B) \subseteq_e A \cap B$. Hence A∩B is y-ec-closed submodule of B. The converse by the same way.

Proposition (1.8): Let M = A⊕B be an R-module, if A is an y-ec-closed submodule of M, then B is nonsingular

Proof: Assume that A is y-ec-closed of M, then A is y-closed submodule of M by Remark 1.2(3). Therefore B is nonsingular by [4, prop.2.1.7].

Proposition (1.9): Let A and B are y-ec-closed submodule of an R-module M, then A∩B is y-ec-closed of M.

Proof: Since A and B are y-ec-closed, then A and B are y-closed by Remark 1.2(3), then A∩B is y-closed by [4, prop.2.1.8]. But A and B are y-ec-closed, then there exists a∈A and b∈B such that $<a> \subseteq_e A$ and $<b> \subseteq_e B$, then $<a> \cap <b> \subseteq_e A\cap B$ by [1, prop.1.1, p.16]. Hence A∩B is y-ec-closed of M.

Proposition (1.10): Let M be an R-module, and let $\{B\alpha | \alpha \in \xi\}$ be an independent family of submodules of M. If $\{A\alpha | \alpha \in \xi\}$ is a family of submodules of M such that $A\alpha \subseteq B\alpha$, $\forall \alpha \in \xi$, then $\oplus A\alpha$ is y-ec-closed submodule of $\oplus B\alpha, \alpha \in \xi$ if and only if $A\alpha$ is y-ec-closed submodule of $B\alpha$, $\forall \alpha \in \xi$.

Proof: → by Remark 1.2(3) and [prop.2.1.20, p.24], we get if $\oplus A\alpha$ is y-ec-closed submodule of $\oplus B\alpha, \alpha \in \xi$ then $A\alpha$ is y-ec-closed submodule of $B\alpha$, $\forall \alpha \in \xi$. Since $\oplus A\alpha$is y-ec-closed, then there exists $a\alpha \in A\alpha, \forall \alpha \in \xi$, such that $\oplus < a\alpha > \subseteq_e \oplus A\alpha, \forall \alpha \in \xi$ then $< a\alpha > \subseteq_e A\alpha, \forall \alpha \in \xi$ by [1, prop.1.10]. Hence $A\alpha$ is y-ec-closed of $B\alpha$.

Conversely, let $A\alpha$be y-ec-closed of $B\alpha$, $\forall \alpha \in \xi$, then $A\alpha$ is y-ec-closed submodule of $B\alpha, \forall \alpha \in \xi$by Remark 1.2(3), then $\oplus A\alpha$ is y-ec-closed submodule of $\oplus B\alpha, \alpha \in \xi$ by [4, prop.2.1.20, p.29]. Since $A\alpha$ is y-ec-closed submodule of $B\alpha$, then there exists $a\alpha \in A\alpha, \forall \alpha \in \xi$ such that $< a\alpha > \subseteq_e A\alpha, \forall \alpha \in \xi$ then $\oplus < a\alpha > \subseteq_e \oplus A\alpha, \forall \alpha \in \xi$ by (prop.1.1(d), p.16), then $\oplus A\alpha$ is y-ec-closed submodule of $\oplus B\alpha, \alpha \in \xi$.

Proposition (1.12): Let M be an R-module and N be y-ec-closed submodule of M, then [N:M] is y-ec-closed ideal of R.

Proof: Let N be an y-ec-closed submodule of M, then N is y-closed of M by Remark 1.2(3). Thus, [N:M] is y-closed ideal of R by [, prop.2.1.21, p.30]. Since N is y-ec-closed submodule of M, then there exists n∈N such that $<n> \subseteq_e N$, $[<n> : <x>] \subseteq_e [N:M]$, ∀ x∈M by [5, prop.3.13, p.59]. Hence [N:M] is y-ec-closed of R.

2. EC-CLS- module:

In this section we introduce the concept of EC-CLS-module and discuss some of basic properties of these modules.

Definition (2.1): An R-module M is called EC-CLS-module, if every y-ec-closed submodule of M is a direct summand of M.

Remark (2.2): Every CLS-module is EC-CLS-module.

Proof: Let M be an R-module, and let A be y-ec-closed submodule of M. Then A is y-closed submodule of M by Remark 1.2(3). But M is CLS-module, then A is a direct summand of M. Therefore M is EC-CLS-module.

Remarks and Examples (2.3):

1. Z as Z-module is EC-CLS-module.

2. It is clear that Z_6 as Z_6-module is EC-CLS-module.

3. Every singular R-module is EC-CLS-module.

4. Every CS-module is EC-CLS-module, but the converse is not true in general. For example, consider the module $M = Z_8 \oplus Z_2$ as Z-module. Since Z_8 and Z_2 are singular, then M is singular by [1, prop.1.22, p.32] and hence M is CLS-module. Therefore M is EC-CLS-module by Remark 2.2. But M is not CS-module by [6, p.56].

5. Every nonsingular finite uniform dimension, then M is EC-CLS-R-module if R is CS-module.

Proof: Let A be any maximal uniform submodule of M, clearly A is an ec-closed submodule in M. But M is nonsingular, then A is y-ec-closed of M. Since M is EC-CLS-module, then A is a direct summand of M. Hence M is CS. The converse is clear by the above Remark

Lemma (2.4): Any direct summand of an EC-CLS-module is an EC-CLS-module.

Proof: Let M = A⊕B be an EC-CLS-module. To show that A is EC-CLS-module, let K be an y-ec-closed submodule of A. By third and second isomorphism theorems, we have $\frac{M}{K\oplus B} = \frac{A\oplus B}{K\oplus B} \cong \frac{\frac{A\oplus B}{B}}{\frac{K\oplus B}{B}} \cong \frac{\frac{A}{A\cap B}}{\frac{K}{K\cap B}} = \frac{A}{K}$. Since K is y-ec-closed of A, then $\frac{A}{K}$ is nonsingular. Thus, K⊕B is an y-ec-closed submodule of M. But M is EC-CLS-module, therefore K⊕B is a direct summand of M. So, M = K⊕B⊕D for some D a submodule of M. Since K is a direct summand of M and K⊆A, then K is a direct summand of A.

Proposition (2.5): Every y-ec-closed of an EC-CLS-module is an EC-CLS-module.

Proof: Let M be an EC-CLS-module, and N be y-ec-closed submodule of M. Let A ne y-ec-closed submodule of N. Then by prop.1.5(4), A is y-ec-closed submodule of M. But M is an EC-CLS-module, therefore A is a direct summand of M. Hence A is a direct summand of N.

Proposition (2.6): Let A and B be submodules of an R-module M, if B is EC-CLS-module and A is an y-ec-closed submodule of M, then A∩B is a direct summand of B.

Proof: Assume that A is y-ec-closed of M, and B is EC-CLS-module. By the second isomorphism theorem $\frac{A}{A\cap B} \cong \frac{A+B}{B}$. Since $\frac{A+B}{A} \subseteq \frac{M}{A}$, then $\frac{A+B}{A}$ is nonsingular, and hence A∩B is y-ec-closed submodule of B. But B is EC-CLS-module, therefore A∩B is a direct summand of B.

Proposition (2.7): Let A be a submodule of an R-module M, if M is an EC-CLS-module, then $\frac{M}{A}$ is an EC-CLS-module.

Proof: Let $\frac{B}{A}$ be an y-ec-closed submodule of $\frac{M}{A}$, then by prop.1.5(2) B is y-ec-closed submodule of M. But M is EC-CLS-module, then B is a direct summand of M. Thus M = B⊕K, for some submodule K of M. Since A⊆B, then $\frac{M}{A} = \frac{B}{A} \oplus \frac{K+A}{A}$, thus $\frac{M}{A}$ is EC-CLS-module.

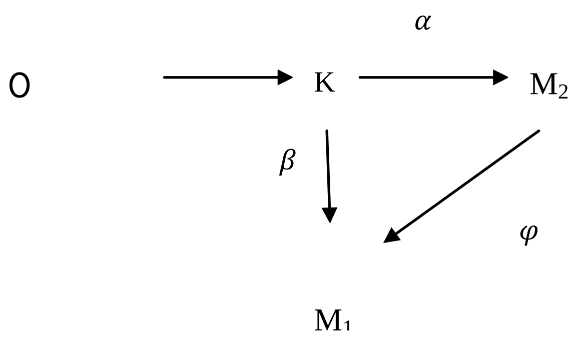
3. The direct sums of EC-CLS-modules.

A direct sum of EC-CLS-modules need not EC-CLS-module in general. Hence, we look for conditions under which this property is valid.

Example (3.1): Each of Z_2 and Z_8 are EC-CLS-modules, but M = $Z_2 \oplus Z_8$ is not EC-CLS-module.

Theorem (3.2): Let M = $M_1 \oplus M_2$ be an R-module such that M_1 is M_2-injective, where M_1 and M_2 are EC-CLS-modules, then M is an EC-CLS-module.

Proof: Let A be an y-ec-closed submodule of M, then $\frac{M}{A}$ is nonsingular. By the second isomorphism theorem $\frac{M1}{A\cap M1} \cong \frac{M1+A}{A} \subseteq \frac{M}{A}$. So, A∩$M_1$ is an y-ec-closed submodule of M_1. ButM_1 is EC-CLS-module, therefore A∩M_1 is a direct summand of M_1. Hence A∩M_1 is a direct summand of M. It follows that A∩M_1 is a direct summand of A, then A = (A∩M_1) ⊕ K, for submodule K of A. Let π_i: M→M_i, i=1,2 be the projective maps. Now, consider the following diagram.



Where $\alpha = \pi_2|_K$ and $\beta = \pi_1|_K$. Since α is a monomorphism and M_1 is M_2-injective, then there exists a homomorphism φ: $M_2 \to M_1$ such that $\varphi \circ \alpha = \beta$. Let L = {x + φ(x): x∈M_2}. One can easily check that L is a submodule of M and L≅M_2. Moreover, M = M_1⊕L. To show that, let x∈M = $M_1 \oplus M_2$, then x = m_1+m_2, where $m_1 \in M_1$ and $m_2 \in M_2$. Thus x =$m_1+m_2 + \varphi(m_2)- \varphi(m_2) = (m_1- \varphi(m_2))+((m_2+ \varphi(m_2)) \in M_1+L$. Now, let x∈$M_1$+L. Since x∈L, then x = y+ φ(y), y∈M_2, thus y∈$M_1 \cap M_2$ = 0 and then x = 0. Now, let k∈K, then k = m_1+m_2, for some,$m_1 \in M_1$ and $m_2 \in M_2$. Then $m_1= \beta(k) = \varphi \circ \alpha(k) = \varphi(m_2)$. This implies that k =$m_2 + \varphi(m_2)$ ∈L. Thus k⊆L. Since $\frac{M}{A} = \frac{M1 \oplus L}{(A \cap M1) \oplus K} \cong \frac{M1}{(A \cap M1)} \oplus \frac{L}{K}$, then $\frac{L}{K}$ is nonsingular, and K is an y-ec-closed submodule of L. But L≅M_2 and M_2 is EC-CLS-module, then K is a summand of L. Thus L = K⊕D, for some submodule D of L. Now, since A∩M_1 is a direct summand of M_1, then M_1 = (A∩M_1) ⊕B, for some B of M_1. So, M =M_1⊕L = (A∩M_1) ⊕B ⊕K⊕D = A⊕B⊕D, then A is a direct summand of M. Hence M is EC-CLS-module.

Proposition (3.3): Let R be a ring and M be an R-module such that M = ⊕M_i , (i=1,…,n)is finite direct sum of relatively modules M_i,(i=1,…,n). Then M is EC-CLS-module if and only if M_i is an EC-CLS-module for each(i=1,…,n).

Proposition (3.4): LetM_1 and M_2 be EC-CLS-modules such that annM_1+ annM_2 = R, then M = $M_1 \oplus M_2$ is EC-CLS-module.

Proof: Let A be an y-ec-closed submodule of $M_1 \oplus M_2$. Since annM_1+ annM_2 = R, then by the same way of the prove [7, prop.2.2, ch.2], A = C⊕D, where C is a submodule of$M_1$and D is a submodule ofM_2. Since A =C ⊕D is y-ec-closed submodule of M = $M_1 \oplus M_2$, then C is y-ec-closed submoduleM_1andD is y-ec-closed submoduleM_2 by prop.1.11. But $M_1$and M_2 are EC-CLS-modules, then C is summand of M_1 and D is a summand of M_2. So, A = C ⊕D is a summand of M = $M_1 \oplus M_2$. Hence M is an EC-CLS-module.

Proposition (3.5): Let M = ⊕M_i, (i=1,…,n) be an R-module such that every y-ec-closed submodule of M is fully invariant, then M is an EC-CLS-module if and only if M_i is EC-CLS-module,(i=1,…,n).

Proof: → Clear by lemma 2.4.

← Let A be an y-ec-closed submodule of M. For each (i=1,…,n), let π_i: M→M_i, be the projection map. Now, let x∈A, then x = $\sum_{i=0}^{n}$ mi , $m_i \in M_i$and m_i = 0, for all except a finite of i = 1,…,n). Clearly that $\pi_i(x) = m_i \forall$ i∈I Since A isis y-ec-closed submodule then by our assumption A is fully invariant and hence $\pi_i(x) = m_i \in A \cap M_i$. So, x∈ ⊕(A∩$M_i$), thus A ⊆ ⊕(A∩$M_i$). But ⊕(A∩$M_i$) ⊆ A_i, then ⊕(A∩M_i)= A_i, since A is y-ec-closed submodule of M, therefore (A∩M_i) is y-ec-closed submodule of M_i ,∀ i, (i=1,…,n), by prop.1.8. But M_i is EC-CLS-module ∀ i, (i=1,…,n), then (A∩M_i) is a direct summand ofM_i. Thus, A is a direct summand of M.

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